Seismic ray theory in spherical Earth

Both $\alpha(r)$ and $\beta(r)$ fcns only of radius.

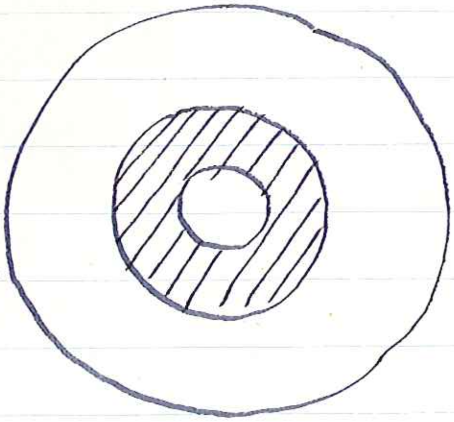
An immediate consequence of Fermat: ray paths in this case always lie in planes passing through center of $\oplus$.

Problem thus reduced to 2 dimensions.

Why is it useful to speak of P and S rays separately?

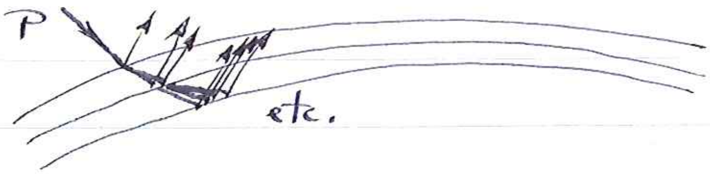
Earth has several major discontinuities. Two most important are inner core-outer core and core-mantle. The Moho is a less profound discontinuity than these two, there are also near disconts. or sharp gradient zones near 400 and 650 km depth in mantle.

also $\oplus$'s surface, obviously.

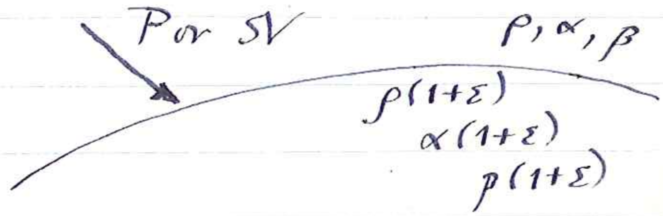


These major disconts. give rise to reflections and converted phases.

Can think of mantle as composed of many layers. Each will give rise to reflections and conversions



Consider problem:



Small contrast in properties, of <u>order</u> $\varepsilon \ll 1$.

Can be shown that if energy of incoming phase is E_{inc}, then:

$$\boxed{\begin{aligned} E_{\substack{\text{transmitted as}\\ \text{same type}}} &= E_{inc}\left[1 - O(\varepsilon^2)\right] \\ E_{\text{all other types}} &= O(\varepsilon^2) \end{aligned}}$$

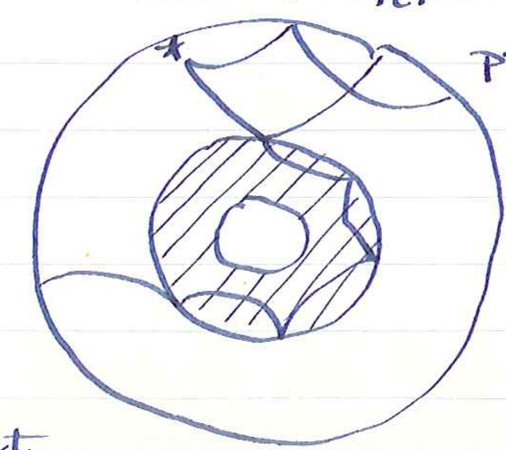
This justifies speaking of pure P and pure SV rays in regions without major discontinuities. Energy not transmitted as same type is of <u>second order</u>.

Body wave nomenclature

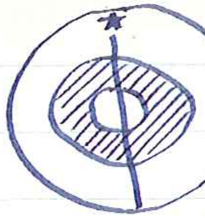
After a quake energy can arrive at a given station by a variety of stationary time paths. The conventional notation for the various paths is as follows:

P : P wave in mantle
S : S wave in mantle
K : P wave in outer core (no S wave in outer core)
I : P wave in inner core
J : S wave in inner core
c : external reflection off CM boundary
i : external reflection off IC-OC bdry
p, s : reflection off free surface

Examples:

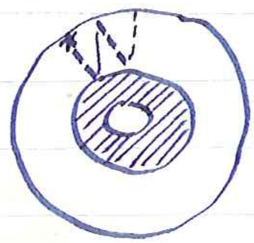


Note again: ray paths curve upward throughout most of ⊕ since $\alpha(r)$, $\beta(r)$ decrease with increasing r.

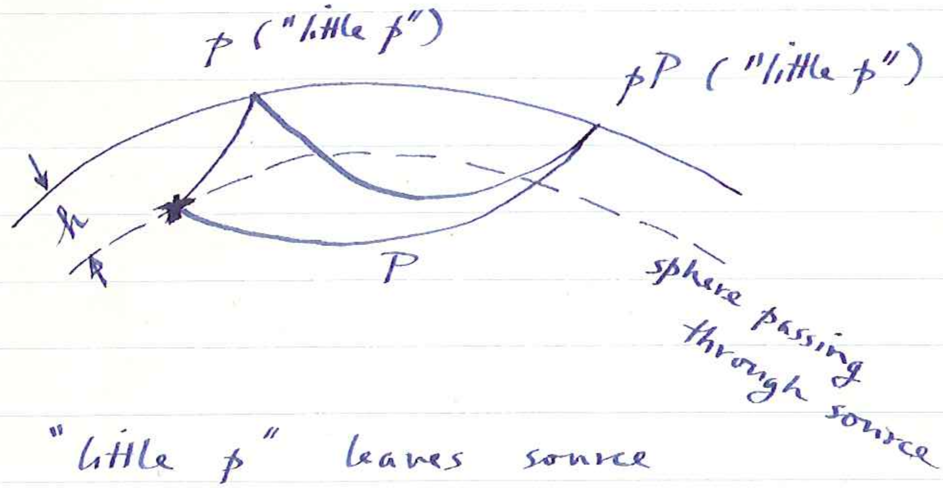


PKIKP straight through ⊕
(T = 22 m for surface
focus quake)

ScS ScSScS or
ScS2 for short



(T = 15.5 m for surface
focus, $\Delta = 0$, ScS)

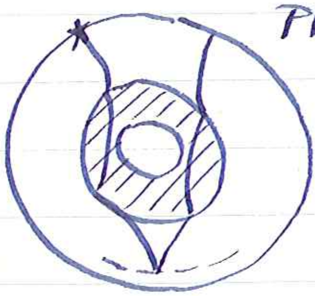


"little p" leaves source
going <u>up</u> rather than down
(out top 1/2 of focal sphere)

Comparing arrival times of pP and P
an excellent way to determine depth
h of a quake, relatively insensitive
to epicentral distance Δ. 1st lab exercise.

Some other notation has been suggested, not in as common a use as above, e.g.

d : reflections off discontinuities in upper mantle

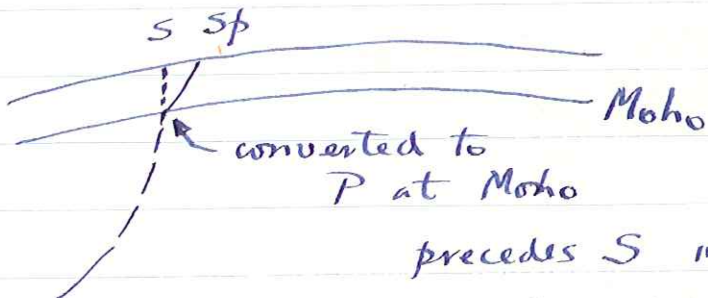


PKPdPKP or P'dP'

$P' \equiv PKP$

also in use : P'400P' for reflection off 400 km discont.

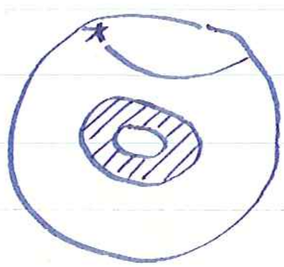
Sp :



precedes S in time, a way to determine Moho depth.

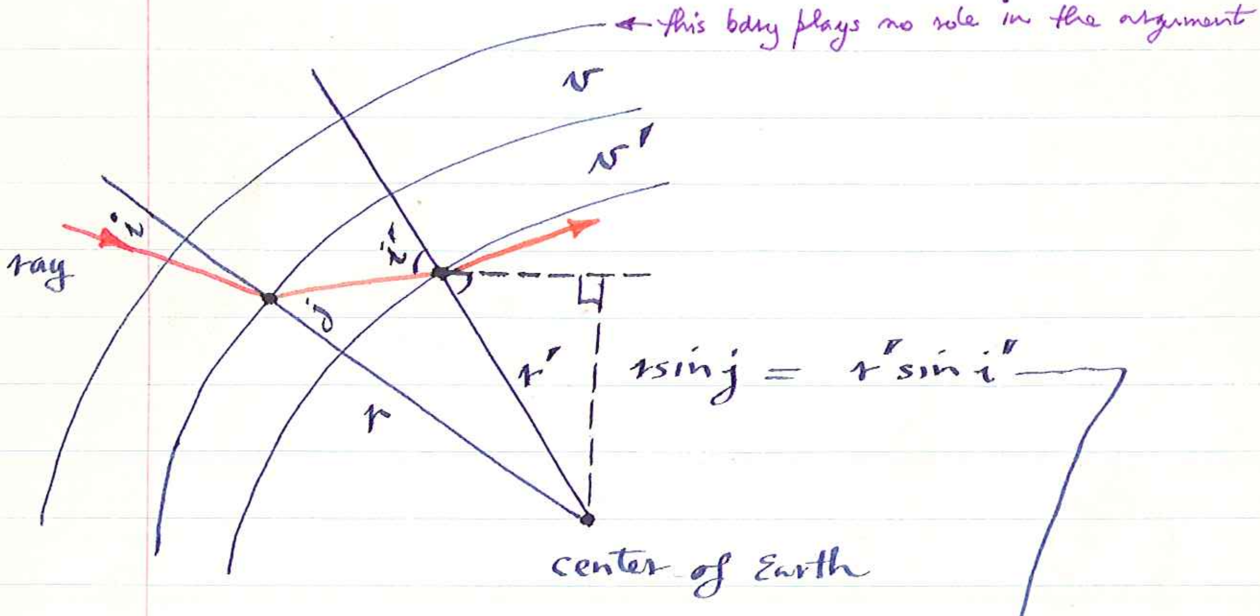
The ray parameter:

Probably the most important concept in seismology. Consider a ray, P or S, in the $\oplus$:



$\alpha(r)$ and $\beta(r)$ are, except for LVZ, increasing fns of depth, so the rays dive down as shown to arrive along least time paths.

Treat mantle (or core) as limit of ∞ number of very thin concentric layers. Consider two such layers, velocities v, v', radii to base r, r'



$$r \sin j = r' \sin i'$$

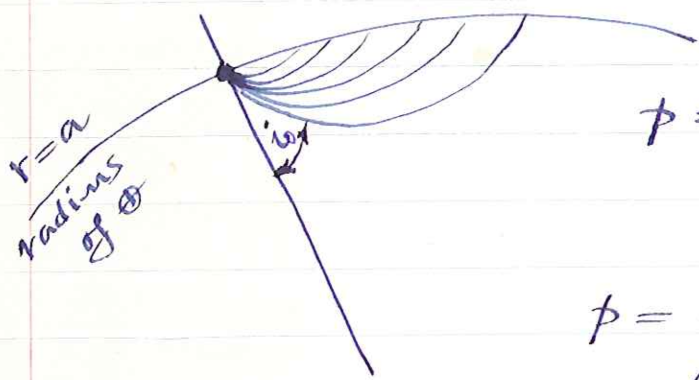
same

Snell's law: $$\frac{r \sin i}{v} = \frac{r \sin j}{v'} = \frac{r' \sin i'}{v'}$$

Thus we see that:

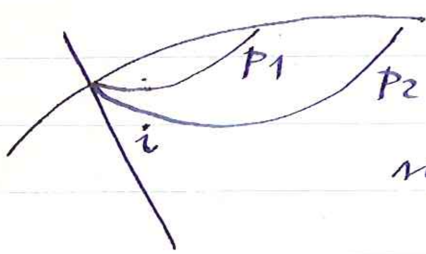
$$\boxed{p = \frac{r \sin i}{v} \text{ is \underline{constant} along the ray}}$$

Called the <u>parameter</u> of the ray. The various rays radiating from the source have different ray parameters. For a <u>surface focus quake</u>



$$p = \frac{a \sin i_0}{v_{crust}}$$

$p = 0$ corresponds to $i = 0$
ray going <u>straight down</u> from source

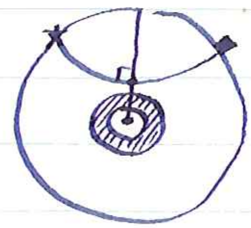


ray parameter $p_2 < p_1$
(actually as we shall soon see this picture is too simple, because of upper mantle <u>triplications</u>.

Units of p (sec per radian).
Conventional to multiply by $\pi/180°$ and

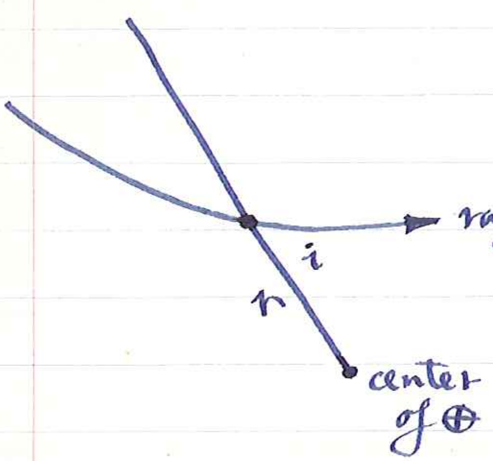
measure in sec / degree, epicentral distances Δ usually measured in degrees. Range of p for P in mantle is 5 - 15 sec / deg.

At bottoming point of ray, radius r_p,
$i = 90°$,
$\sin i = 1$



$$\boxed{p = r_p / v(r_p)} \; *$$

Note that $1/p$ is simply the angular velocity of the ray or wave



linear velocity of wave at pt. • is

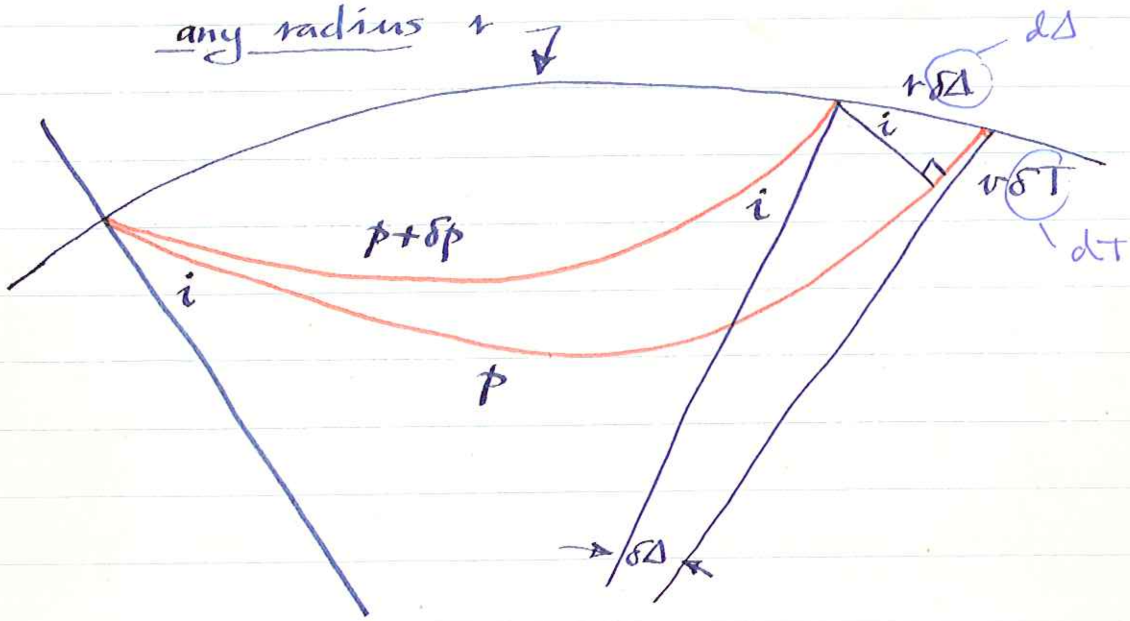
$\underline{v} = \underline{\omega} \times \underline{r}$
$\underline{\omega}$ = angular velocity

But $|\underline{v}| = v$, so $v = \omega r \sin i$, or

$$\boxed{\frac{1}{p} = \frac{v}{r \sin i} = \omega, \text{ angular velocity}}$$

Note from * that if p can be measured, it provides in principle a means of determining v at the bottoming or turning point r_p.

Can p be measured? Yes.



p is the angular slowness

Consider 2 nearby rays p, $p+\delta p$ from same source, at a radius r

Clearly

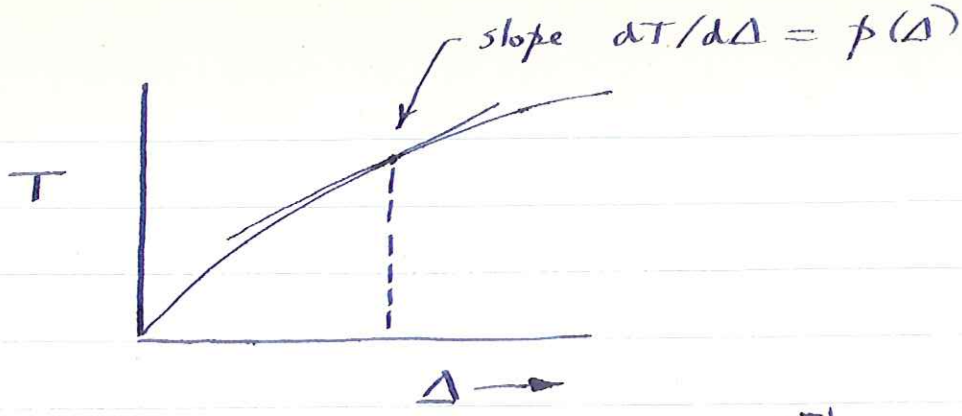
$$\sin i = \frac{v\,\delta T}{r\,\delta\Delta}, \quad \text{or}$$

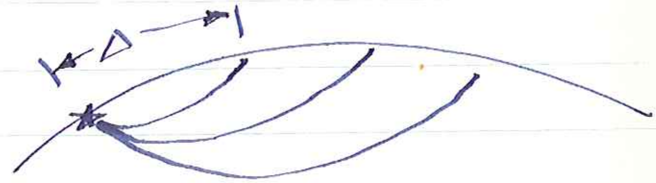
dT, $d\Delta$

$$r\sin i / v = \cancel{\delta T/\delta\Delta}\ dT/d\Delta$$

$$\boxed{p = dT/d\Delta}$$

$p(\Delta)$ is the slope of the $T(\Delta)$ travel time curve.

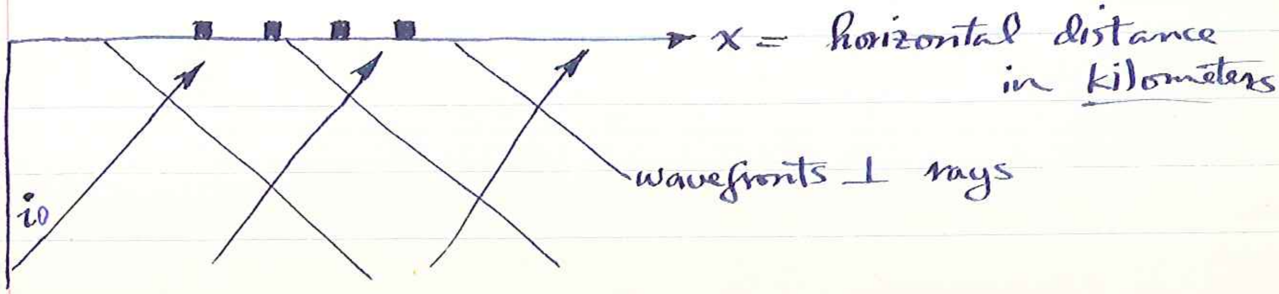




Δ = epicentral distance
(note: epicenter vs.
hypocenter)

Note: again we have $1/p = d\Delta/dT$, the angular velocity of the ray.

With an array of seismometers p can be measured directly



$$p = \frac{a \sin i_0}{v_{crust}} = \frac{dT}{d\Delta} = a\,\frac{dT}{a\,d\Delta} = a\,\frac{dT}{dx}$$

$$\boxed{dT/dx = \frac{\sin i_0}{v_{crust}}}$$

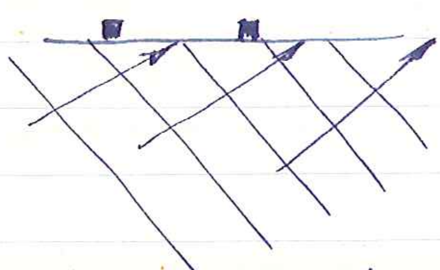
dT/dx can be measured directly with an array

$$\boxed{p = a\,\frac{dT}{dx}}$$

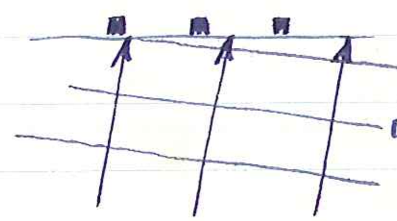
$\frac{dT}{dx}$ = horizontal slowness

dx/dT = apparent velocity of wavefronts as they pass through a horizontal array.

Note the steeper the incoming ray, larger is dx/dT and the smaller is p, $p = 0$ is coming straight up



wavefronts



wavefronts

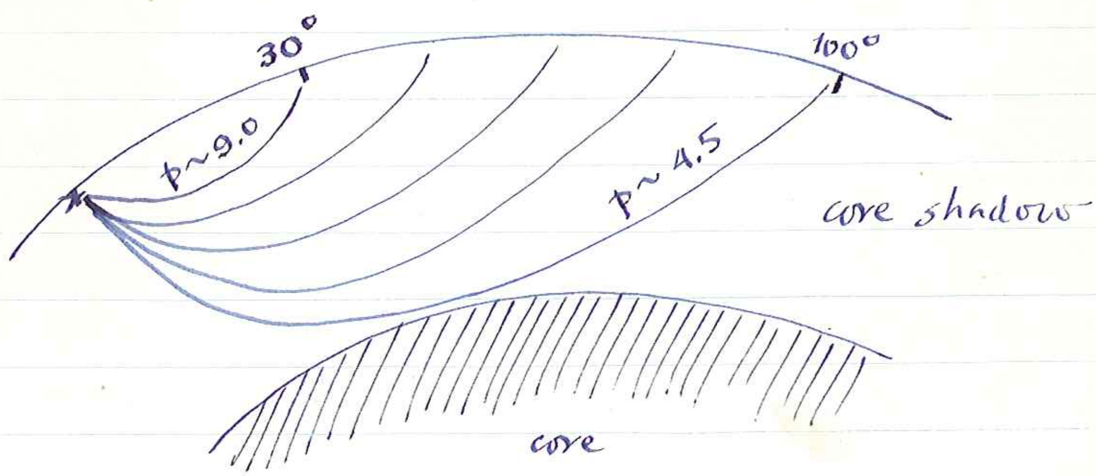
Numerous arrays: LASA, TFSO, NORSAR, S. California, etc.

Alternatively $p(\Delta)$ can be obtained by numerical differentiation of $T(\Delta)$.

Lane Johnson performed a now classical study of $p(\Delta)$ using the TFSO array in Arizona.

Fig. 2 shows location of array, and quakes $\Delta > 30°$ used in study.

Fig. 3 shows <u>measured</u> $p(\Delta)$, smoothly decreasing from about 9 sec/deg at $\Delta = 30°$ to about 4.5 sec/deg at about $\Delta \approx 100°$, near core shadow.



The <u>smooth</u> variation of $p(\Delta)$ for $\Delta > 30°$ indicative of smooth velocity profile below ~ 650 km depth, α increases <u>smoothly</u> from ~11 to ~13.5 km/sec.

Fig. 9 shows α vs. depth below ~ 650 km depth, with bottoming depths of rays indicated, i.e. ray emerging at 70° from surface source bottoms at about 2000 km depth.

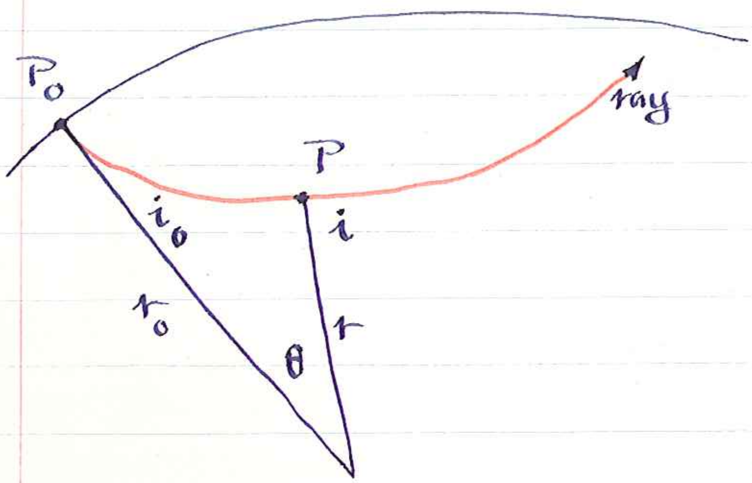
Behaviour of $p(\Delta)$ for $\Delta < 30°$ in contrast very complicated, multiple-valued, see Fig. 4, this due to steep gradients of α vs. r, *phase transitions* in upper mantle.

Fig. 5 shows location of array in more detail.

The forward problem of body wave seismology:

Given an ⊕ model $v(r)$ how do we find $T(\Delta)$ or $p(\Delta)$?

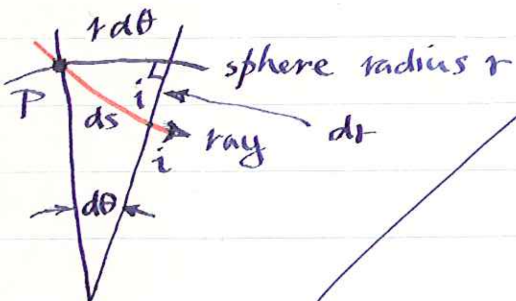
This problem is easy.



i_0: initial incidence ∠

i: incidence ∠ at pt. P (arbitrary pt. on ray)

r, θ : polar coords



$$\sin i = \frac{r\,d\theta}{ds}$$

ds = an incremental distance along ray

$$\left.\begin{aligned} ds &= \frac{r}{\sin i}\,d\theta = \frac{r^2}{vp}\,d\theta \\ ds^2 &= dr^2 + r^2\,d\theta^2 \end{aligned}\right\} \text{two equations}$$

$$\boxed{\begin{aligned} ds &= \frac{r^2}{vp}\,d\theta \\ ds^2 &= dr^2 + r^2\,d\theta^2 \end{aligned}}$$

eliminate $d\theta$ from these two.

$$\frac{p^2 v^2}{r^2} = r^2 \left(\frac{d\theta}{ds}\right)^2 = 1 - \left(\frac{dr}{ds}\right)^2$$

$$\left(\frac{dr}{ds}\right)^2 = 1 - \frac{p^2 v^2}{r^2}$$

We shall need $r/v(r)$ a lot. Call it:

$$\boxed{\eta(r) \equiv r/v(r)}$$ — a fn. only of radius.

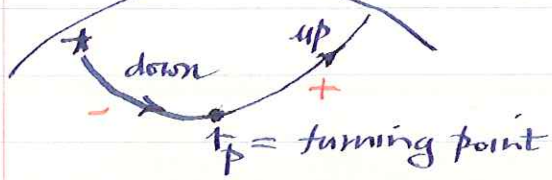
$$\left(\frac{dr}{ds}\right)^2 = \frac{\eta^2 - p^2}{\eta^2} \qquad \left(\frac{ds}{dr}\right)^2 = \frac{\eta^2}{\eta^2 - p^2}$$

$$ds = \pm \eta \left(\eta^2 - p^2\right)^{-1/2} dr$$

The time to travel ds is

$$dT = \frac{ds}{v} = \pm \frac{\eta^2}{r} \left(\eta^2 - p^2\right)^{-1/2} dr$$

sign depends on whether wave is travelling up ($dr > 0$) or down ($dr < 0$)



$\eta = \frac{r}{v}$ always $> p = \frac{r}{v} \sin i$

+ : wave going up

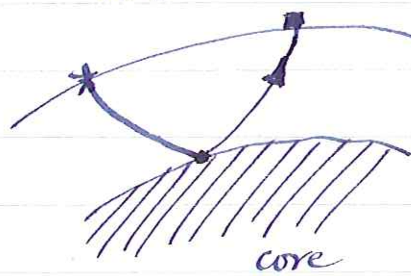
− : wave going down

Total travel time: integrate segments, i.e.

$$T = \int_{ray} dT = \int \frac{\eta^2 \, dr}{r(\eta^2 - p^2)^{1/2}}$$

What are the limits? Upper limit clearly radius of source or receiver.
Lower limit: two cases.

1. ray reflected off an interface, e.g. PcP

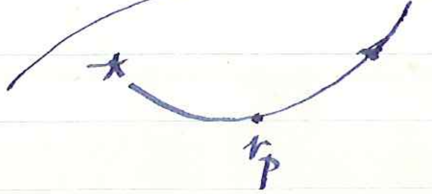


time to travel from $r = b$ to $r = a$ is

$$T = \int_b^a \frac{\eta^2 \, dr}{r(\eta^2 - p^2)^{1/2}}$$

Note: PcP time is twice this (down and back)

2. direct P is not reflected, merely turned around.



at turning point
$i = 90°$
$p = \eta(r_p)$,
wave is travelling horizontally.

Suppose source at surface $r = a$ as well as receiver, simplest case, then both ray segments are the same.

$$\boxed{T = 2 \int_{r_p}^{a} \frac{\eta^2 \, dr}{r(\eta^2 - p^2)^{1/2}}}$$

surface focus quake, travel time for ray of parameter p.

We found T by eliminating $d\theta$ from 2 eqns. Let's eliminate ds instead. Find:

$$d\theta = \pm \frac{p}{r} (\eta^2 - p^2)^{-1/2} \, dr$$

$+$: wave going up

$-$: wave going down

$$ds^2 - r^2 d\theta^2 = dr^2$$

$$r^2 \left(\frac{\eta^2}{p^2} - 1\right) d\theta^2 = dr^2$$

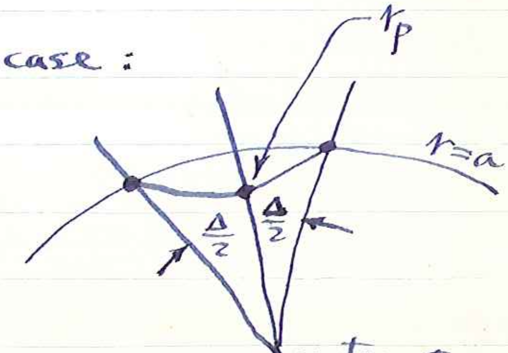
$$r^2 (\eta^2 - p^2) \, d\theta^2 = p^2 \, dr^2$$

We can use this to find Δ, since

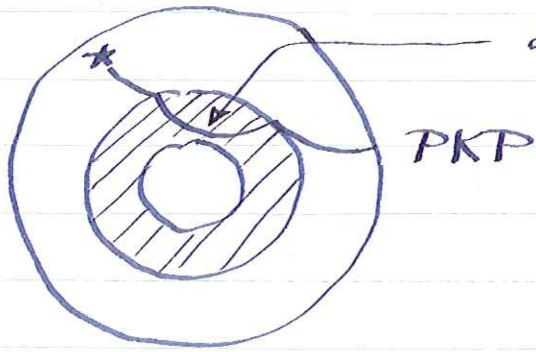
$$\Delta = \int_{ray} d\theta$$

Thus for the surface focus case:

$$\boxed{\Delta = 2 \int_{r_p}^{a} \frac{p \, dr}{r(\eta^2 - p^2)^{1/2}}}$$



If focus not at surface, must write integral for each segment separately, must do the same for refracted phases such as PKP.



all three segments have <u>same ray parameter</u> p, even if there are strong refractions, reflections or conversions.

This solves the <u>direct</u> or <u>forward problem</u>: given an $\oplus$ model $v(r)$, we can calculate $T(p)$ and $\Delta(p)$, thus $T(\Delta)$ and/or $p(\Delta)$

these the theoretical data or observed quantities for a given model.

<u>Inverse problem</u>: given observations of $T(\Delta)$ or $p(\Delta)$, find $v(r)$ or $\eta(r)$.