Can problems in the geosciences inspire fundamental research in the mathematical and statistical sciences?

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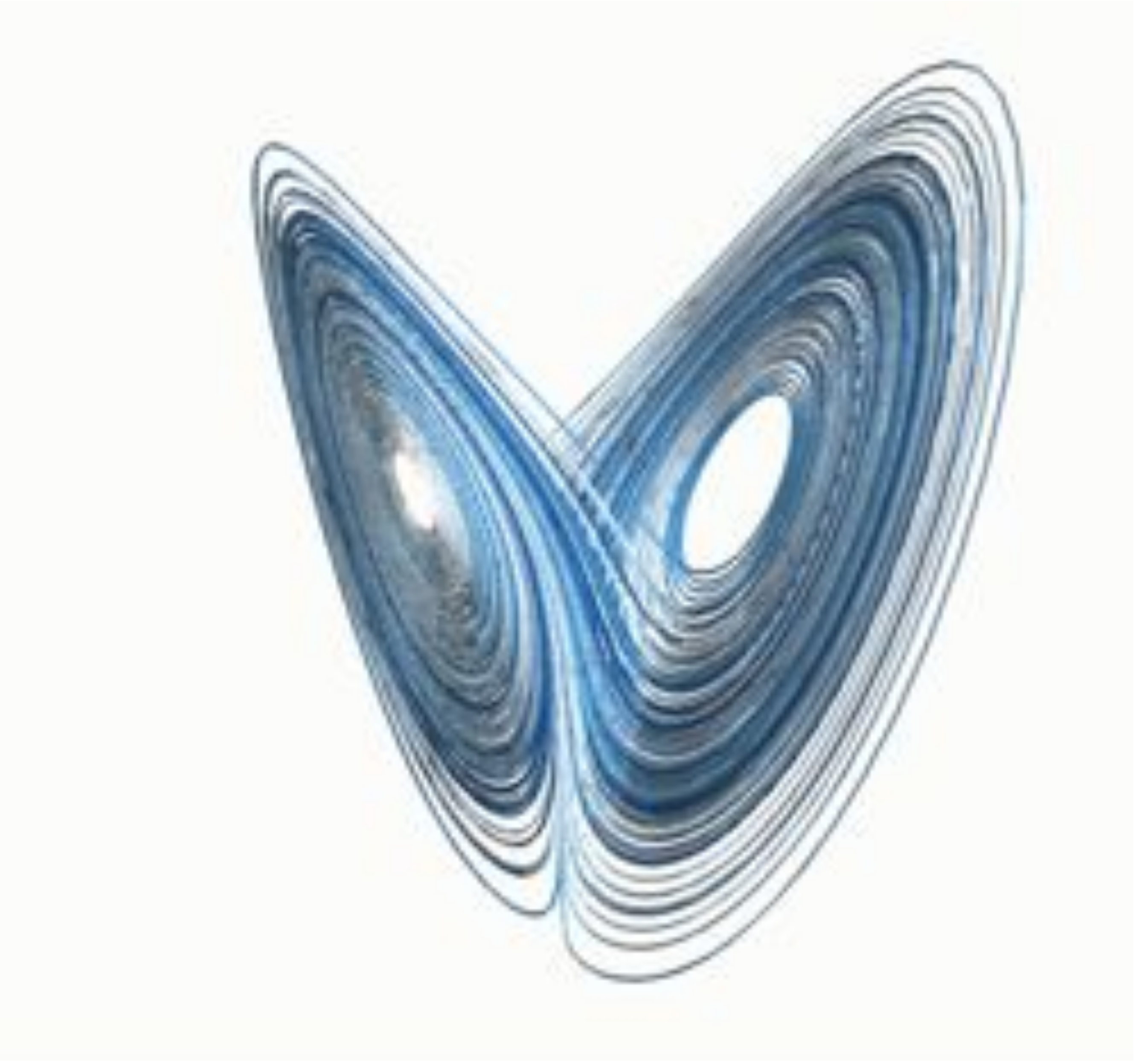
My opinion: Yes!



Questions for discussion

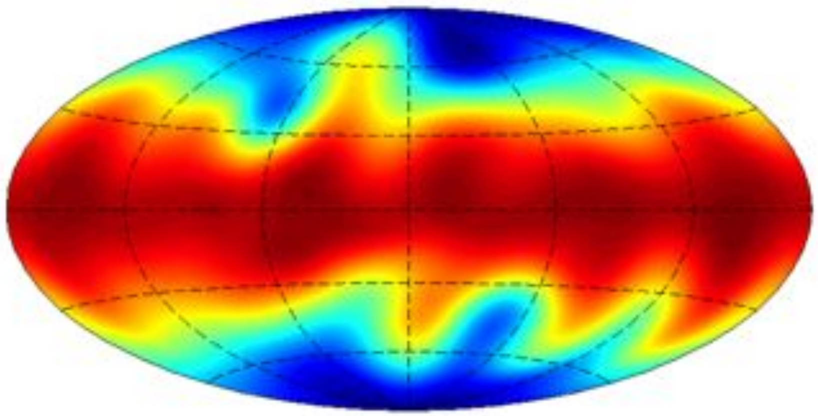
1. How can problems in geosciences inspire fundamental research in mathematics and statistics?
   i. Examples?
   ii. How did the inspiration happen?
   iii. Is this unique to the geosciences?
   iv. How have breakthroughs lead to advances in other fields?
   v. Devils advocates?

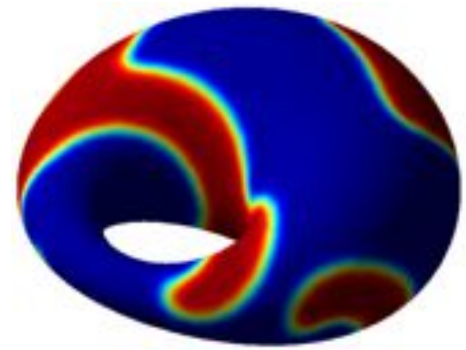
2. How can we foster these innovations?
   i. Is this possible through an "institute" type setting?
   ii. Can it be done virtually?
   iii. How do we get students involved?

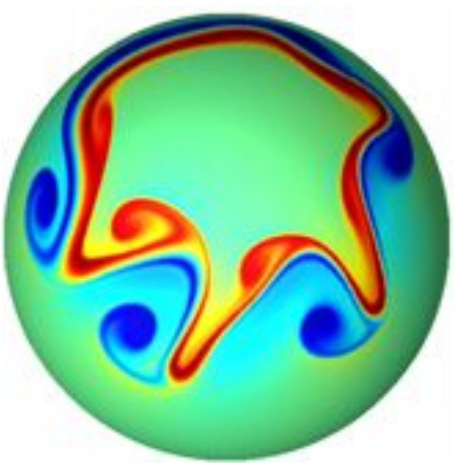
3. Can fundamental research in mathematics and statistics lean to fundamental research in the geosciences?

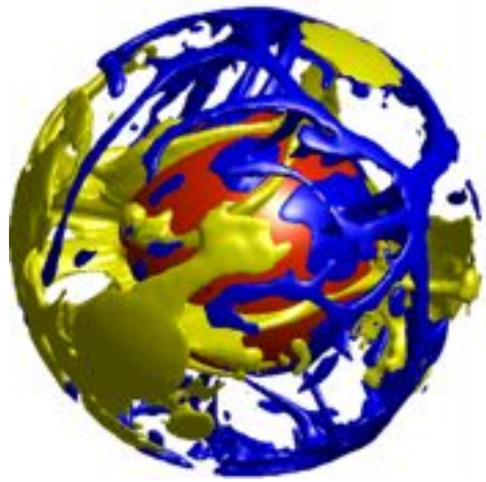
My background

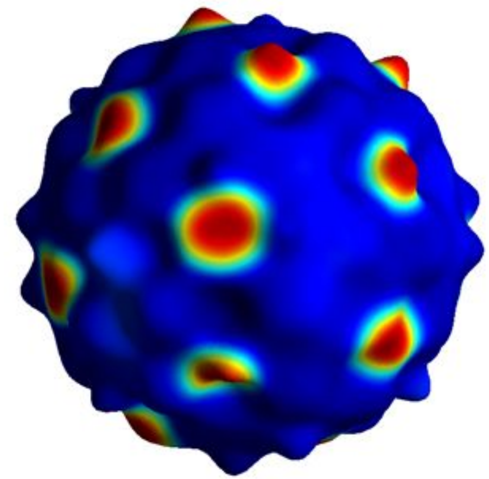
Joint work with Natasha Flyer, supported by NSF-CMG 0801309 and 0934581.







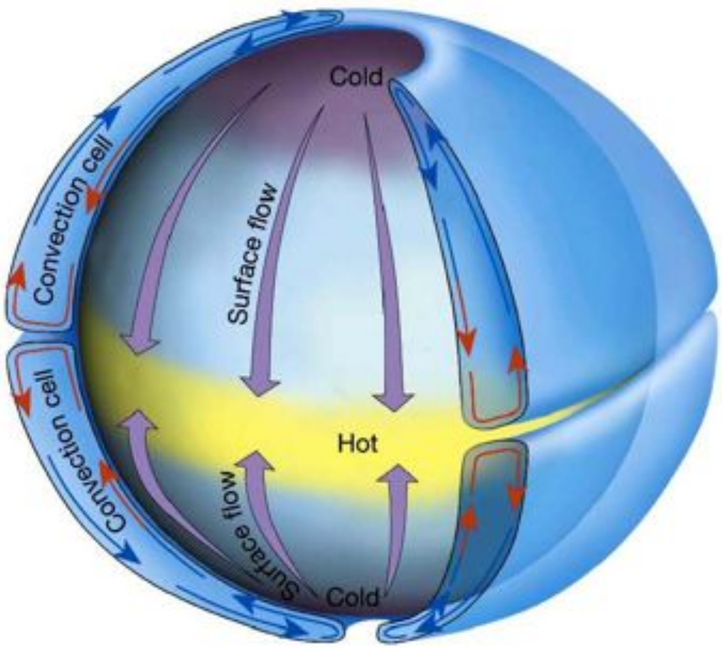


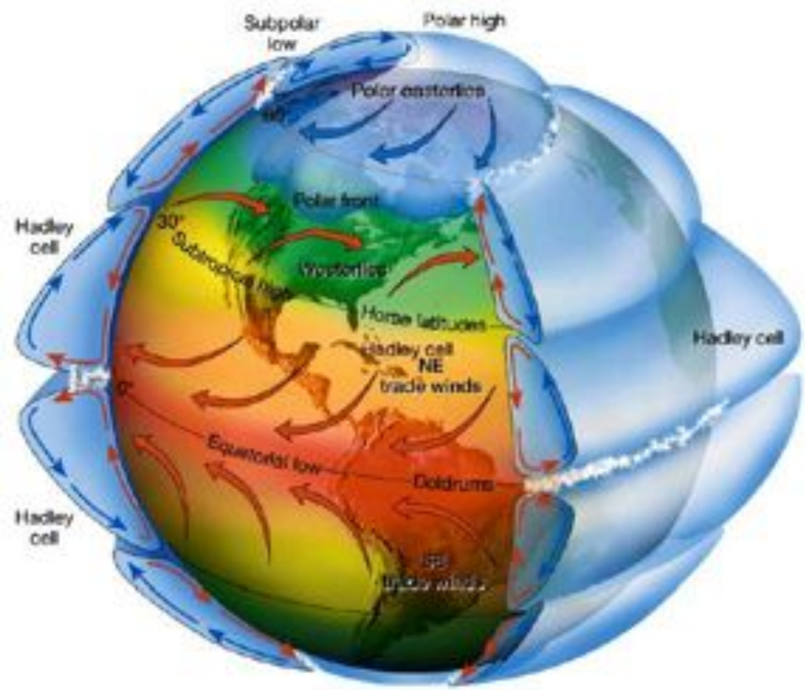


Other collaborators: E. J. Fuselier, E. Lehto, L.H. Kellogg, D.A. Yuen.

Some new geo-inspired work

- Joint work with Uwe Harlander, Department Aerodynamics and Fluid Mechanics, BTU Cottbus.
- DFG program MetStröm: Multiple Scales in Fluid Mechanics and Meteorology
- Atmospheric general circulation:

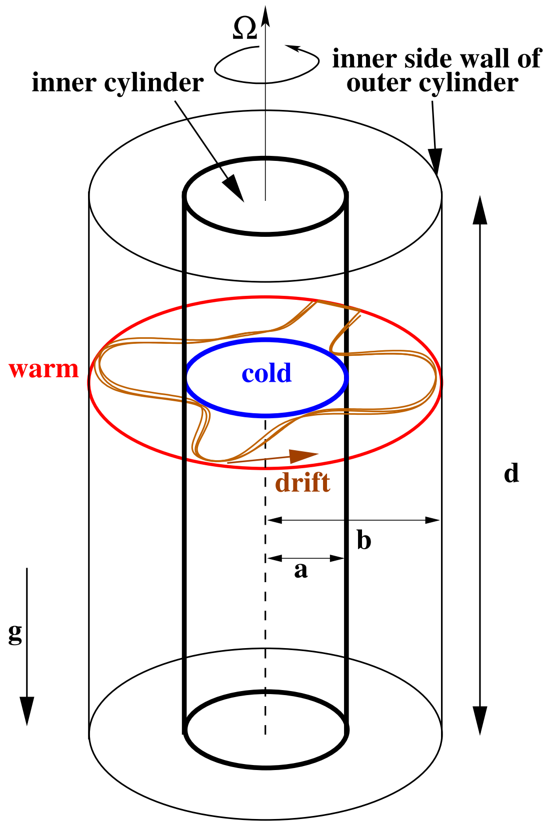




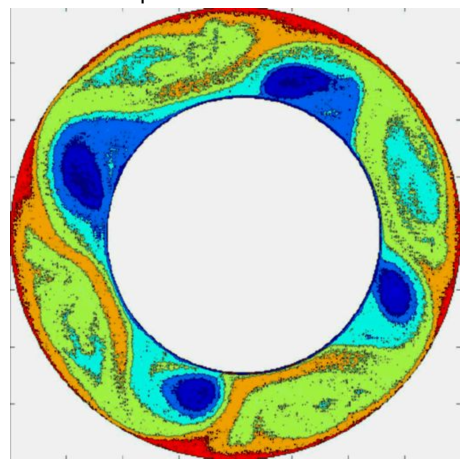
Source: Vera Schlanger - Hungarian Meteorological Service

Differentially heated rotating annulus

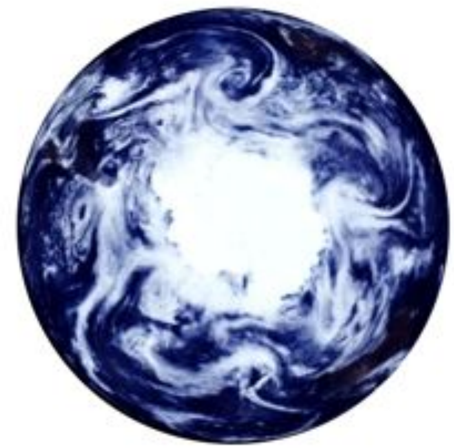
- Accepted laboratory experiment for large scale flow in the mid-latitudes.



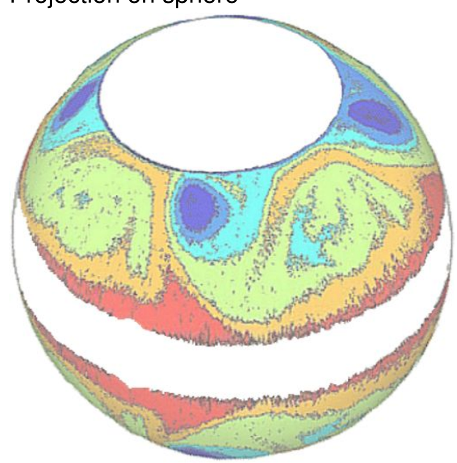
surface temperature



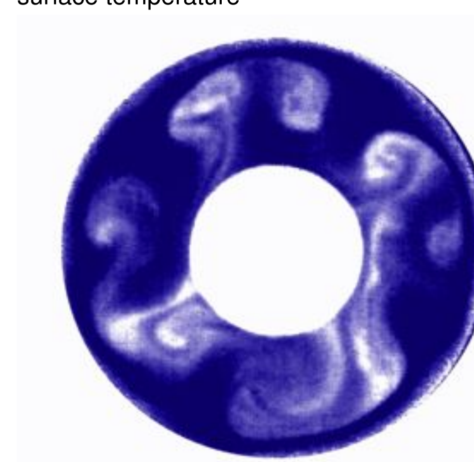
cloud cover



Projection on sphere



surface temperature



Parameters:

$$\begin{aligned} \Omega &= \text{rotation rate} \\ \Delta T &= T_{\text{outer}} - T_{\text{inner}} \\ b - a &= \text{width annulus} \\ d &= \text{height annulus} \end{aligned}$$

Equations, numbers, experimental setup

Governing equations:

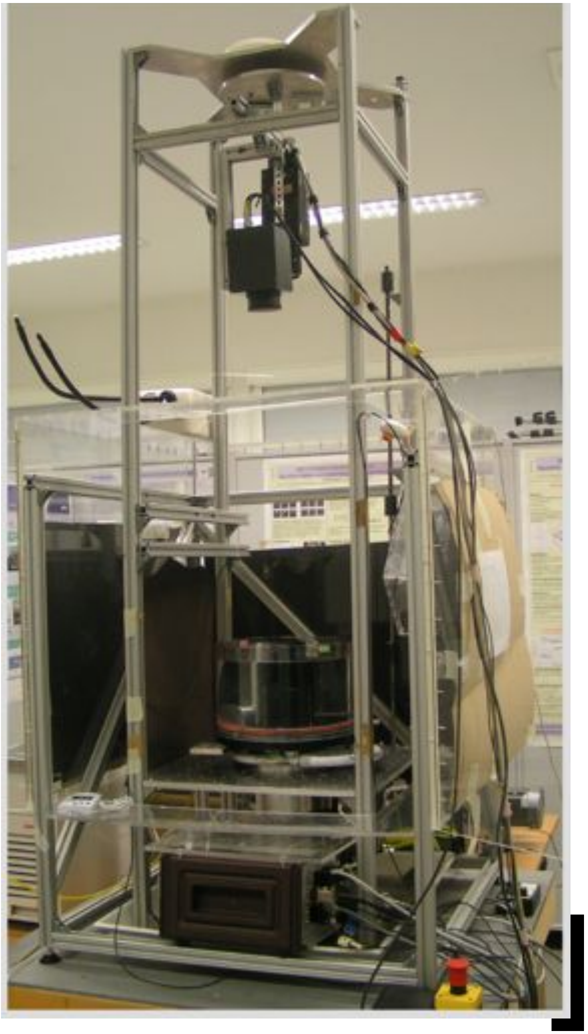
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \Delta\mathbf{u} - \mathrm{Ra}\,\phi\,\mathbf{k} - \mathrm{Ta}^{1/2}\mathbf{k}\times\mathbf{u}$$

$$\frac{\partial \phi}{\partial t} + \mathbf{u}\cdot\nabla\phi = \frac{1}{\mathrm{Pa}}\Delta\phi$$

$$\nabla\cdot\mathbf{u} = 0$$

Non-dimensional numbers

Taylor	Ta	$\frac{4\Omega^2(b-a)^4}{\nu^2}$
Rayleigh	Ra	$\frac{g\alpha\Delta T(b-a)^3}{\kappa\nu}$
Prandtl	Pr	$\frac{\nu}{\kappa}$
Modified Taylor	Ta'	$\frac{b-a}{d}\mathrm{Ta}$
Thermal Rosby	Ro	$4\frac{\mathrm{Ra}}{\mathrm{Pr}\,\mathrm{Ta}}$



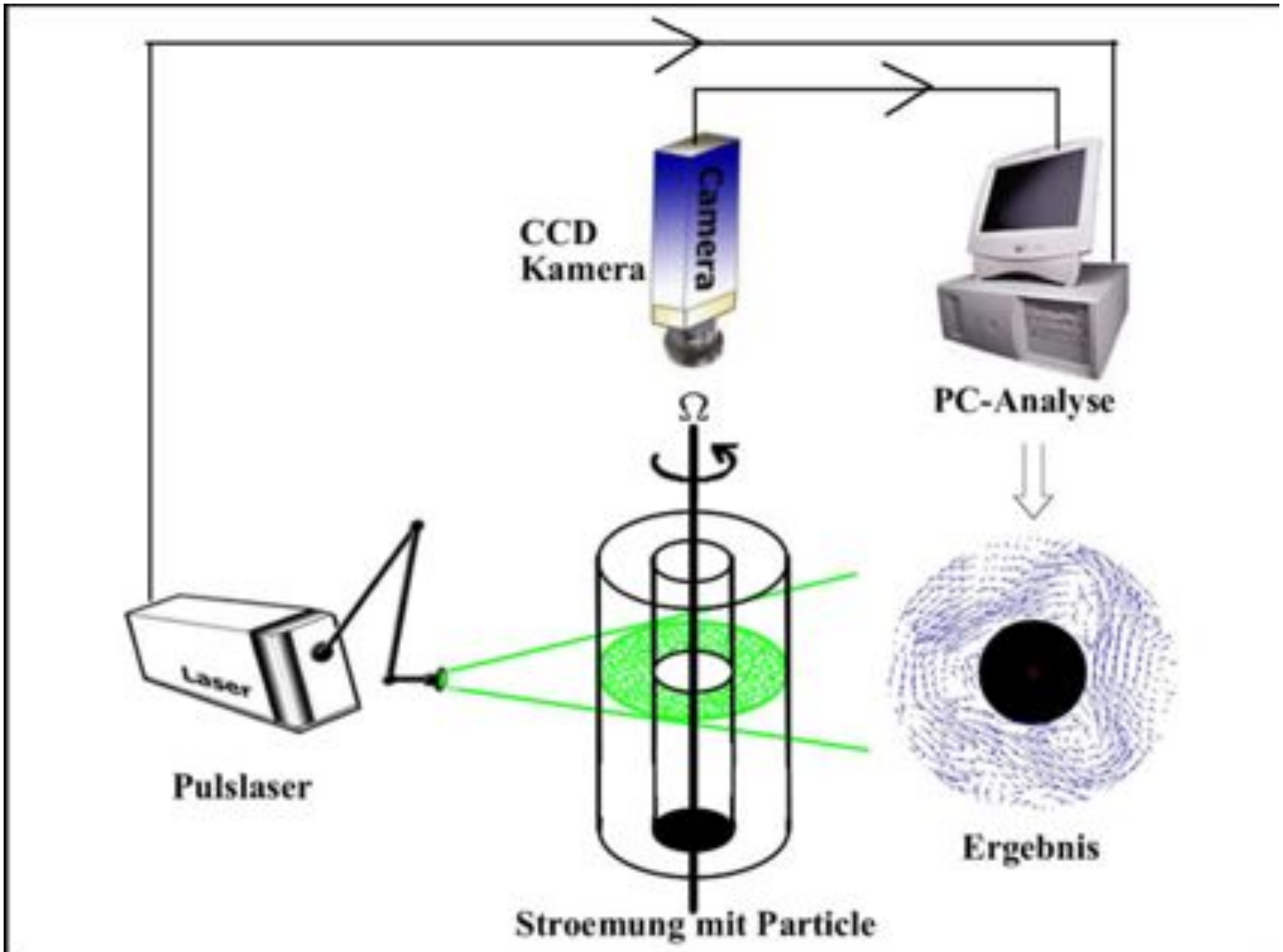
Actual experiment at BTU Cottbus.

Movie of spin-up to a 3-wave pattern

	Ω [rpm]	ΔT [K]	Ta	Ro_{th}	$b - a$ [mm]	d [mm]
exp	6	5.1	$1.55 \cdot 10^7$	0.79	75	135

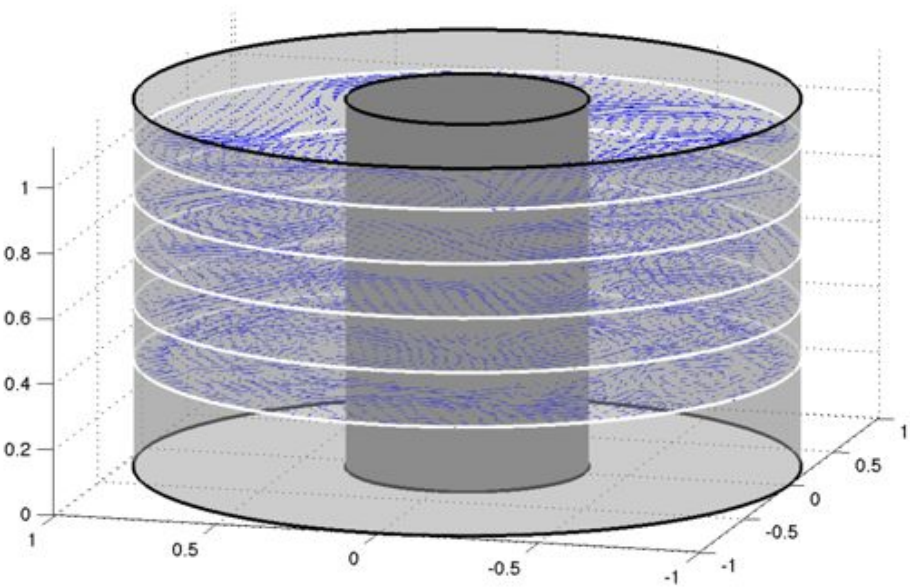
Fluid velocity measurements: PIV

- At various levels of the tank, horizontal velocity measurements of the fluid are obtained by Particle Image Velocimetry (PIV).

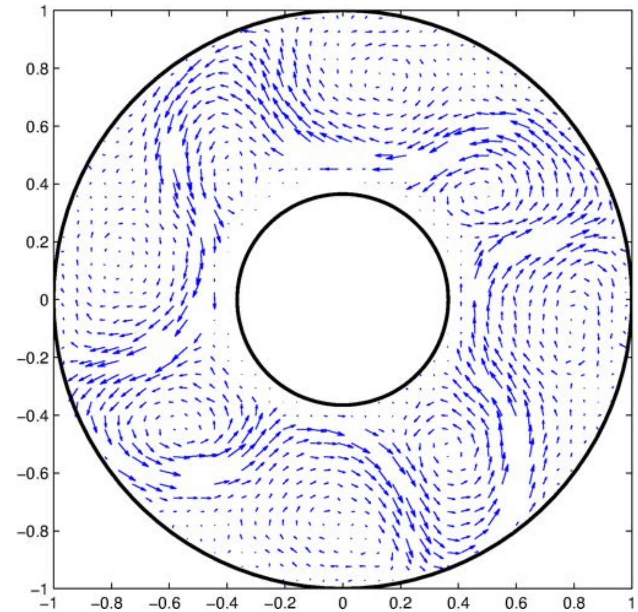


Data analysis problems

Velocity data at all levels:



Velocity data at top level:



First problem:

Reconstruct the full 3D field from the various horizontal slices.

a) Reconstruct the 2D field on each horizontal slice.
b) Compute divergence of the reconstructed field.
c) Use incompressibility of the full 3D field to recover the vertical velocity component:

$$w_z = -(u_x + v_y) \Longrightarrow w = -\int_{z_0}^{z} (u_x + v_y)\,dz$$

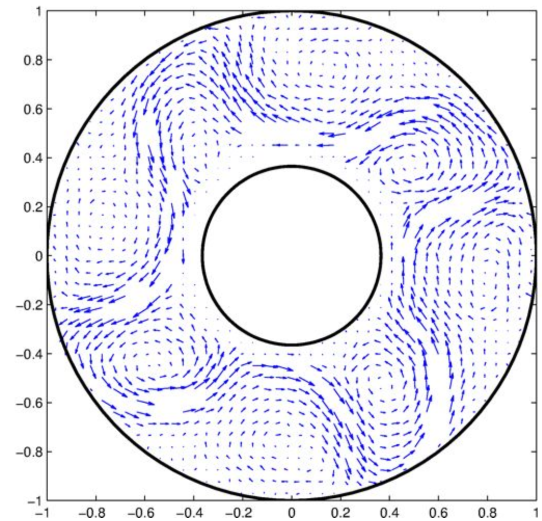
Challenge: Data is scattered and contains noise.

Data analysis problems

Second (and more interesting) problem:

Compute the Helmholtz-Hodge decomposition of the 2D horizontal fields.

Velocity data at top level:



Recall: Helmholtz-Hodge theorem states any sufficiently smooth vector field $\mathbf{u}$ can be decomposed as follows:

$$\begin{aligned}\mathbf{u} &= \mathbf{u}_{\text{div}} + \mathbf{u}_{\text{curl}} \\ &= \nabla \times (\psi \mathbf{k}) + \nabla \chi \quad \text{(for 2D field)}\end{aligned}$$

Decomposition is unique if appropriate boundary conditions applied.

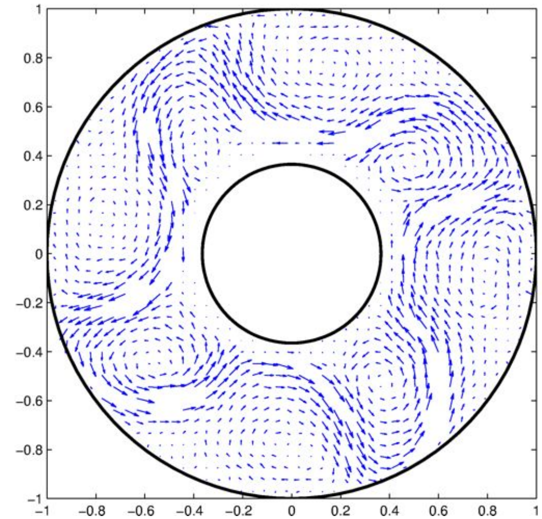
Importance: These two fields can be used to discriminate different wave-types:

Baroclinic and Rosby waves are div-free, while inertial-gravity waves are not.

Challenge: Data is scattered and contains noise.

Property conserving approximations

Key idea:
Use a reconstruction that mimics the Helmholtz-Hodge decomposition theorem.



Notation:

$\mathbf{u}_j = (u_j, v_j)$ is measured field at $\mathbf{x}_j$, $j = 1, \ldots, N$

$\boldsymbol{\xi}_k$ = nodes on the boundary, $k = 1, \ldots, M$

$\mathbf{n}_k$ = unit outward normal at $\boldsymbol{\xi}_k$

Property conserving approximation:

$$\mathbf{s}(\mathbf{x}) = \underbrace{\sum_{j=1}^{N} \Phi_{\text{div}}(\mathbf{x}, \mathbf{x}_j)\mathbf{a}_j + \sum_{k=1}^{M} [\Phi_{\text{div}}(\mathbf{x}, \boldsymbol{\xi}_j)\mathbf{n}_k] d_j}_{:= \mathbf{s}_{\text{div}}} + \underbrace{\sum_{j=1}^{N} \Phi_{\text{curl}}(\mathbf{x}, \mathbf{x}_j)\mathbf{a}_j}_{:= \mathbf{s}_{\text{curl}}}$$

- Φ_{div} and Φ_{curl} are positive-definite, *matrix-valued* kernels
- Columns of Φ_{div} are divergence-free
- Columns of Φ_{curl} are curl-free

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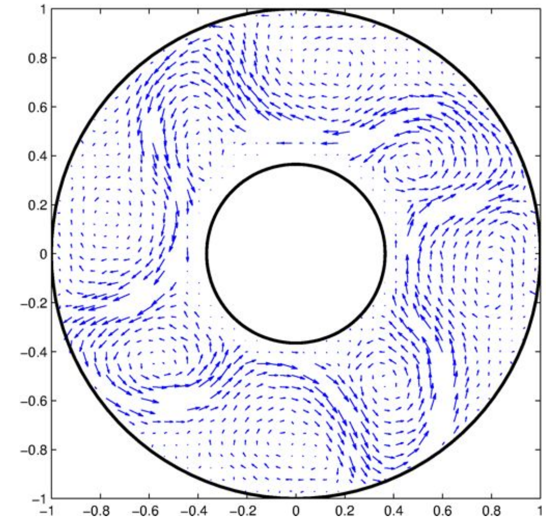

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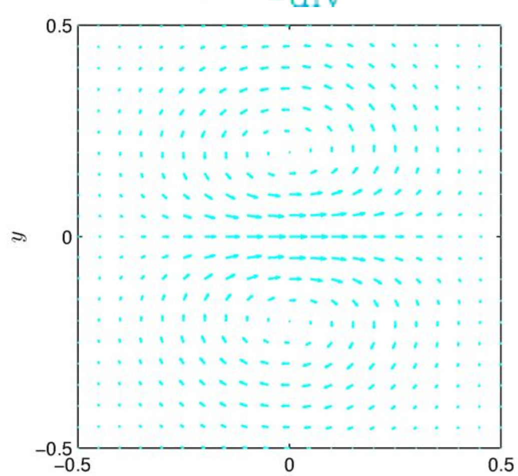
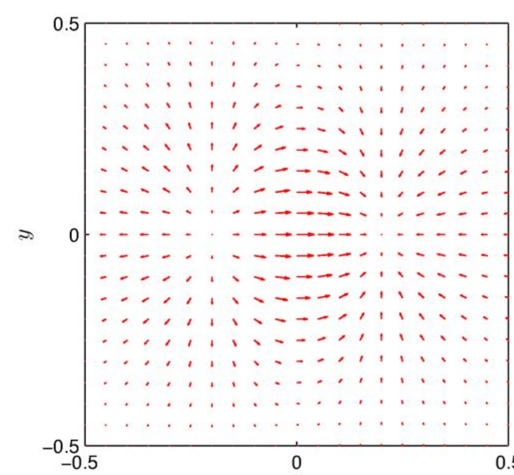
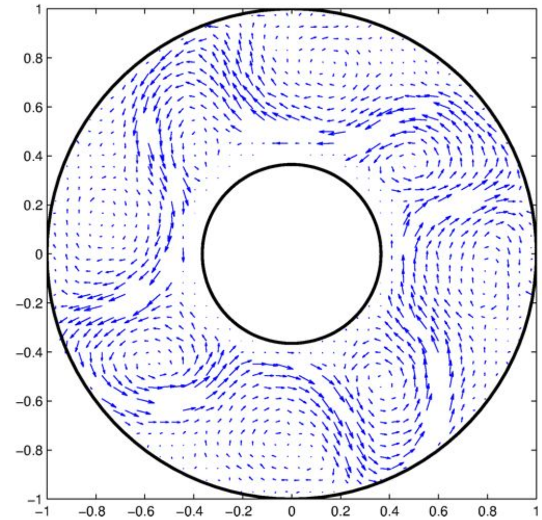




Property conserving approximations



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Constraints for interpolation:

$$\begin{aligned} \mathbf{s}(\mathbf{x}_i) &= \mathbf{u}_i,\ i = 1, \ldots, N, \\ \mathbf{s}_{\text{div}}(\boldsymbol{\xi}_i) \cdot \mathbf{n}_i &= 0,\ i = 1, \ldots, M, \end{aligned}$$

(Leads to a positive-definite linear system of equations)

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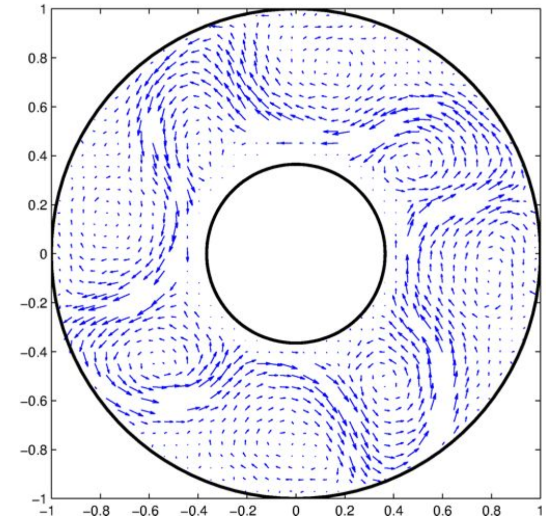

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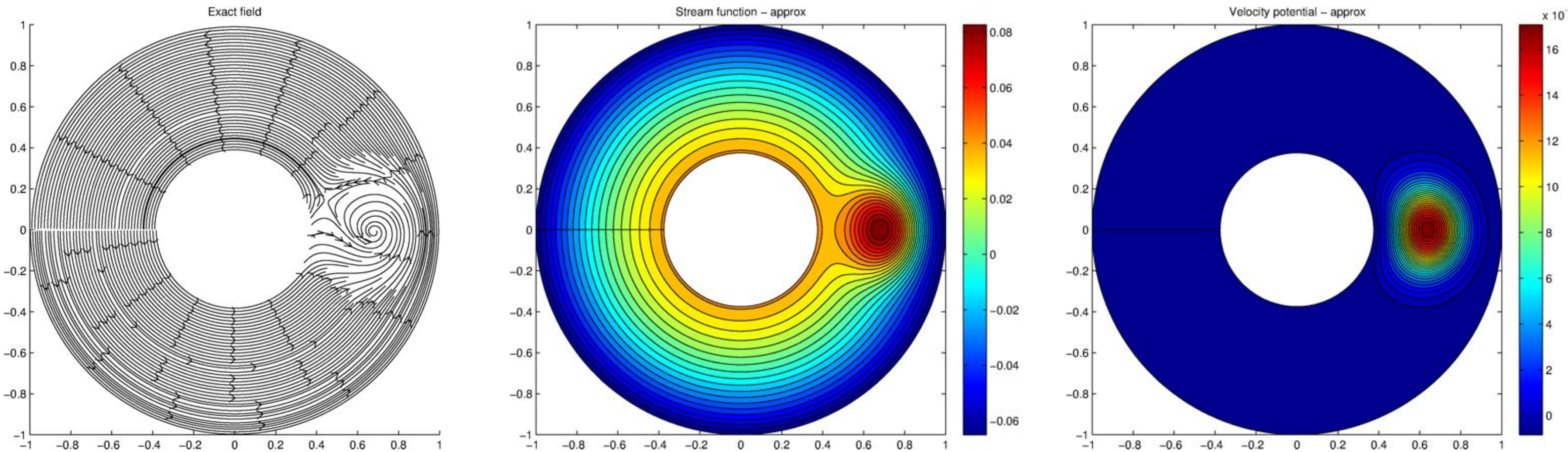
Results:

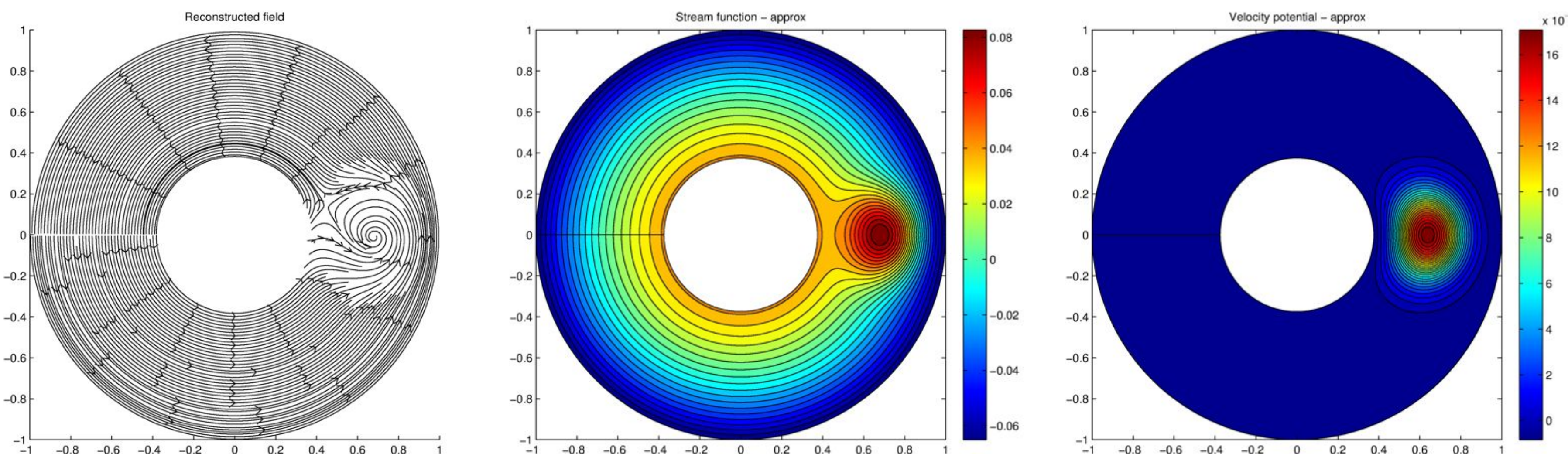
1. $\mathbf{s} \approx \mathbf{u}$
2. $\mathbf{s}_{\text{div}} \approx \mathbf{u}_{\text{div}}$
3. $\mathbf{s}_{\text{curl}} \approx \mathbf{u}_{\text{curl}}$
4. Can recover a stream-function and a velocity potential for the fields

Example with exact data

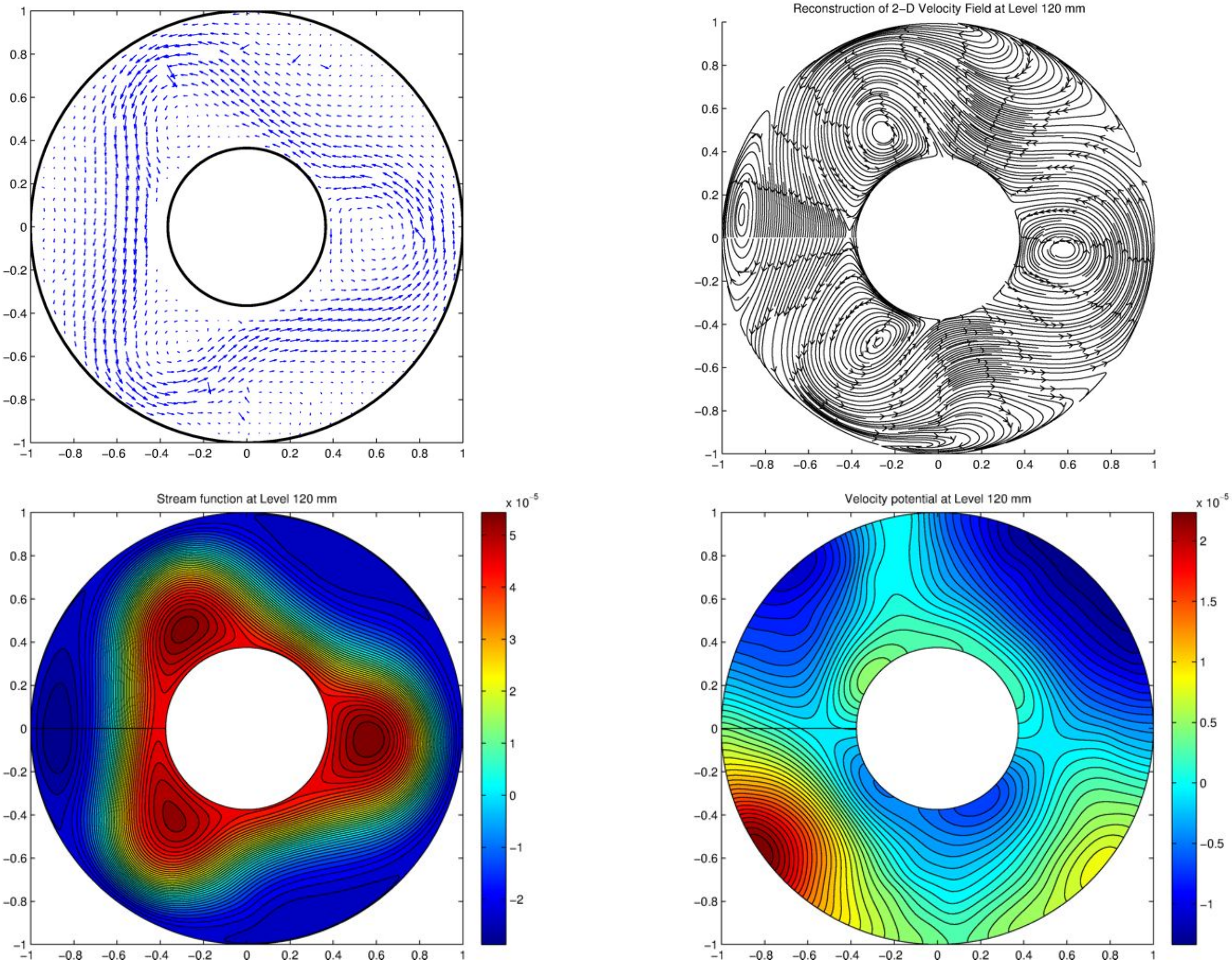
- Exact field, stream function, and velocity potential



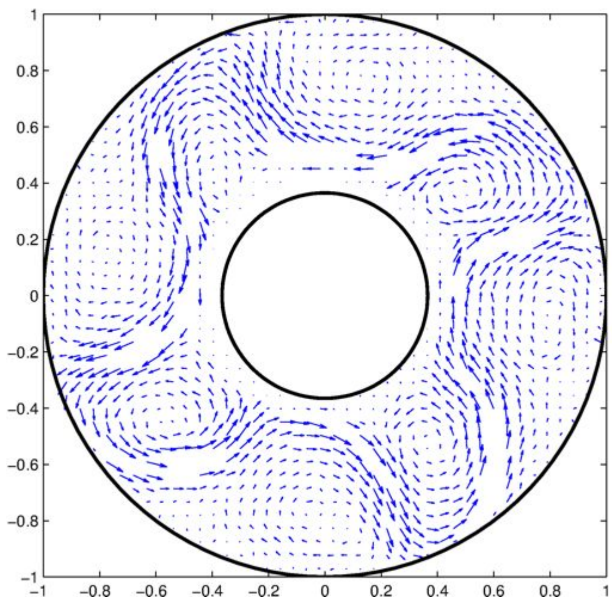
- Reconstructed field, stream function and velocity potential

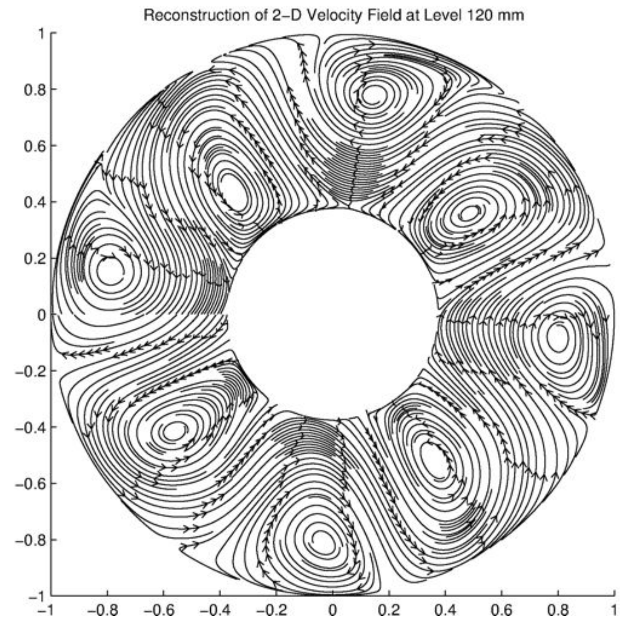


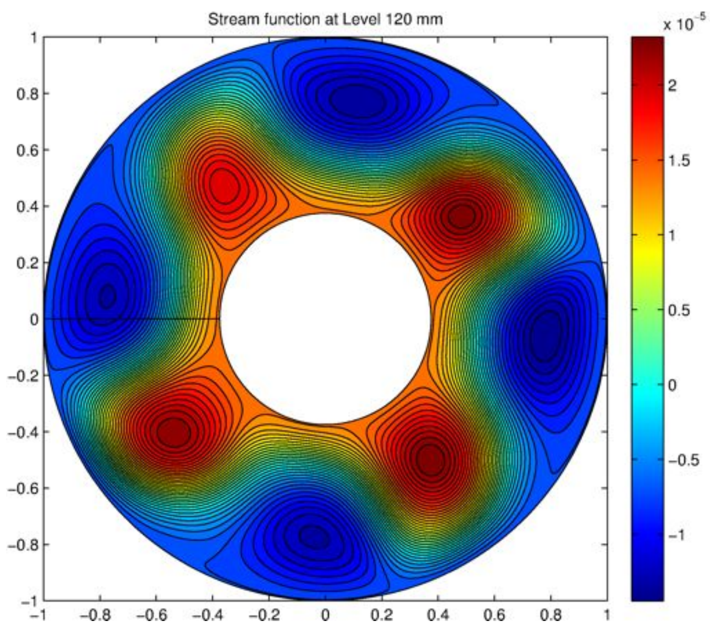
Decompositions of annulus data: 3-wave

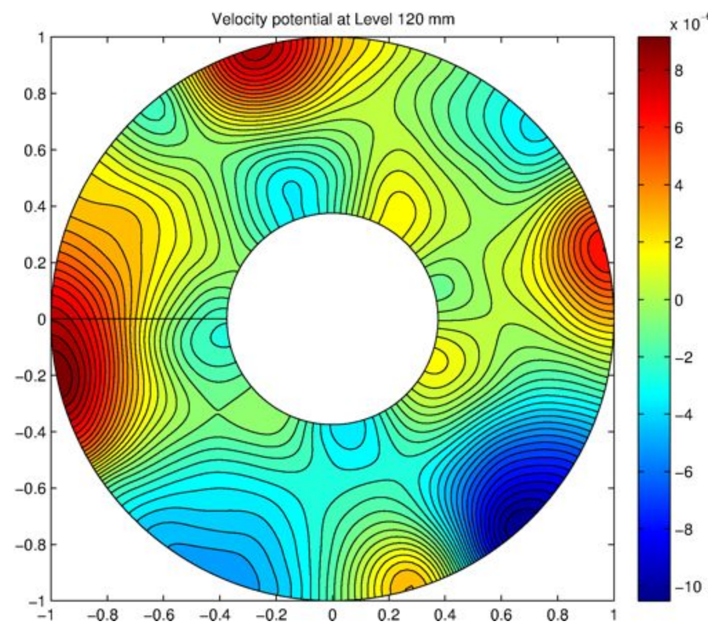


Decompositions of annulus data: 4-wave



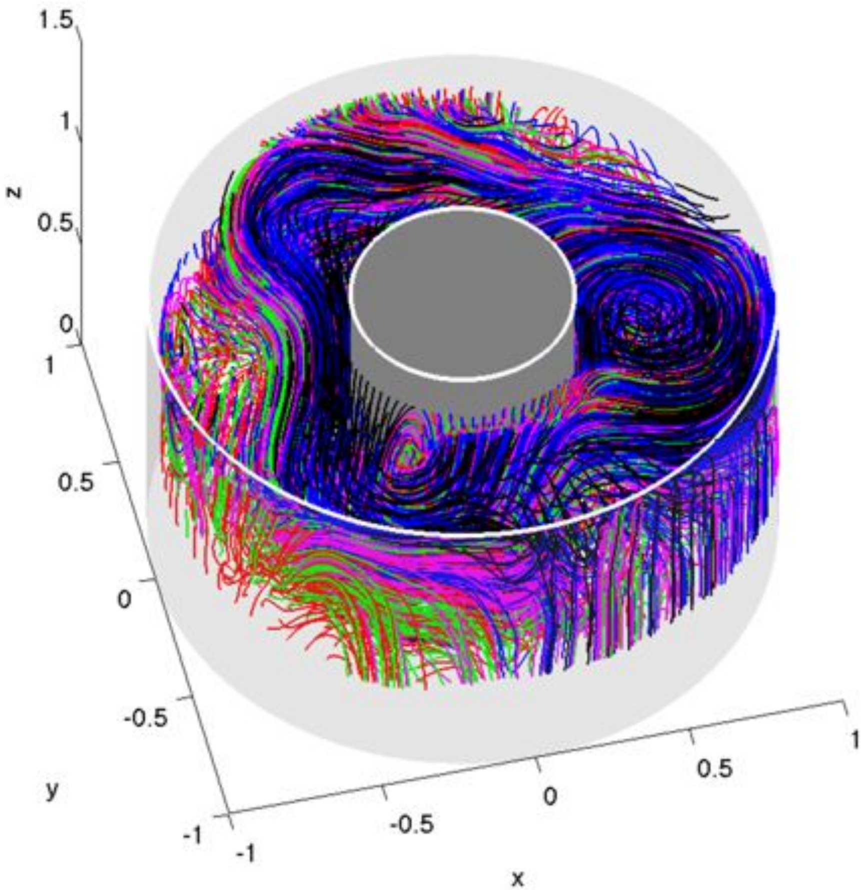




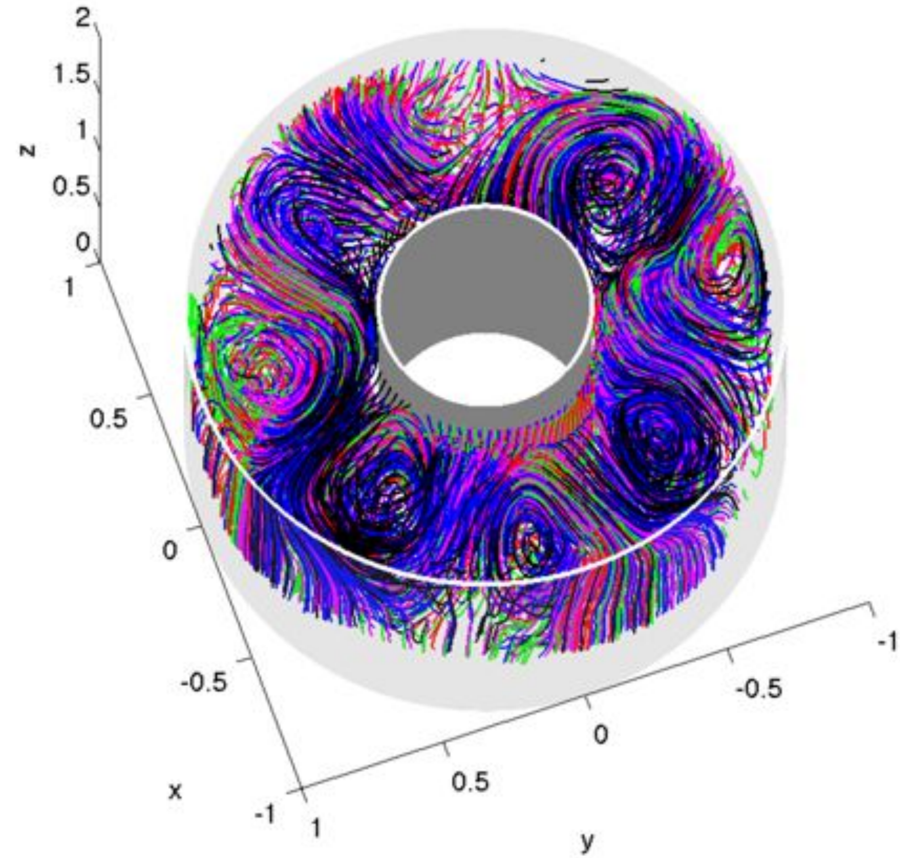


Full 3D reconstructions of the velocity

<u>3-wave pattern</u>



<u>4-wave pattern</u>



The colors of the streamlines correspond to the vertical levels of the streamline seeds:
red=40mm, green=60mm, magenta=80mm, blue=100mm, black=120mm

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Don't forget...