Interferometric waveform inversion

– or –

Geophysical imaging meets spectral graph theory

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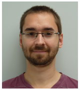
October 2012 – Princeton, Bridging the gap

Where does it fit?

I.
II.
III.
IV. Inverse problems / optimization
  Recovery principles from indirect data
  Dimensionality reduction
  Convex and semidefinite relaxation, regularization
Va.
Vb.

Why is CMG++ timely?



Interferometric waveform inversion
w/ Vincent Jugnon

Model-robust imaging, linearized reflection regime

Problem: find reflectors $m(x)$ in

$$\frac{1}{c_0^2(x)}\frac{\partial^2 u}{\partial t^2} - \Delta u = m(x)\frac{\partial^2 u_0}{\partial t^2},$$

when $c_0(x)$ is poorly known. Data: $u(r, t; s)$ with good angular coverage.

- Typical approach: optimize over $c_0(x)$
  ... *lacks convexity*

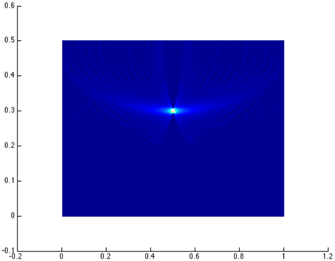
Interferometric waveform inversion

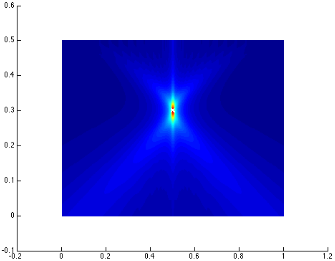
- Goal: robust imaging without inverting for $c_0(x)$
- Idea: use *correlograms* in an optimization framework

	Traveltime	Waveform
Absolute	τ_r	$d_r(\omega)$
Relative	$\tau_{r_1} - \tau_{r_2}$	$d_{r_1}(\omega)\overline{d_{r_2}(\omega)}$

Interferometric waveform inversion

Rationale: robustness to errors in $c_0(x)$, e.g., shift

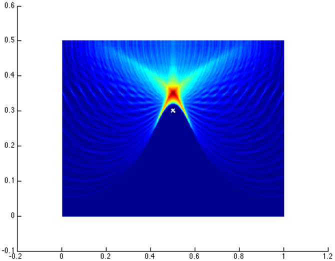




$c_0 = 1$ (exact)

Interferometric waveform inversion

Rationale: robustness to errors in $c_0(x)$, e.g., shift



$c_0 = .9 \qquad (10\% \text{ error})$

Interferometric waveform inversion

Linearized inversion for $m(x)$:

- Absolute times:

$$\min_m \sum_i |d_i - (Fm)_i|^2 \qquad i = (r, s, \omega)$$

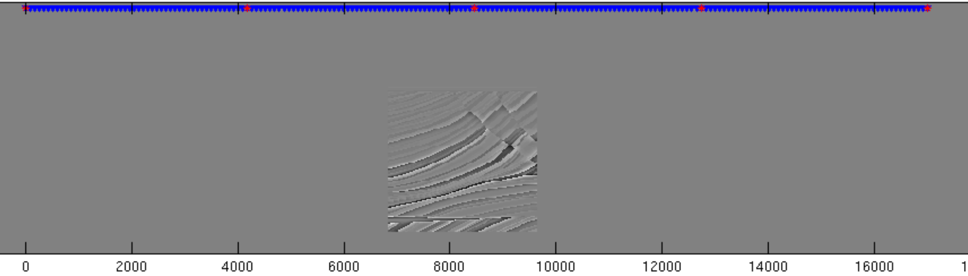
$\Rightarrow m = F^{-1} d$ (inversion) or $m = F^* d$ (migration).

- Relative times:

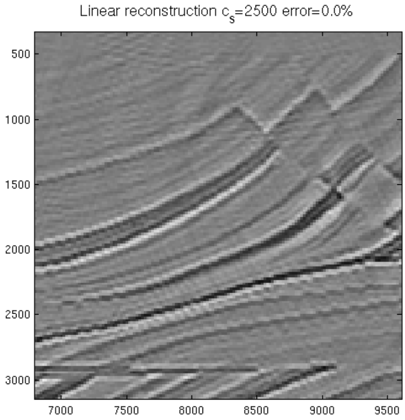
$$\min_m \sum_{i,j} |d_i \overline{d_j} - (Fm)_i \overline{(Fm)_j}|^2, \qquad i = (r, s, \omega),\ j = (r', s', \omega')$$

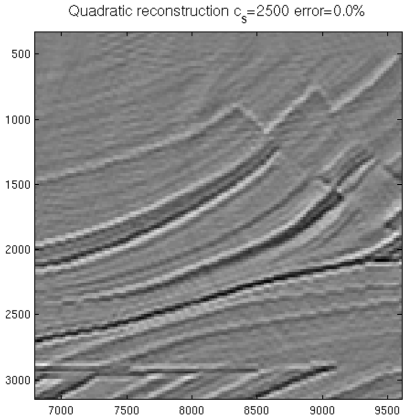
Linear vs. interferometric inversion

- Linear data model, uniform medium
- Wide-aperture array
- Noiseless, but fixed regularization level

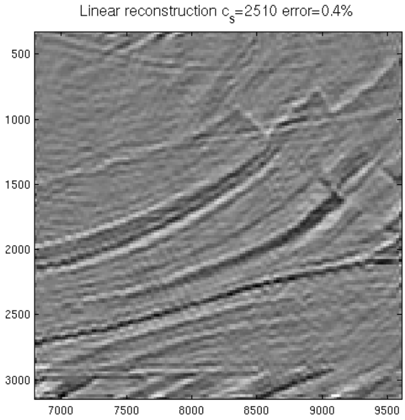


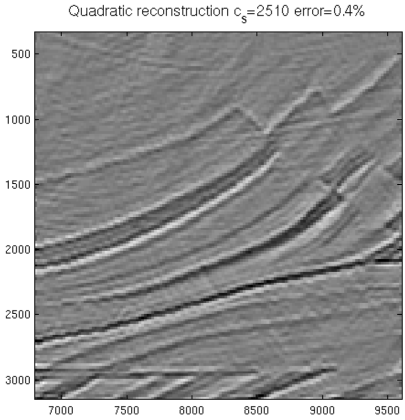
Linear vs. interferometric inversion





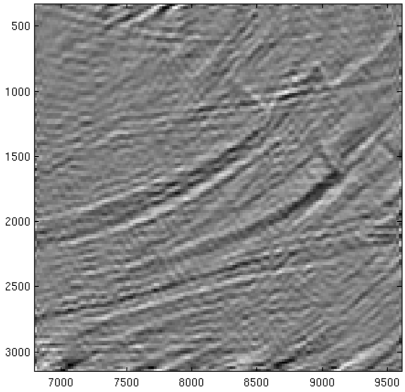
Linear vs. interferometric inversion



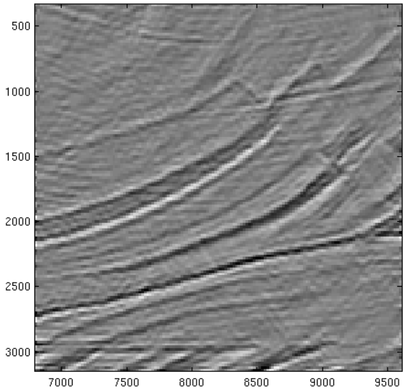


Linear vs. interferometric inversion

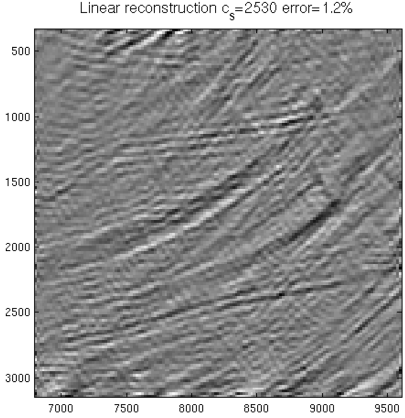
Linear reconstruction c_s=2520 error=0.8%

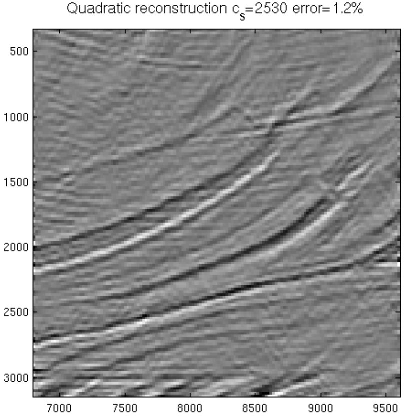


Quadratic reconstruction c_s=2520 error=0.8%

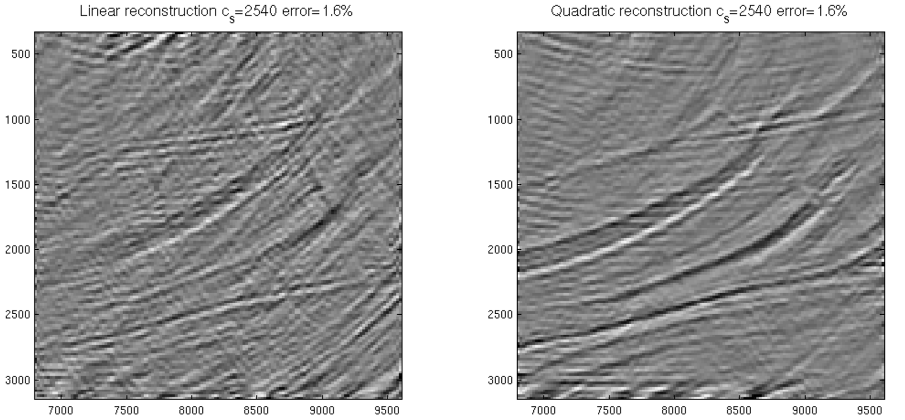


Linear vs. interferometric inversion

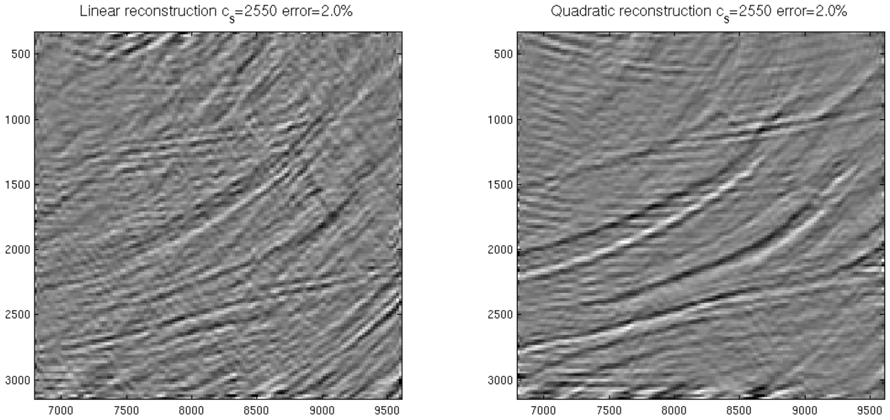




Linear vs. interferometric inversion



Linear vs. interferometric inversion



Issues with interferometric inversion

$$\min_m \sum_{i,j} |d_i \overline{d_j} - (Fm)_i \overline{(Fm)_j}|^2$$

Problems:

1. Quartic objective with spurious local minima
2. Too many products: N^2 vs. N.

Convexification by relaxation

Lifting:

- New unknown: $M = m \otimes m$, i.e., $M(\mathbf{x}_1, \mathbf{x}_2) = m(\mathbf{x}_1)m(\mathbf{x}_2)$.
- Constraint linear in M:

$$(Fm)_i\overline{(Fm)_j} = L(M)_{ij}$$

- Quadratic objective, convex:

$$\min_M \sum_{i,j} |d_i\overline{d_j} - L(M)_{ij}|^2$$

- M is rank 1; get m as leading eigenvector of M.

Convexification by relaxation

Semidefinite relaxation:

- Underdetermined system: keep a small subset of $d_i\overline{d_j}$.
- Constraint search to $M \succeq 0$.
- Enforce low-rank solution: NP-hard problem

$$\min_{M\succeq 0} \sum_{i,j} |d_i\overline{d_j} - L(M)_{ij}|^2 + \lambda\,\mathrm{rank}(M)$$

- Trace relaxation: convex problem

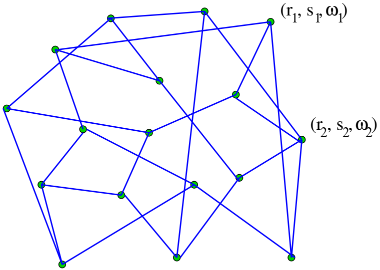
$$\min_{M\succeq 0} \sum_{i,j} |d_i\overline{d_j} - L(M)_{ij}|^2 + \lambda\,\mathrm{trace}(M)$$

Interferometric waveform inversion

How many products $d_i \overline{d_j}$ are needed for well-posedness?

Define a graph $G = (V, E)$ where

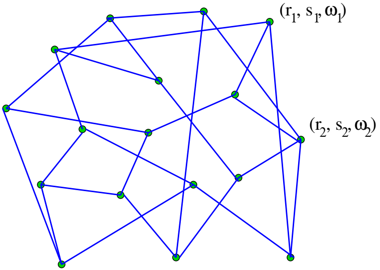
- Each data point $d_i = d_{r,s,\omega}$ is a vertex i in V,
- Each active $d_i \overline{d_j}$ is an edge $(i, j) \in E$.



Matrix of magnitudes: $D_{ij} = |d_i||d_j|$ if $(i, j) \in E$, zero otherwise.
(Weighted adjacency matrix.)

Interferometric waveform inversion

Necessary: G is connected, and all its vertices have loops.
Sufficient: in addition, G should be very well connected.



Definition

G is an expander graph if its adjacency matrix $A(G)$ has a large spectral gap:

$$\lambda_1 - \lambda_2$$

is a non-negligible fraction of $\lambda_1 - \lambda_N$.

Interferometric waveform inversion

Let $\lambda_1 > \lambda_2 > \ldots$ be the eigenvalues of D.

Theorem

Consider noiseless data $d_i \overline{d_j} = L(M_0)_{ij}$, $(i,j) \in E$, and $M_0 = m_0 \otimes m_0$. Assume the forward map F is invertible. Then any method that imposes $d_i \overline{d_j} = L(M)_{ij}$ with $M \succeq 0$ returns a model m obeying

$$\|m - m_0\|^2 \leq C \frac{\lambda_1 - \mu}{\lambda_1 - \lambda_2} \|m_0\|^2, \qquad \mu = \frac{\lambda_1^2 + \lambda_2^2 + \ldots}{\lambda_1 + \lambda_2 + \ldots}$$

Eigenvector method (Amit Singer, 2012): $m = m_0$.

$\Rightarrow$ A good expansion property of G is sufficient for well-posedness of linearized interferometric inversion.