Inverse Problems for Scanning Magnetic Microscopy

Laurent Baratchart[1], Doug Hardin[2], Eduardo Lima[3], Ed Saff[2] and Ben Weiss[3]

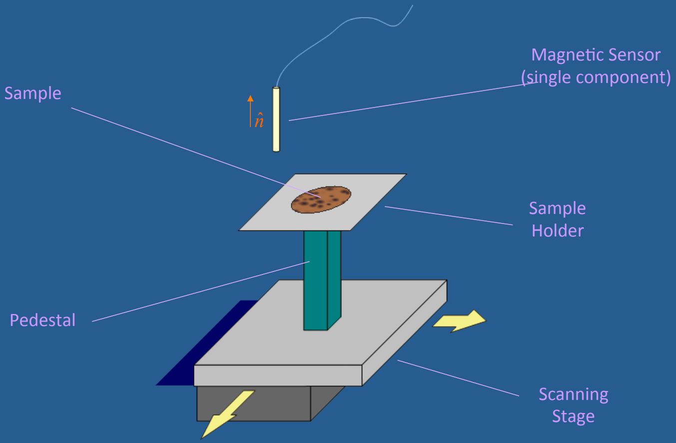
[1]INRIA-Sophia Antipolis

[2]Department of Mathematics
Vanderbilt University

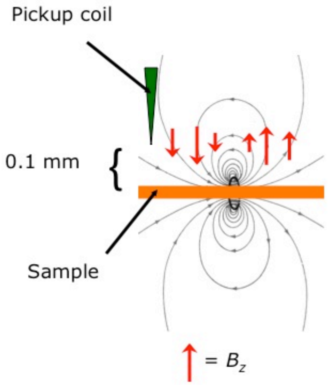
[3]Department of Earth, Atmospheric and Planetary Sciences
Massachusetts Institute of Technology

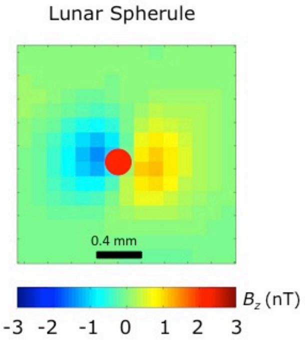
Bridging the Gap Workshop, Princeton, October 1–2, 2012

Scanning Magnetic Microscope



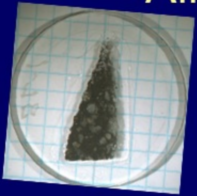
What the SQUID Microscope Measures

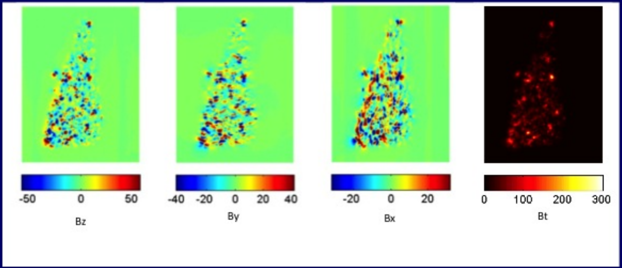




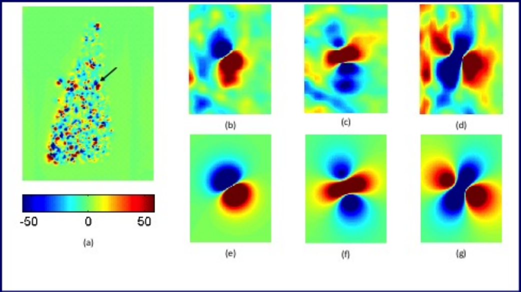
Moment=10^{-13} Am^2 !!

Allende Meteorite





Dipolar Features in Allende



Constitutive Relations

- Given a quasi-static $\mathbf{R}^3$-valued magnetization $\mathbf{M}$,
- the magnetic-flux density $\mathbf{B}$ and the magnetic field $\mathbf{H}$ satisfy

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}). \tag{1}$$

- Maxwell's equations give $\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{0}$ and $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$.
- Hence $\mathbf{H} = -\boldsymbol{\nabla}\phi$ where ϕ is the *magnetic scalar potential*, and taking divergence in (1)

$$\Delta\phi = \boldsymbol{\nabla} \cdot \mathbf{M} \tag{2}$$

Potentials and Magnetizations

- which can be recast in the form

$$\phi(\mathbf{r}) = \frac{1}{4\pi} \iiint \frac{\mathbf{M}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d\mathbf{r}', \quad \mathbf{r} \notin \text{supp } \mathbf{M}. \qquad (3)$$

- SCA: Assume support of $\mathbf{M}$ is a distribution of the form

$$\mathbf{M}(\mathbf{x}, z) = \mathbf{m}(\mathbf{x})\delta_0(z) =: (\mathbf{m}_T(\mathbf{x}), m_3(\mathbf{x}))\delta_0(z),$$

where $\mathbf{m}_T = (m_1, m_2)$ and $m_1, m_2, m_3 \in L^p(\mathbf{R}^2)$. Then

$$\phi(\mathbf{x}, z) = \frac{1}{4\pi} \iint \left(\frac{\mathbf{m}_T(\mathbf{x}') \cdot (\mathbf{x} - \mathbf{x}')}{(|\mathbf{x} - \mathbf{x}'|^2 + z^2)^{3/2}} + \frac{m_3(\mathbf{x}')z}{(|\mathbf{x} - \mathbf{x}'|^2 + z^2)^{3/2}} \right) d\mathbf{x}',$$

for all $(\mathbf{x}, z)$ such that either $z \neq 0$ or $\mathbf{x} \notin$ supp. $\mathbf{m}$.

Thin plate potentials as convolutions

Then

$$\begin{aligned}\phi(\mathbf{x}, z) &= \frac{1}{2}\left(\mathbf{H}_z * \mathbf{m}_T(\mathbf{x}) + P_z * m_3(\mathbf{x})\right) \\ &= \frac{1}{2} P_{|z|} * \left(R_1(m_1) + R_2(m_2) + \frac{z}{|z|} m_3\right)(\mathbf{x}),\end{aligned}$$

where

$$P_z(\mathbf{x}) := \frac{1}{2\pi}\frac{z}{(|\mathbf{x}|^2 + z^2)^{3/2}}, \quad \mathbf{H}_z(\mathbf{x}) := \frac{1}{2\pi}\frac{\mathbf{x}}{(|\mathbf{x}|^2 + z^2)^{3/2}}$$

and R_1 and R_2 are *Riesz transforms*.

Riesz transforms

- For $f \in L^p(\mathbf{R}^2)$, $p \in (1, \infty)$, the *Riesz transforms of f*, denoted by $R_1(f)$ and $R_2(f)$, are defined by

$$R_j(f)(\mathbf{x}) := \lim_{\epsilon \to 0} \frac{1}{2\pi} \iint_{\mathbf{R}^2 \setminus B(\mathbf{x},\epsilon)} f(\mathbf{x}') \frac{(x_j - x_j')}{|\mathbf{x} - \mathbf{x}'|^3} \, d\mathbf{x}'. \qquad (4)$$

- The limit (4) exists a.e. and R_j continuously maps $L^p(\mathbf{R}^2)$ into itself.
- In the Fourier domain $\widehat{R_j f}(\boldsymbol{\kappa}) = -i \frac{\kappa_j}{|\boldsymbol{\kappa}|} \hat{f}(\boldsymbol{\kappa})$.

Silent sources

- A magnetization $\mathbf{m}$ is *silent from above* (resp. *below*) if it is equivalent from above (resp. below) to the null magnetization. It is *silent* if it is silent from above and below.
- Since the Poisson transform is injective, the magnetization $\mathbf{m}$ is silent from above if and only if $R_1(m_1) + R_2(m_2) + m_3 = 0$ and silent from below if and only if $R_1(m_1) + R_2(m_2) - m_3 = 0$.
- Hence, $\mathbf{m}$ is silent if and only if $R_1(m_1) + R_2(m_2) = 0$ and $m_3 = 0$, i.e., if and only if $m_3 = 0$ and $\mathbf{m}_T$ is divergence free.

The Hardy-Hodge decomposition

Let

$H^+ := \{(R_1(f), R_2(f), f) : f \in L^p\}$ and $H^- := \{(-R_1(f), -R_2(f), f) : f$

Theorem

For $p > 1$, we have the following direct sum:

$$\left(L^p(\mathbf{R}^2)\right)^3 = H^+ \oplus H^- \oplus \mathrm{S}.$$

The sum is orthogonal sum when $p = 2$.
Specifically, $\mathbf{m} = (m_1, m_2, m_3) = P_{H^+}(\mathbf{m}) + P_{H^-}(\mathbf{m}) + P_{\mathrm{S}}(\mathbf{m})$,
where

$$P_{H^+}(\mathbf{m}) = \left(R_1(m^+), R_2(m^+), m^+\right), \quad 2m^+ := -\Sigma_{j=1}^2 R_j(m_j) + m_3$$

$$P_{H^-}(\mathbf{m}) = \left(-R_1(m^-), -R_2(m^-), m^-\right), \; 2m^- := \Sigma_{j=1}^2 R_j(m_j) + m_3$$

$$P_{\mathrm{S}}(\mathbf{m}) = \left(-R_2(d), R_1(d), 0\right), \quad d := R_2(m_1) - R_1(m_2).$$

Equivalence via Hardy-Hodge decomposition

Theorem

Let $\mathbf{m} \in \left(L^p(\mathbf{R}^2)\right)^3$.

- *The magnetization* $\mathbf{m}$ *is silent from above (resp. below) if and only if* $P_{H^-}(\mathbf{m}) = 0$ *(resp.* $P_{H^+}(\mathbf{m}) = 0$*).*
- *The magnetization* $\mathbf{m}$ *is silent from above and below if and only if it belongs to* S*; that is, if and only if* $\mathbf{m}_T$ *is divergence-free and* $m_3 = 0$.
- *If* $\operatorname{supp} \mathbf{m} \neq \mathbf{R}^2$*, then* $\mathbf{m}$ *is silent from above if and only if it is silent from below.*

Unidirectional Magnetizations

- A magnetization $\mathbf{m}$ is *unidirectional* if $\mathbf{m} = Q\mathbf{u}$ for some fixed $\mathbf{u} \in \mathbf{R}^3$ and some $Q \in L^p(\mathbf{R}^2)$.
- Unidirectional magnetizations occur naturally for materials formed in a uniform external magnetic field.

Unidirectional Magnetizations

Theorem

- *A unidirectional magnetization* $\mathbf{m} \in \left(L^p(\mathbf{R}^2)\right)^3$ *is determined uniquely by its direction and the field it generates from above (or below). In particular,* $\mathbf{m}$ *is silent from above (or below) if, and only if* $\mathbf{m} = 0$.
- *For* $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{R}^3$ *with* $u_3 \neq 0$, *any magnetization in* $\left(L^p(\mathbf{R}^2)\right)^3$ *is equivalent from above to a unidirectional magnetization of the form* $Q(\mathbf{x})\mathbf{u}$.
- *A compactly supported unidirectional magnetization is equivalent from above (or below) to no other compactly supported unidirectional magnetization.*

Compactly supported bidirectional silent sources

Theorem

Suppose $\mathbf{m}(\mathbf{x}) = Q(\mathbf{x})\mathbf{u} + R(\mathbf{x})\mathbf{v}$ *where* $\mathbf{u} = (u_1, u_2, u_3)$ *and* $\mathbf{v} = (v_1, v_2, v_3)$ *are nonzero vectors in* $\mathbf{R}^3$ *while* Q, R *are distributions with compact support.*

1. *If* u_3 *or* v_3 *is nonzero, then* $\mathbf{m}$ *is silent iff* $\mathbf{m} = \mathbf{0}$.
2. *If* $u_3 = v_3 = 0$, *then* $\mathbf{m}$ *is silent iff* $\mathbf{m}_T(\mathbf{x}) = Q(\mathbf{x})(u_1, u_2) + R(\mathbf{x})(v_1, v_2)$ *is divergence free.*

L-R: Ed Saff, Ben Weiss, Laurent Baratchart, Doug Hardin at Ben and Eduardo's (taking the picture) lab at MIT.

GeoMath Wish List

GeoMath Wish List

CMG!

Summary: Generating good point sets

- **Given**: a d-rectifiable set A with $\mathcal{H}_d(A) > 0$ that is contained in a d-regular set and a positive and continuous density $\rho(x)$ on A.
- To distribute points on A according to ρ, choose $s > d$ and

$$w(x, y) := (\rho(x)\rho(y))^{-s/2d},$$

- Compute configurations that (nearly) minimizes the weighted s-energy:

$$E_s^w(\{x_1, x_2, ..., x_N\}) := \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{w(x_i, x_j)}{|x_i - x_j|^s}$$

- Any sequence of such configurations will have limiting distribution ρ and is quasi-uniform.

Summary: Generating good point sets

- **Given**: a d-rectifiable set A with $\mathcal{H}_d(A) > 0$ that is contained in a d-regular set and a positive and continuous density $\rho(x)$ on A.
- To distribute points on A according to ρ, choose $s > d$ and

$$w(x, y) := (\rho(x)\rho(y))^{-s/2d},$$

- Compute configurations that (nearly) minimizes the weighted s-energy:

$$E_s^w(\omega_N) := \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{w(x_i, x_j)}{|x_i - x_j|^s} \Phi\left(\frac{|x_i - x_j|}{r_N}\right)$$

- Any sequence of such configurations will have limiting distribution ρ and is quasi-uniform.

Spherical Shell - for geoscience models

500K points in spherical shell
$.55 < r < 1$, **'low'**
$s = 3.5$ **energy**