Maximum-likelihood Theory for the Inversion of Gravity and Topography

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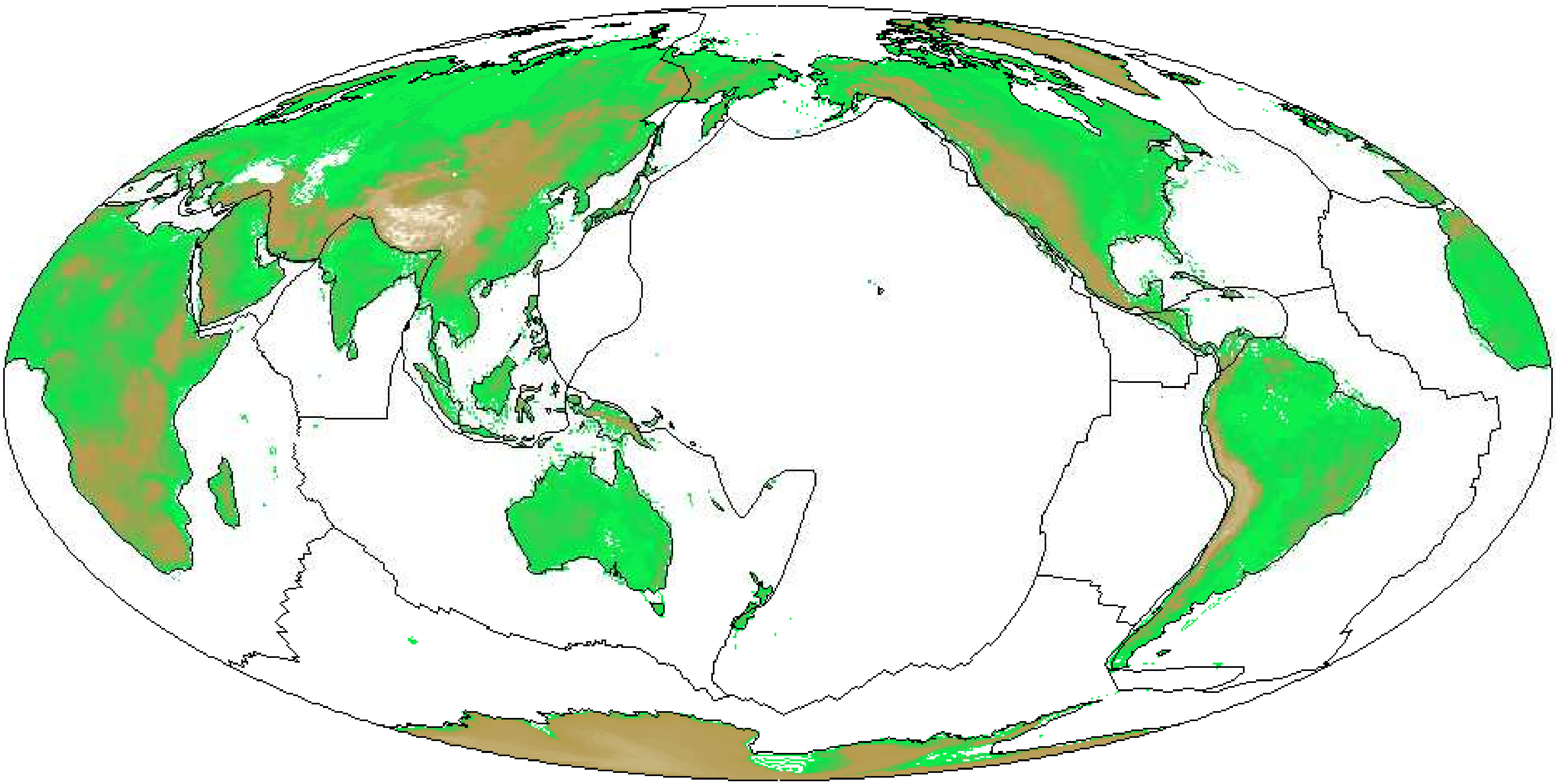
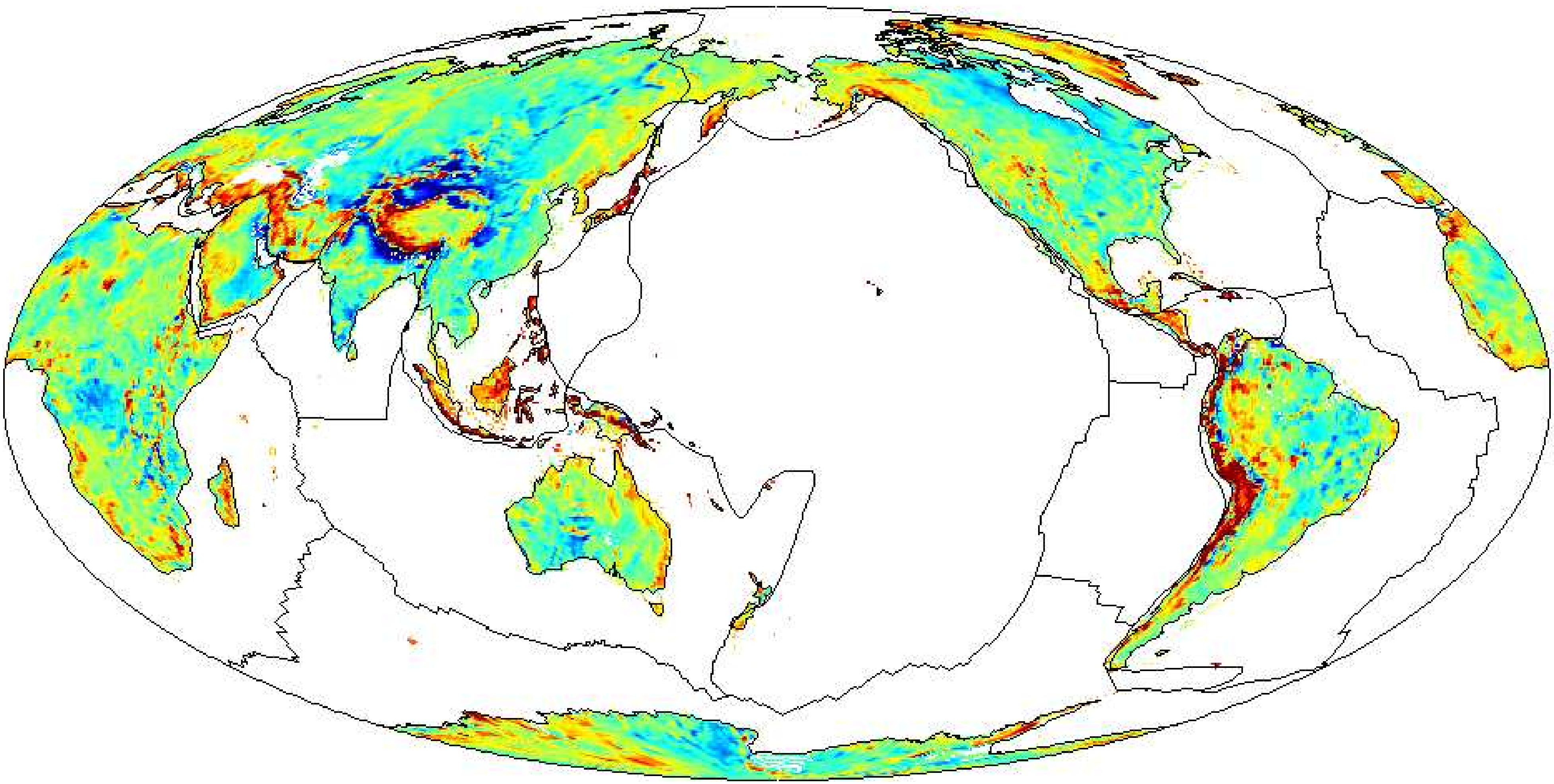
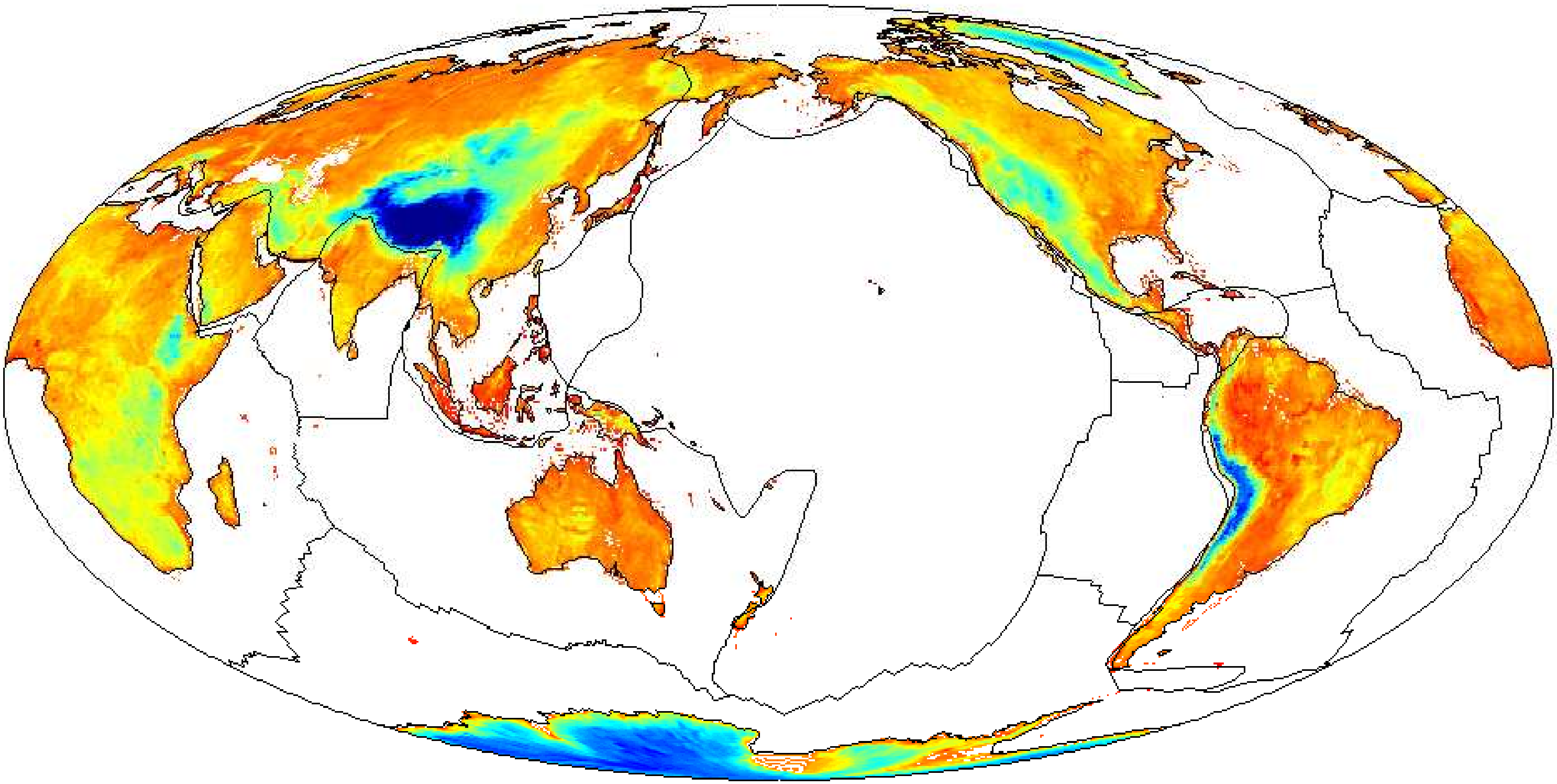
Topography



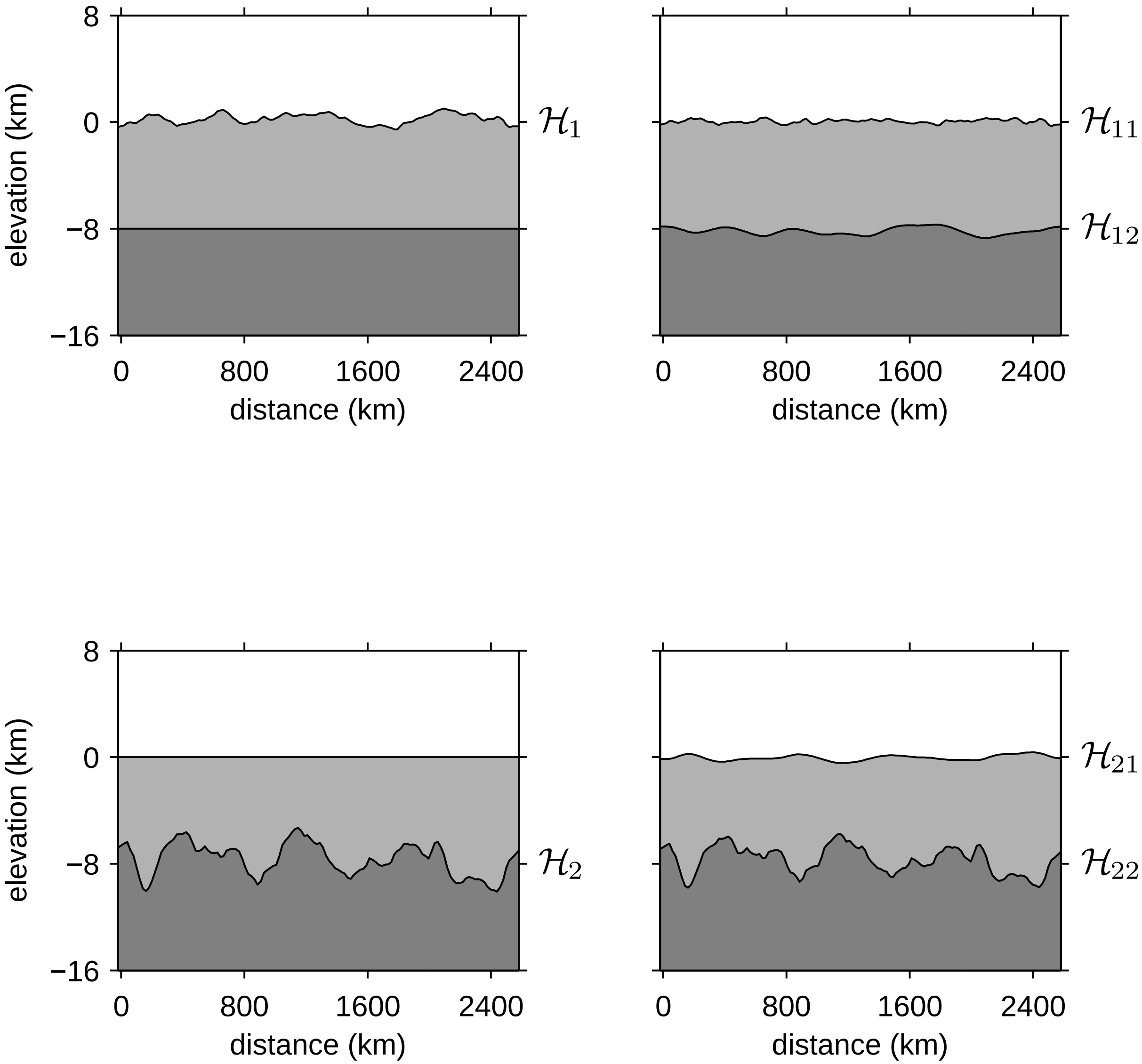
The free-air gravity anomaly

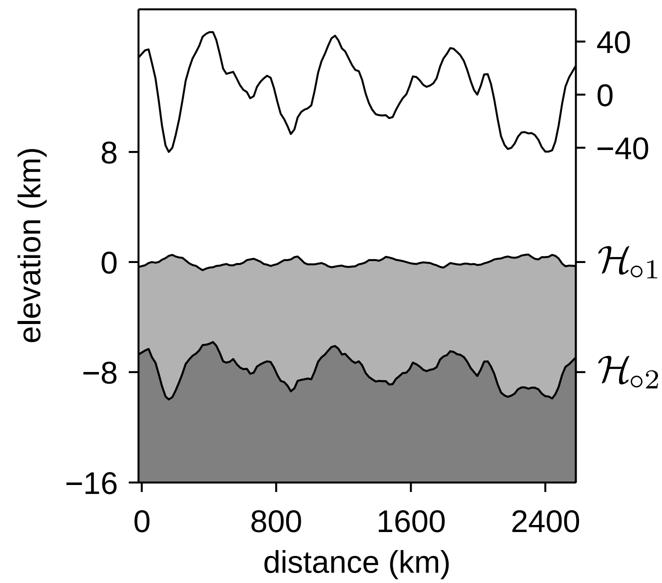


The Bouguer-air gravity anomaly



The standard model





D = 7e+22 ; f^2 = 0.4

σ^2 = 0.0025

ν = 2

ρ = 20000

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Surface loading

Surface-loading topography $\mathcal{H}_{11}$ is in instantaneous **elastic** balance with the subsurface topography $\mathcal{H}_{12}$, according to the **biharmonic equation**:

$$\left(\nabla^4 + \frac{g\Delta_2}{D}\right)\mathcal{H}_{12}(\mathbf{x}) = -\frac{g\Delta_1}{D}\mathcal{H}_{11}(\mathbf{x}). \tag{1}$$

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We want to find D, the **flexural rigidity**.

Turcotte & Schubert (1982)

Gravity anomalies in the spectral domain

Thus the **free-air anomaly** $d\mathcal{G}_{ij}(\mathbf{k})$ due to the topographic perturbation $d\mathcal{H}_{ij}(\mathbf{k})$, at the jth interface resulting from the ith loading process, is given by

$$d\mathcal{G}_{ij}(\mathbf{k}) = 2\pi G \Delta_j d\mathcal{H}_{ij}(\mathbf{k}) e^{kz_j}. \quad (3)$$

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The **observable free-air anomaly** is the sum of all contributions of this kind:

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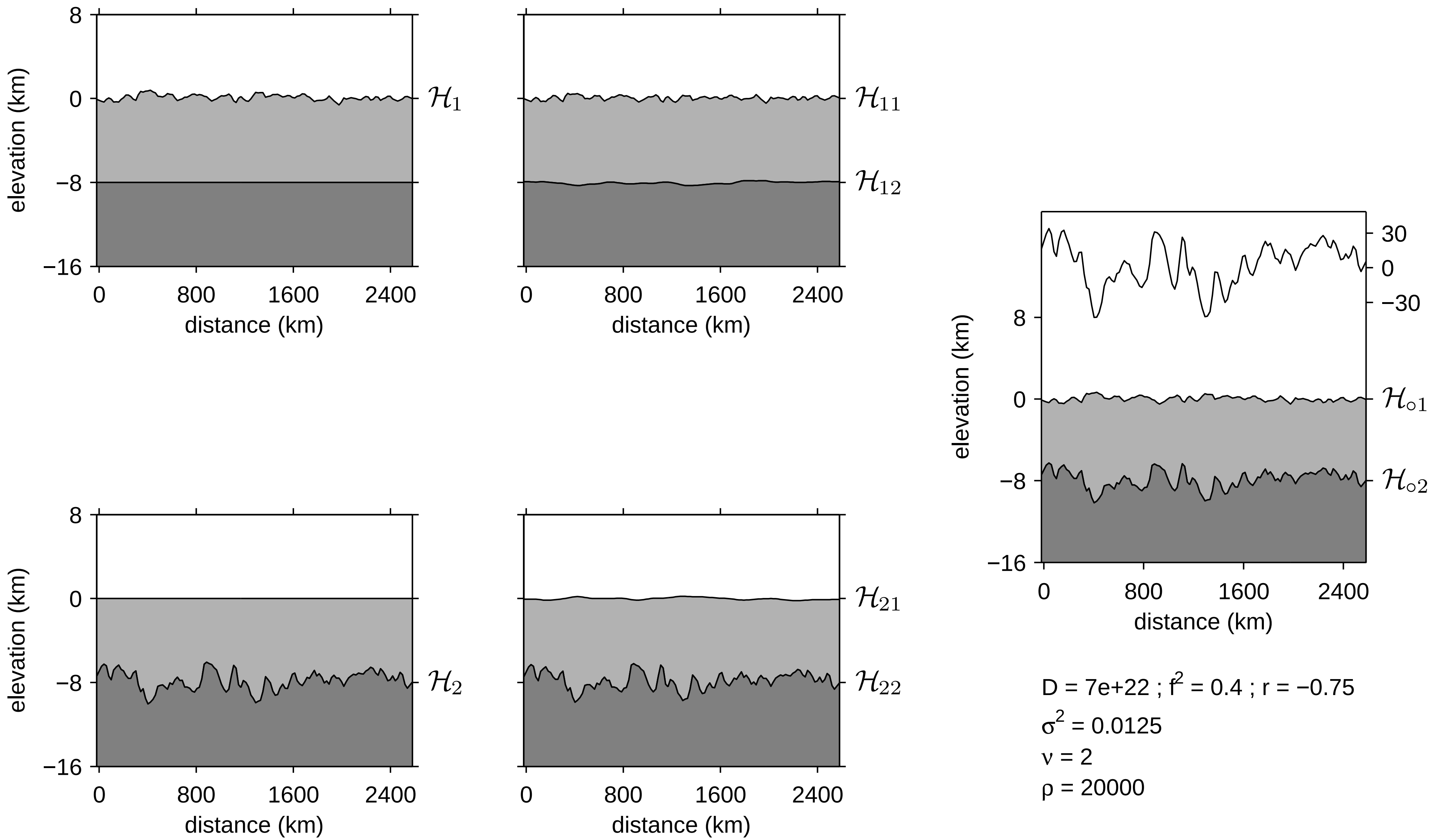
The **Bouguer gravity anomaly** is calculated from the free-air anomaly by subtracting the gravitational effect from the observable surface topography $d\mathcal{H}_{\circ 1}(\mathbf{k})$,

$$d\mathcal{G}_{\circ 2}(\mathbf{k}) = d\mathcal{G}_{12}(\mathbf{k}) + d\mathcal{G}_{22}(\mathbf{k}). \quad (5)$$

The standard model — again



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- We allow ourselves to assume a certain **proportionality** between the power-spectral densities of the inputs: f^2.
- We specify a joint structure to the initial inputs that also allows for their **correlation**: r.

Admittance and Coherence

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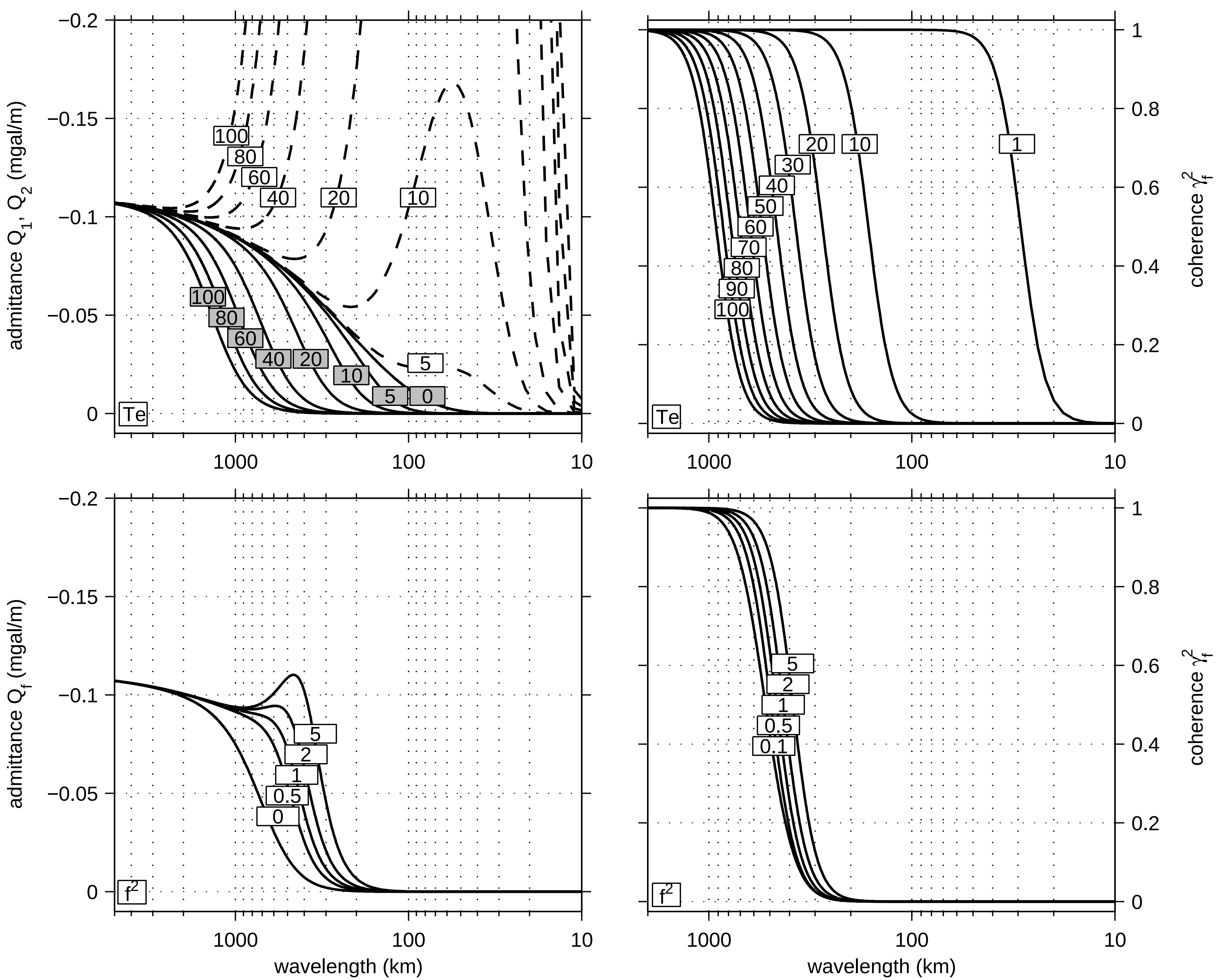
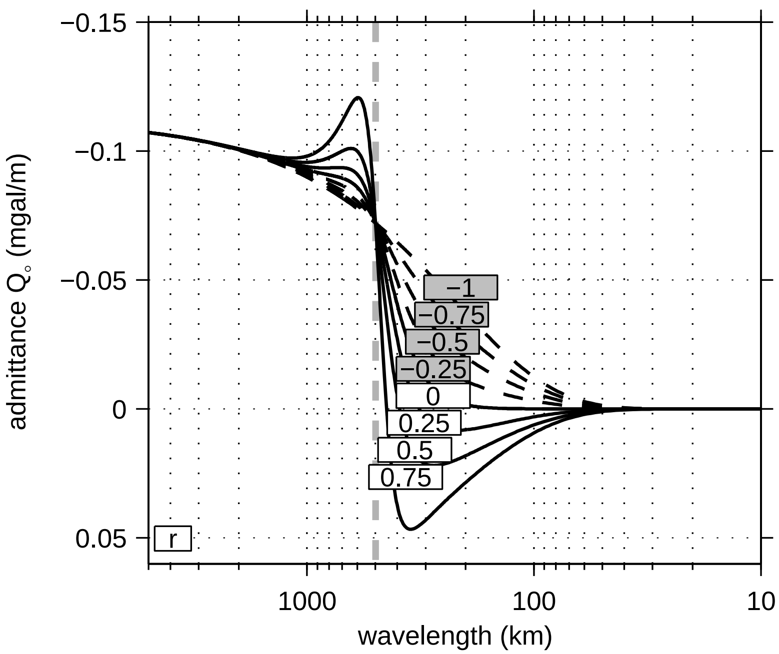
The forward model is a very doable function of D, f^2, and r.

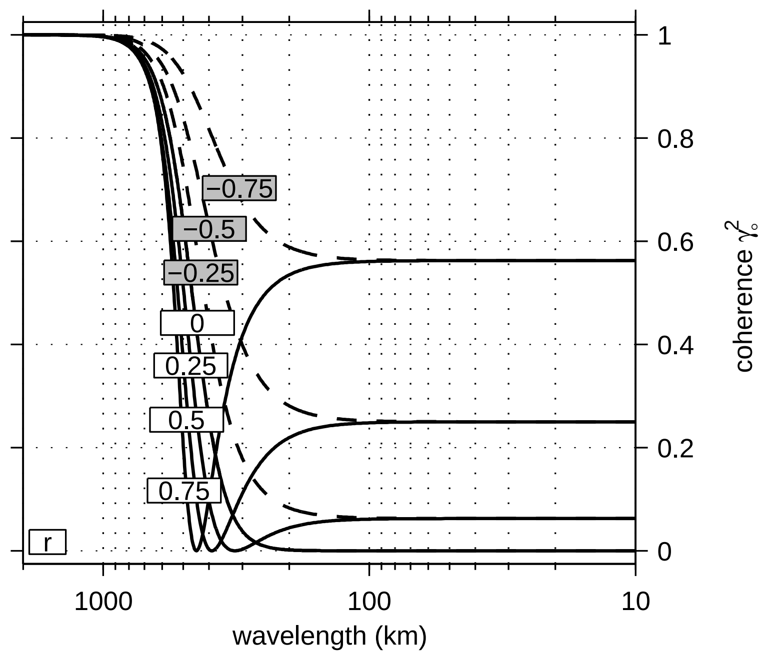
Bendat & Piersol (2000)

Admittance and Coherence — II

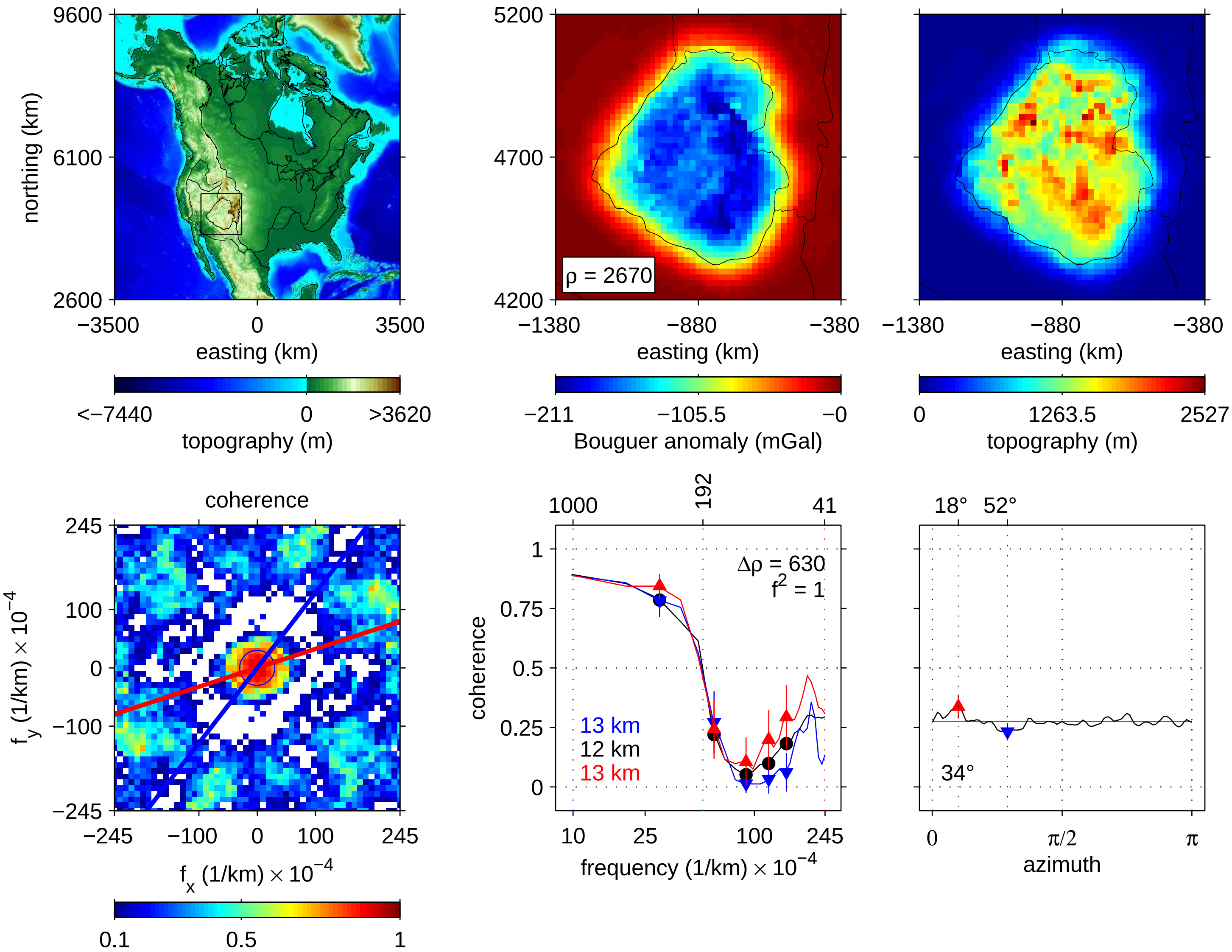


Admittance and Coherence — III

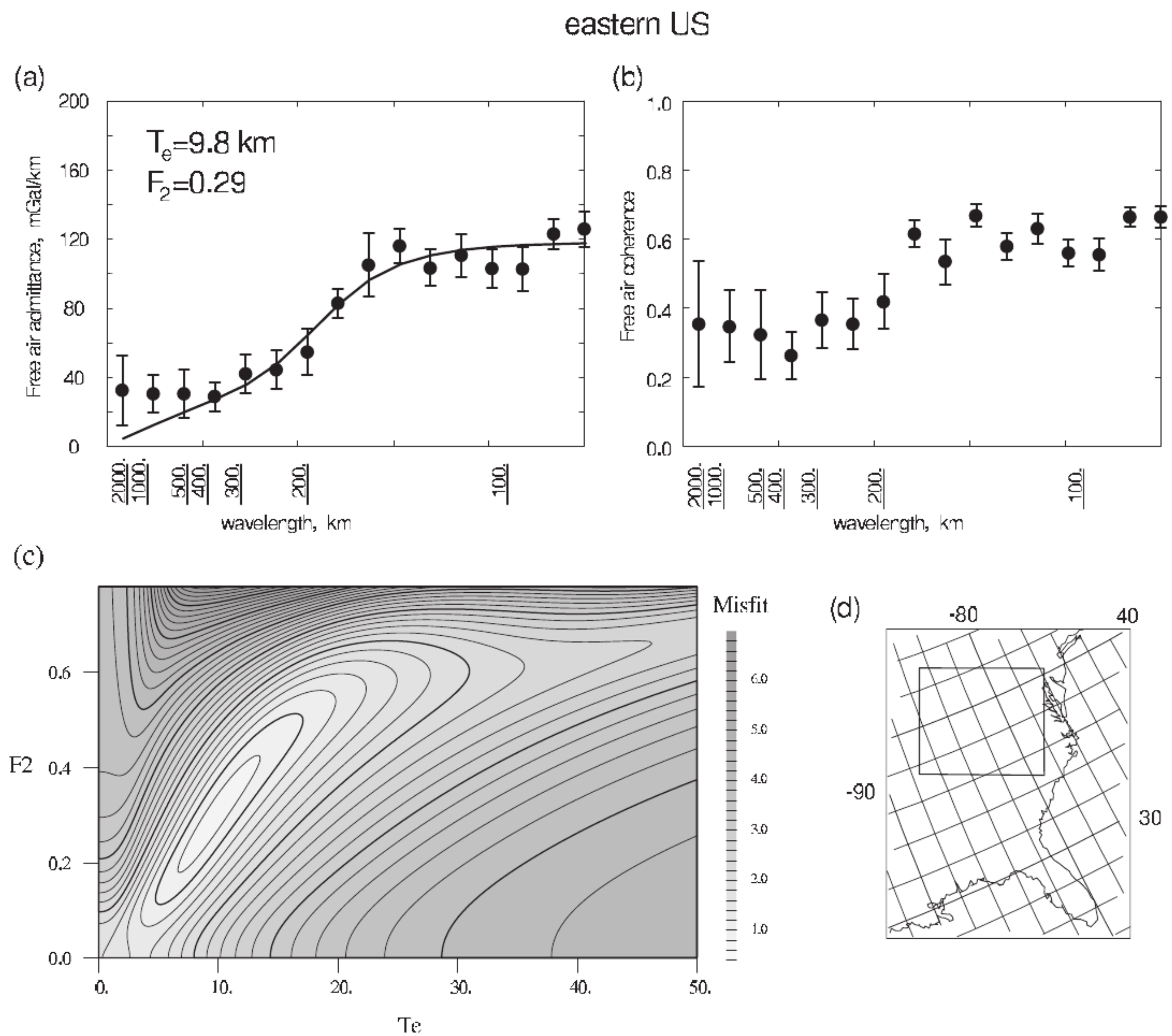




Measurements of coherence



Measurements of admittance



McKenzie, *JGR*, 2003

Synthetic experiments

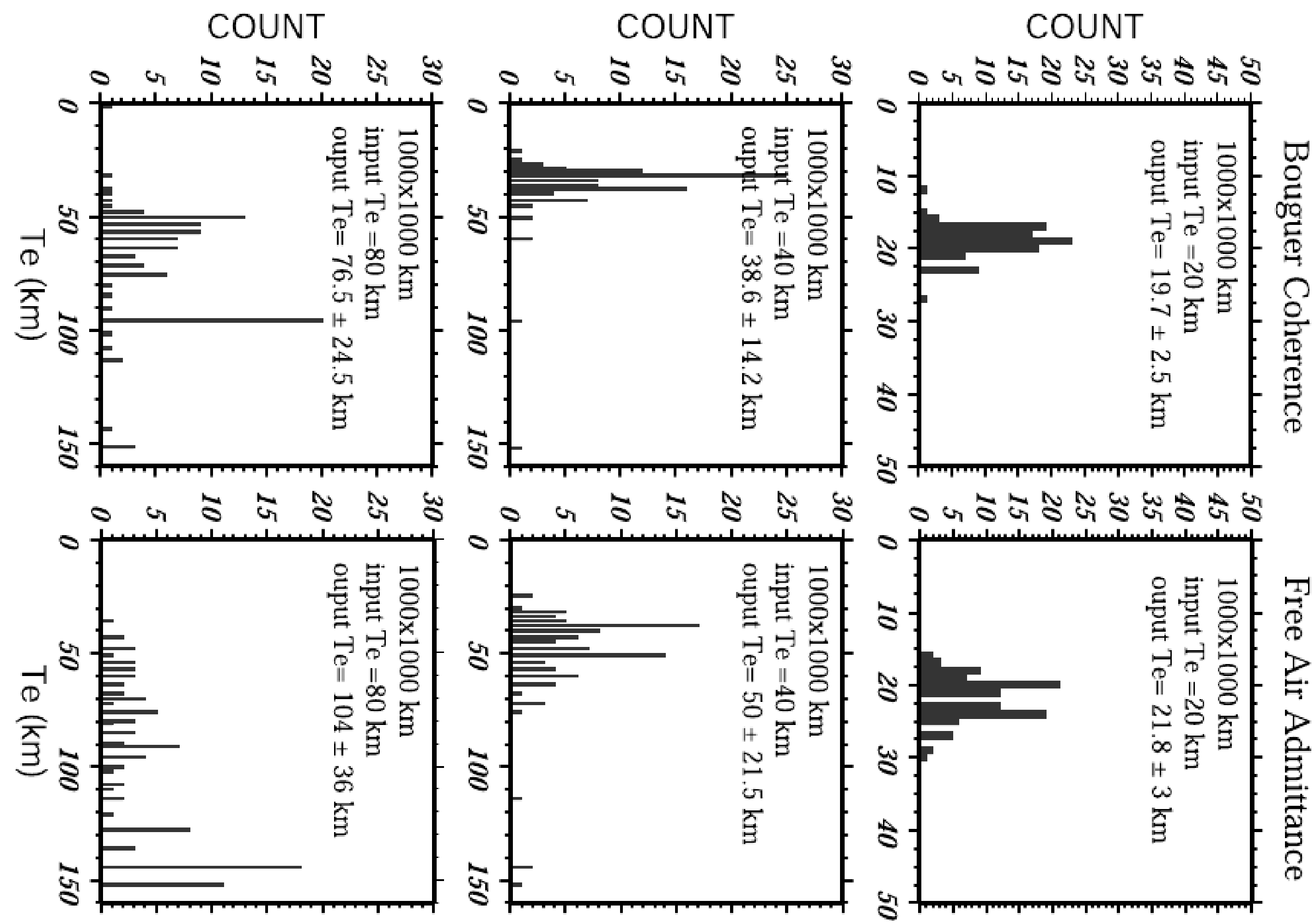


Figure 2. (left) Histograms of 100 Bouguer coherence and (right) free-air admittance analyses of synthetic data generated with input or "true" T_e = 20, 40, and 80 km and an average loading ratio of $f = 1$. The window size used for analysis was 1000 × 1000 km. Output T_e is the mean and standard deviation resulting from the 100 analyses.

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and $\mathbf{S}$ is something that we can perfectly well calculate within the model.

Miller, *SIAM Review*, 1969

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Note that the forms for admittance and coherence are ratios of the terms in this variance matrix. In our formulation, we simply retain the individual terms, which lead to Gaussian forms, rather than forming the spectral ratios, which don't.

A direct maximum-likelihood solver for D, f^2, r

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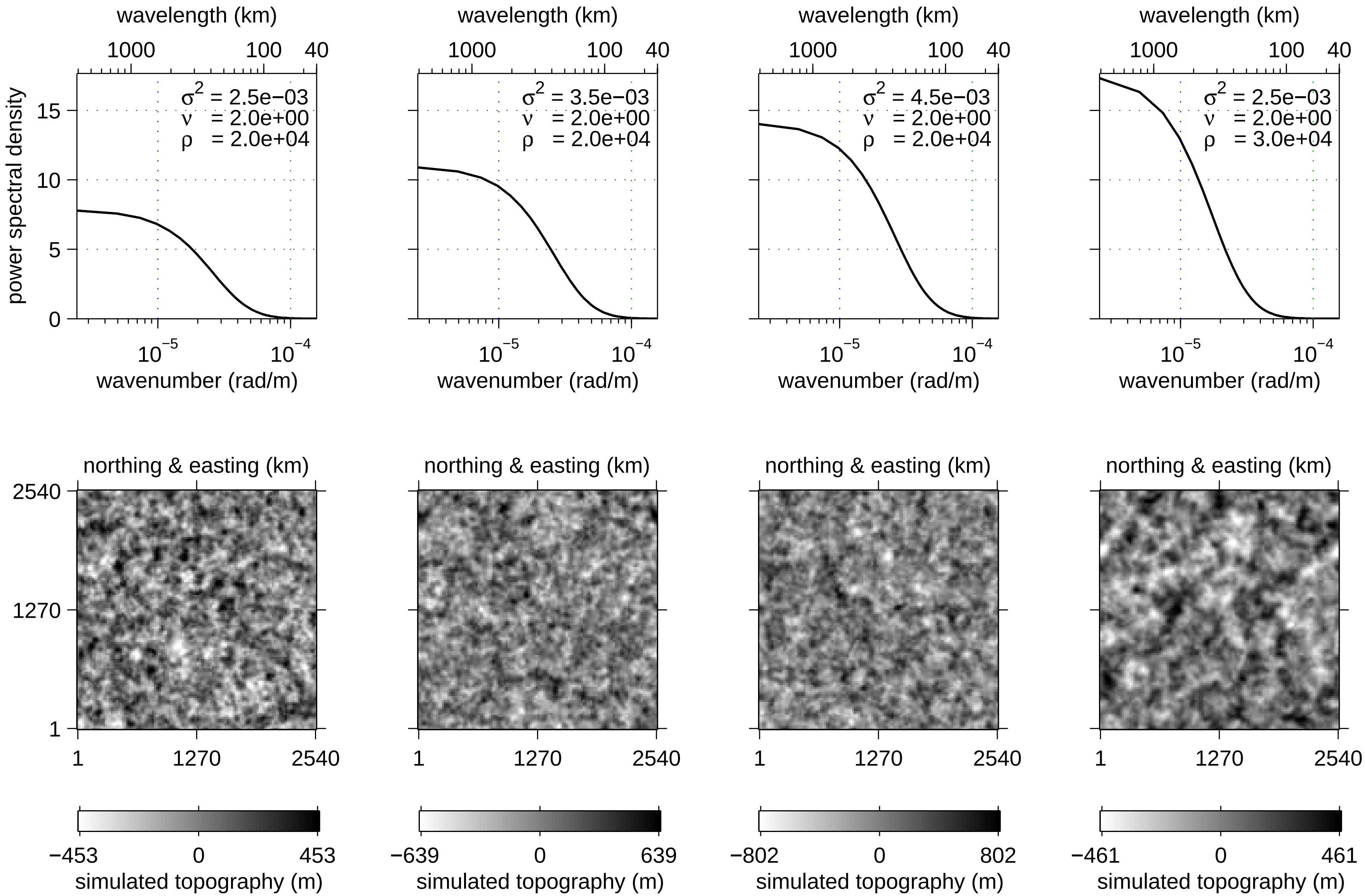
The **maximum-likelihood** is the **best**, minimum-variance, unbiased estimate of the new (three lithospheric, three spectral) parameter vector:

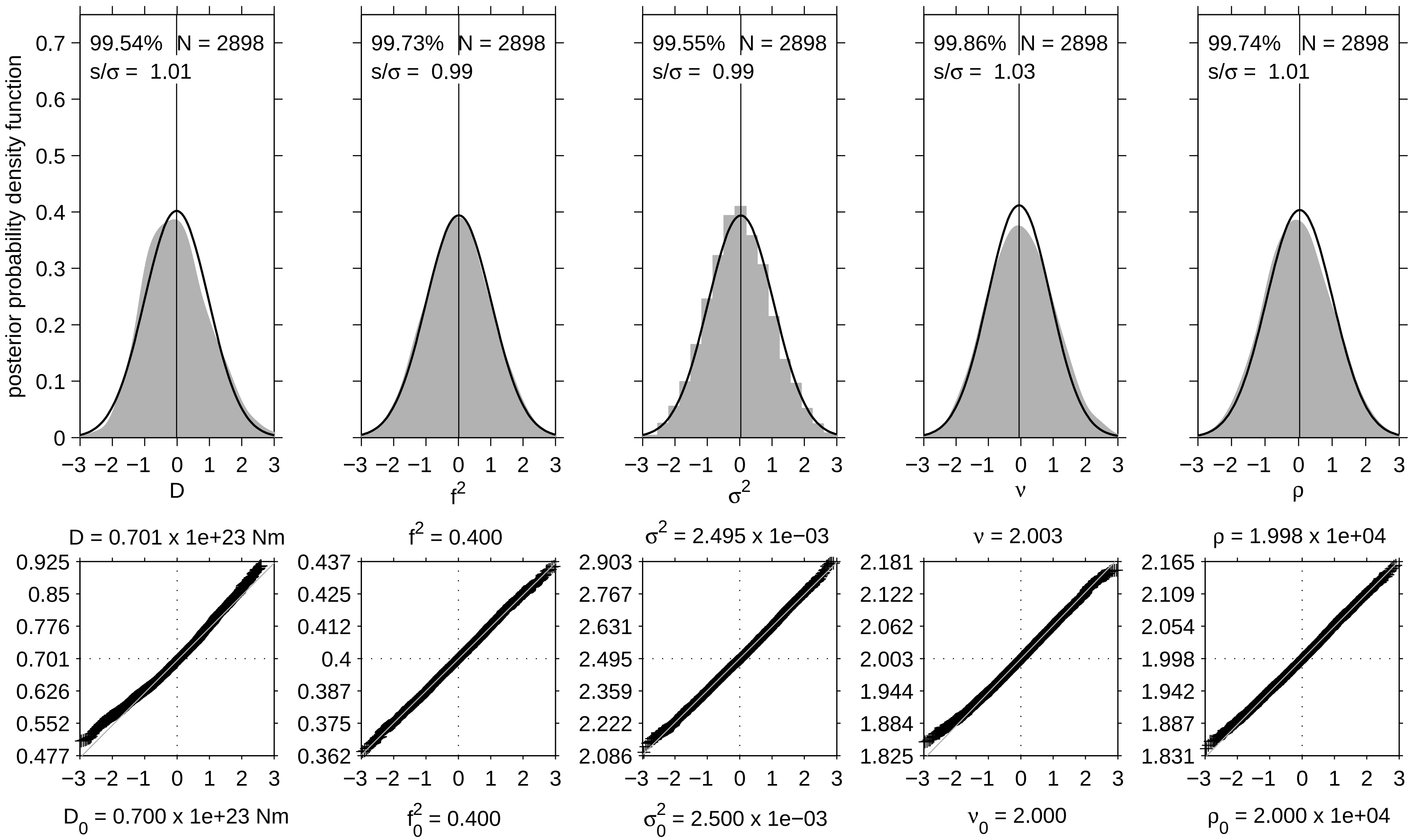
$$\boxed{\boldsymbol{\theta} = [D \;\; f^2 \;\; r \;\; \sigma^2 \;\; \nu \;\; \rho]^T.}$$

The (isotropic) Matérn spectral class



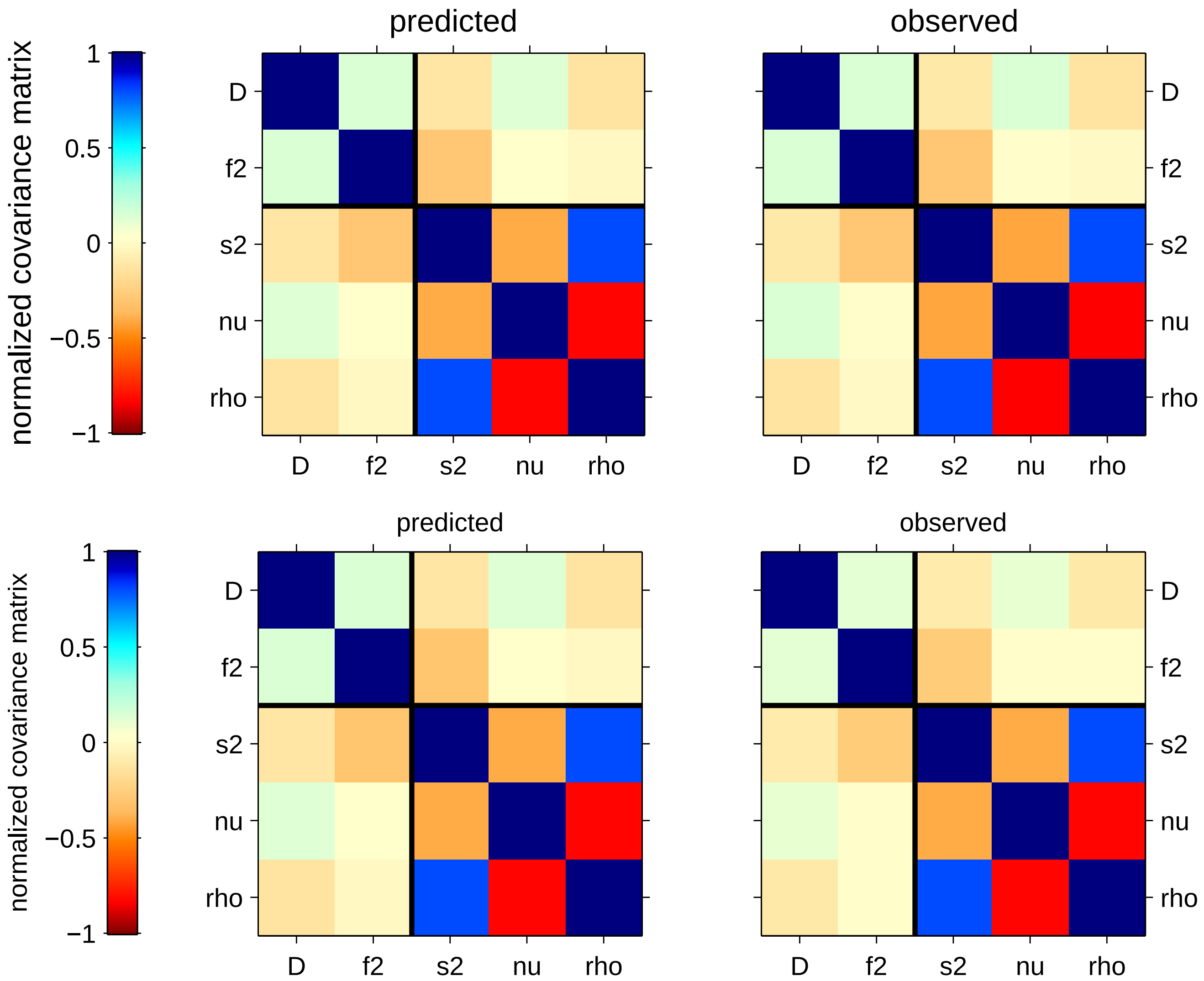
Does it work? Results

MLE Simulations with Δ_1 = 2670 ; Δ_2 = 630 kg m^{-3} ; z_2 = 35 km ; 64x64 grid ; 1280x1280 km



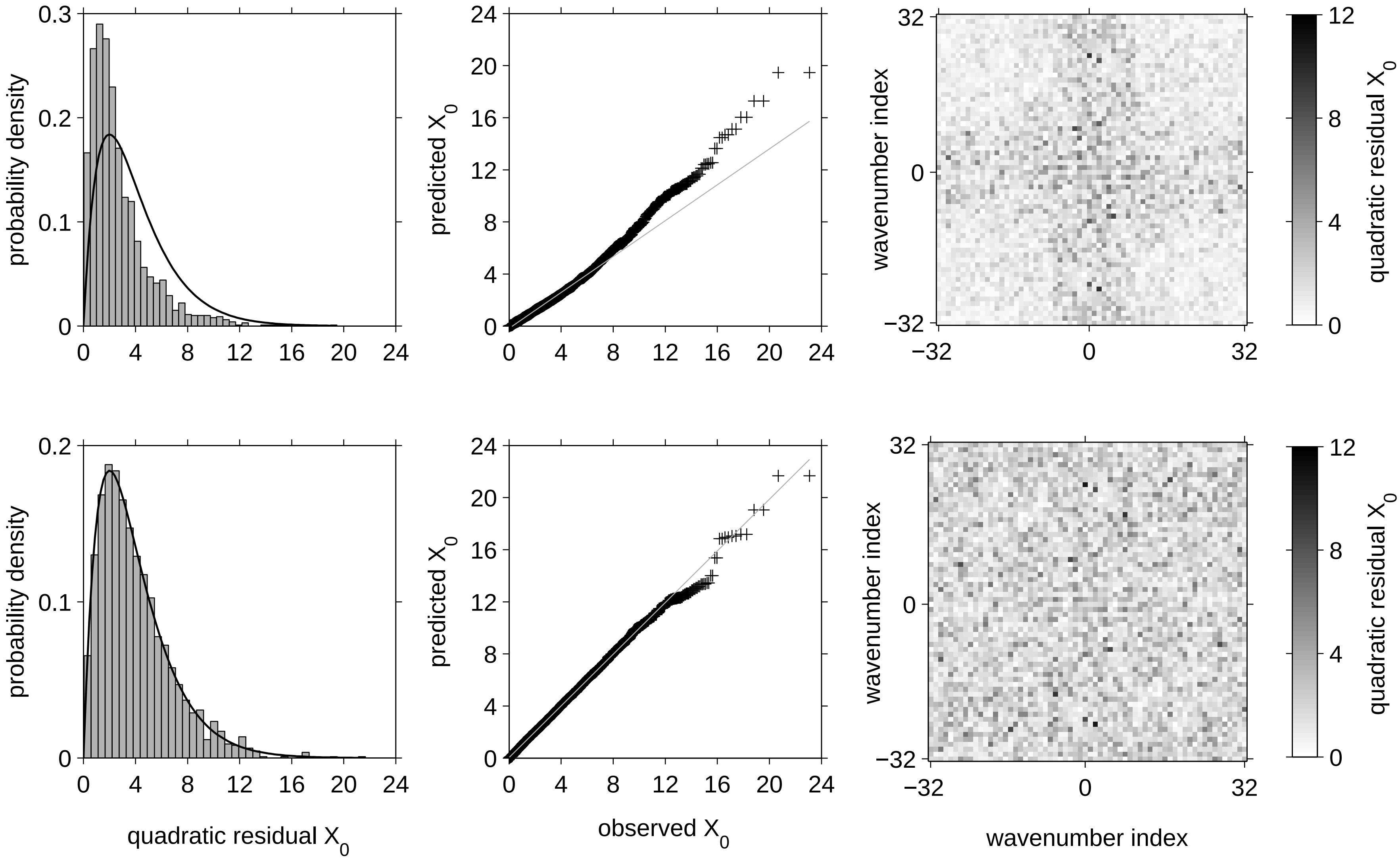
With E = 1.4e+11 ; ν = 0.25 ; T_e = 18 ± 8 km ; f^2 = 0.4 ± 0.013

Does it work? Trade-offs



Simons & Olhede, *in revision*, 2012

Does it work? Modeling the residuals



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- The *full* problem: **anisotropic** inputs, **anisotropic** responses, **multilayer** systems... is still out there.
- The connections in marrying **deterministic** forward models with **stochastic** inputs and observables are fruitful and widespread.

Conclusions

- Reducing gravity and topography to **coherence** or **admittance** estimated at a handful of wavenumbers, and then inverting for rigidity D and loading ratio f^2 is a *very* non-optimal thing to do. A **robust method** keeps the power of the abundance of data by forming a **Whittle maximum-likelihood estimator**.

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- Simulations verify that the estimates are normally distributed, and unbiased and closely track the variance predicted by the **fully analytic theory**, which is minimized. Confidence intervals are symmetric, and covariance between the estimated D and f^2 is small. Though no longer needed, predicting admittance and coherence in retrospect vastly tightens their error bars.

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- Our current method successfully takes into account r-**correlated** loads, small data sets, irregular domains. **Anisotropy** in both the loading and the response is further down the line. *Then*, the statistics will be interesting.