

PROBLEMS and TECHNIQUES

This issue's Problems and Techniques section contains, as usual, three papers on very different subjects: integration of singular functions by quasi-Monte Carlo methods; functions on the unit sphere with limited frequency range and maximal spatial concentration; and an approximation for a "cardiac restitution curve" that may aid in predicting the occurrence of heart disease.

The first paper, by Art Owen, is concerned with integrating real-valued functions $f(x)$ over the d -dimensional unit cube, that is, the computation of

$$I = \int_{[0,1]^d} f(x) dx.$$

This is done by means of quasi-Monte Carlo integration, which approximates the integral I by a finite-dimensional sum

$$S = \frac{1}{n} \sum_{i=1}^n f(x_i),$$

where the x_i are points in the cube $[0, 1]^d$. If $f(x)$ and its partial derivatives are appropriately bounded ("bounded variation") on the unit cube, and if the points x_i are appropriately chosen, then the approximation S converges to the integral I , in the sense that the asymptotic error $|S - I| \sim 1/n^{1-\epsilon}$ as $n \rightarrow \infty$ for all $\epsilon > 0$.

However, if $f(x)$ has a singularity and goes to $\pm\infty$ at a boundary of the unit cube, as happens with certain functions in computational finance, then the approximation S diverges—unless one can manage to keep the points x_i away from the boundary. Art Owen shows that this is possible for functions $f(x)$ with a singularity at the origin, provided their partial derivatives do not grow too fast near the boundary. In this case particular sequences of points x_i , so-called Halton sequences, avoid the origin (hence the title of the paper) and give rise to converging approximations S whose errors $|S - I|$ can be bounded in the same form as above. The approach is based on defining a well-behaved ("low-variation") extension of $f(x)$ close to the origin.

The second paper, by Frederik Simons, Anthony Dahlen, and Mark Wiecezorek, is based on doing the FFT on a sphere. This is important in a variety of applications, such as geophysics, where locally flat approximations are not adequate due to the curvature of the earth.

Let's start with the one-dimensional version of the *time-frequency concentration* problem: Find a function $g(t)$ whose FFT $G(\omega)$ vanishes for all frequencies ω outside the interval $[-W, W]$ (such a g is called bandlimited) but is *optimally concentrated in the time interval* $[-T, T]$. "Optimally concentrated" here means that $g(t)$ has minimal energy outside $[-T, T]$, i.e.,

$$\lambda = \int_{-T}^T g^2(t) dt / \int_{-\infty}^{\infty} g^2(t) dt \quad \text{is maximal.}$$

The ratio λ , which satisfies $0 < \lambda < 1$, measures the spatial concentration of the function g .

Now let's look at the sphere. The counterpart of the FFT is a decomposition into spherical harmonics, "frequencies" are now quantities associated with longitude and latitude, and time is replaced by a region on the sphere. The problem is to find bandlimited functions that are optimally concentrated in space. Again, "optimally concentrated" is

defined by a variational problem as the one above for λ . The authors show that λ can be expressed as the largest eigenvalue of a symmetric positive-definite eigenvalue problem, and that the eigenfunction associated with λ represents a bandlimited function with the the best spatial concentration.

In the third paper, John Cain and David Schaeffer derive an approximation for a function that may help in predicting abnormal heartbeats.

The main pumping action of the heart comes from the contraction of the left ventricle, which is stimulated by an electrical pulse, called the *action potential*. In the simplest model of heart dynamics, the electrical voltage behavior is segregated into two kinds of time intervals (think of an EEG). The first time interval is the *action potential duration*, during which the voltage rises rapidly above a threshold. The second time interval is the *diastolic interval*, when the voltage returns to the resting level. During this time the tissue is less sensitive to stimuli, so that the duration and size of the next action potential depend on this delay. The dependence of an action potential duration on the previous diastolic interval is described by the so-called *restitution curve*. The slopes of this curve may be useful in predicting ventricular fibrillation and sudden cardiac death.

The authors derive an asymptotic approximation for the restitution curve in the context of a “two-current ionic model” which is described by two ordinary differential equations.

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