Gravitational field of $\oplus$ and shape of $\oplus$.

The $\oplus$ as a static distribution of mass.
Official name of this subject: geodesy

An old + venerable subject: Eratosthenes, Newton.

Study of $\oplus$ grav. field closely linked to measurement of position on $\oplus$ surface, i.e. surveying. Why?
Use of plumbob (local vertical) in surveying.

Historically main goal of geodesy to determine shape or figure of $\oplus$.

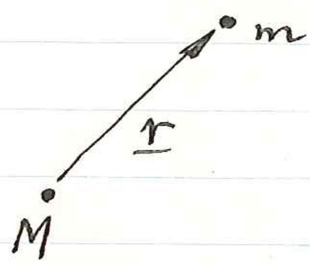
This antique subject recently revolutionized by modern technological developments, esp. analysis of orbits of artificial satellites and even more recently radar altimetry from satellites, and GPS (radio) interferometry.

We shall study both history + modern developments.

Relevant sections in Stacey are 3.1, 3.2, 4.1, 4.2 and 4.3. In Garland Chs. 9, 10, 11, 12, 13.

A vector field : the gravitational field of the ⊕.

Recall nature of grav. attraction 2 pt. masses.



$\underline{F}(\underline{r}) \equiv$ force exerted by M on "test mass" m at $\underline{r}$

$$\boxed{\underline{F}(\underline{r}) = -\hat{r}\,\frac{GMm}{r^2}}$$

inverse square law of attraction (Newton)

$G =$ Newton's constant
$= 6.67 \cdot 10^{-11}\ m^3\,kg^{-1}\,sec^{-2}$
$(N m^2/kg^2)$

We define $\underline{g}(\underline{r}) \equiv$ grav. field produced throughout space by particle M.

$\underline{g}(\underline{r}) = -\hat{r}\,\frac{GM}{r^2}$. Then $F(\underline{r}) = m\underline{g}(\underline{r})$

$\underline{g}(\underline{r}) \equiv$ force / unit mass on a test particle at $\underline{r}$

Called gravitational field or gravitational acceleration. Vector field.

Special case of a so-called conservative field. If we define

$$V(\underline{r}) = -\frac{GM}{r} = -\frac{GM}{|\underline{r}|}. \quad \text{Then}$$

$$\boxed{\underline{g}(\underline{r}) = -\nabla V(\underline{r})} \qquad |\underline{r}| = \sqrt{x^2+y^2+z^2}$$

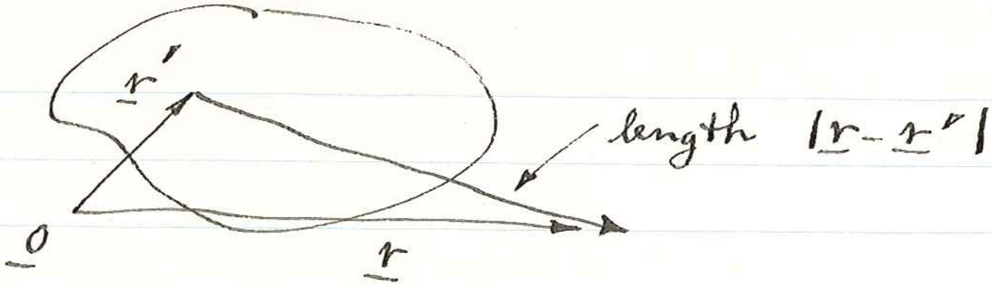
Much more convenient to work with scalar field : called gravitational potential.

Grav. pot. of a pt. mass $= -GM/r$.

Conservative because takes no net work to move a test mass in a closed circuit.

$$\text{Work} = m\int_A^B g(r)\cdot dr = -m\int_A^B \nabla V\cdot dr = 0 \text{ around a closed loop}$$
$$= -m\left[V(B)-V(A)\right]$$

To determine grav. pot. of ⊕, can consider it to consist of ∞ number of pt. masses.



$\rho(\underline{r}')$ mass density throughout ⊕

Then for $\underline{r} \in V$ or $\notin V$

mass element $dM' = \rho(\underline{r}')\,dV'$

$$V(\underline{r}) = -G \int_{\text{Earth}} \frac{dM'}{|\underline{r}-\underline{r}'|} \quad \text{or}$$

$$\boxed{V(\underline{r}) = -G \int_{\oplus} \frac{\rho(r')\,dV'}{|\underline{r}-\underline{r}'|}}$$

perhaps better to call $dV' = d^3r'$

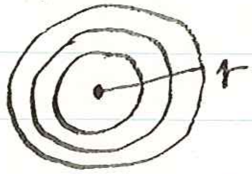
If we knew ⊕'s $\rho(\underline{r})$ simple matter to find grav. pot. $V(\underline{r})$ just sum of pt. masses.

This is <u>direct</u> problem (so-called) of study of ⊕ gravity.

Real geophysical problem is rather the <u>inverse</u> <u>problem</u> (this a very typical situation).

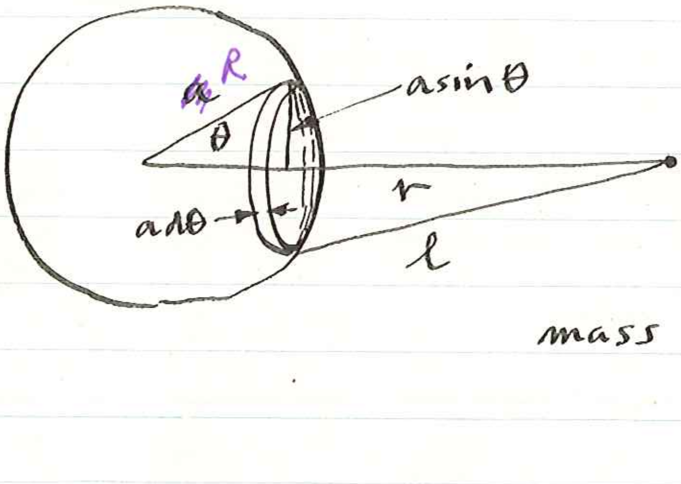
Stated generally, say we could measure $V(\underline{r})$ or $\underline{g}(\underline{r}) = -V(\underline{r})$ on or near ⊕'s surface. What can we infer about density $\rho(\underline{r})$, a quantity of interest?

Answer, unfortunately, very little.
An example shows this clearly.
Say ⊕ is spherically symmetric or radially stratified $\rho(\underline{r}) = \rho(r)$ only



What is $V(\underline{r})$ outside?

Break sphere up into shells.



Consider a single shell thickness t and density ρ. Total mass $M = 4\pi a^2 t\rho$

Consider ring width $a\,d\theta$, all particles same distance l from obs. pt.

Volume of ring = $2\pi \overset{R}{a} \sin\theta \cdot \overset{R}{a}\, d\theta \cdot t$
Mass of ring $= dM = 2\pi t \rho \underset{R^2}{a^2} \sin\theta \, d\theta$

Potential due to ring $= dV = -\dfrac{G\,dM}{l}$

$$dV = \frac{-G\,dM}{\sqrt{r^2 + \underset{R^2}{a^2} - 2\overset{Rr}{ar}\cos\theta}}$$

$l^2 = (r - \underset{R}{a}\cos\theta)^2 + (\underset{R}{a}\sin\theta)^2$

↑ this assumes we're <u>outside</u> the shell

<u>no, not so, see below.</u>

Total potential of entire shell

$$V = -2\pi G\rho t\,\underset{R^2}{a^2}\int_0^{\pi}\frac{\sin\theta\,d\theta}{\sqrt{r^2 + \underset{R^2}{a^2} - 2\underset{R}{a}r\cos\theta}}$$

$$= -\frac{GM}{2}\int_0^{\pi}\frac{\sin\theta\,d\theta}{\sqrt{r^2 + \underset{R^2}{a^2} - 2\underset{R}{a}r\cos\theta}}$$

$M = 4\pi \overset{R^2}{a^2}\rho t =$ mass of whole shell

let $x = r^2 + \overset{R^2}{a^2} - 2\overset{R}{a}r\cos\theta$

$dx = 2\underset{R}{a}r\sin\theta\,d\theta$

$$V = -\frac{GM}{4\underset{R}{a}r}\int_{(r-\underset{R}{a})^2}^{(r+\overset{R}{a})^2} x^{-1/2}\,dx$$

← here's the only change if inside: this becomes $(a-r)^2$ leading to $V(r) = -\dfrac{GM}{a}$, constant inside.

$$= -\frac{GM}{2\underset{R}{a}r}\left(r + \overset{R}{a} - r + \overset{R}{a}\right)$$

$$\boxed{V(r) = -\frac{GM}{r}}$$

$-\dfrac{GM}{R}$ inside

A thin hollow shell attracts as if all its mass conc. at center. Same obviously true of any radially stratified sphere.

Every spher. symm. body has same <u>external</u> potential $V(r) = -GM/r$.

Obviously grave consequences for inverse problem. <u>Lack of uniqueness</u> an inherent property of gravity problems. This but one example of it.

If $\oplus$ spherical, measurement of gravity or grav. accel. on surface $r = a$ R plus knowledge of a yields no info. about $\rho(r)$ except total mass

$$M = 4\pi \int_0^{\overset{R}{a}} \rho(r)\, r^2\, dr$$

$$g \equiv g(\underset{R}{a}) = |\underline{g}(\overset{R}{a})| = \frac{GM}{\underset{R^2}{a^2}}$$

This one way of measuring $M_\oplus$ if complications ($\underline{\Omega}$ and not spher. symmetric) are ignored.

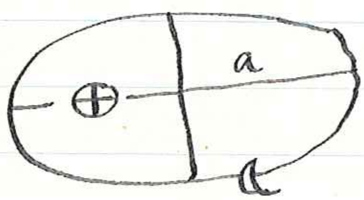
Note however only GM can be so measured

$$GM \cong 980 \text{ cm/s}^2 \times (6.371 \times 10^8)^2 \text{ cm}^2$$
$$\cong 4 \cdot 10^{20} \text{ cm}^3 \text{s}^{-2}$$

Can any other geophysical measurement measure M directly? No.

Much more accurate estimate: <u>Kepler's third law</u>

give, Gauss's proof here next page



a ≡ semi-major axis of lunar orbit
T = period of orbit $\cong$ 28 days

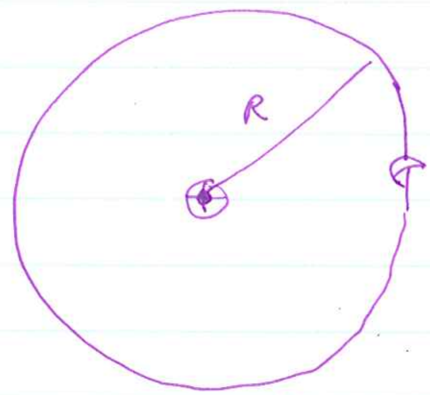
$$\boxed{\frac{4\pi^2 a^3}{T^2} = G(M_\oplus + M_☾)}$$

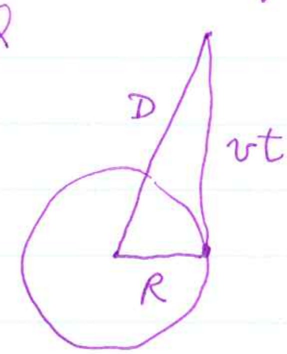
$M_☾ \approx \frac{1}{81.3} M_\oplus$ determined now in same way from lunar orbiters.

Before that, det. from monthly <u>apparent</u> motion of nearby planets due to motion of $\oplus$ w.r.t. c.o.m. of $\oplus$☾ system.

Circular orbit radius R: switch off gravity

T = period





$t = 0$

angular velocity $\frac{2\pi}{T}$

velocity $v = \frac{2\pi R}{T}$

Distance from $\oplus$: $D(t) = \sqrt{R^2 + v^2 t^2}$

Rate of acceleration away from $\oplus$:

$$\ddot{D}(t) = \frac{d^2}{dt^2}\left(R^2 + v^2 t^2\right)^{1/2} = \frac{v^2 R^2}{\left(R^2 + v^2 t^2\right)^{3/2}}$$

$$\dot{D} = \left(R^2 + v^2 t^2\right)^{-1/2} v^2 t$$

$$\ddot{D} = \frac{v^2\left(R^2 + v^2 t^2\right)}{\left(R^2 + v^2 t^2\right)^{3/2}} - \frac{v^4 t^2}{\left(R^2 + v^2 t^2\right)^{3/2}} = \frac{v^2 R^2}{\left(R^2 + v^2 t^2\right)^{3/2}}$$

$\ddot{D}(0) = \frac{v^2}{R}$, radial acceleration in a circular orbit

Balanced by gravity

$$\frac{v^2}{R} = \frac{GM}{R^2} \quad \text{or} \quad \boxed{\frac{4\pi^2 R^3}{T^2} = GM}$$

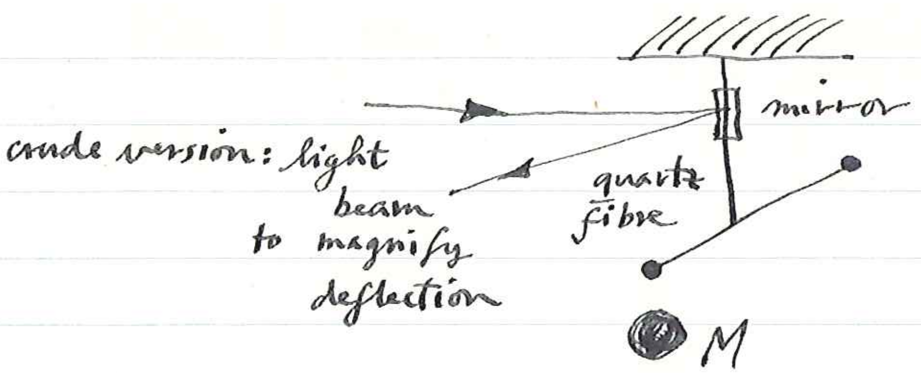
Kepler's third law

Measurement of a, T gives $GM_\oplus$ after correcting for M_{a}, other perturbations etc.

$\exists$ no purely geophysical measurement of $M_\oplus$. To find $M_\oplus$ must measure G in lab.

Gravity very weak force, difficult measurement. One of poorest known physical constants.

Method: Cavendish balance (Lord Cavendish 1798)



small: a few inches

$$G = 6.67 \cdot 10^{-8} \quad cm^3\, gm^{-1}\, sec^{-2}$$

$GM_\oplus$ known much more accurately.
Gaposchkin Dec 1974 JGR

$$GM_\oplus = 3.986013 \cdot 10^{20} \quad cm^3/s^2$$

Corresponding $M_\oplus = 5.98 \cdot 10^{27}$ g
Divide by total volume.

$$\boxed{\text{Mean density } \bar{\rho} = 5.517 \text{ gm}/\text{cm}^3}$$

First det. by Cavendish: first clue that $\rho(r) \nearrow$ with depth since density of surface rocks $\approx 2.5 - 3.3$.

Doppler measurements
- spacecraft in solar system
- extrasolar planets

$\leftarrow T = \frac{1}{f_0} \rightarrow$

freq f_0 

observer

v toward

say it's transmitting spikes 

$1/f_0$

between 2: travels $ds = \frac{v}{f_0}$

second spike arrives $\frac{ds}{c} = \frac{1}{f_0}\frac{v}{c}$ earlier

~~apparent freq:~~ $\cancel{f = \frac{f_0}{1-v/c}}$

time between arrivals $\frac{1}{f_0} - \frac{1}{f_0}\frac{v}{c} = \frac{1}{f_0}\left(1-\frac{v}{c}\right)$

apparent freq: $\boxed{f = \frac{f_0}{1-v/c}}$ $\quad \frac{f_0}{1+v/c}$ toward

Doppler shifted to higher freq if moving toward

train noise at crossing 

spacecraft: angular tracking and Doppler velocity

Extrasolar planets

How far are nearest stars?

~ 10 light years

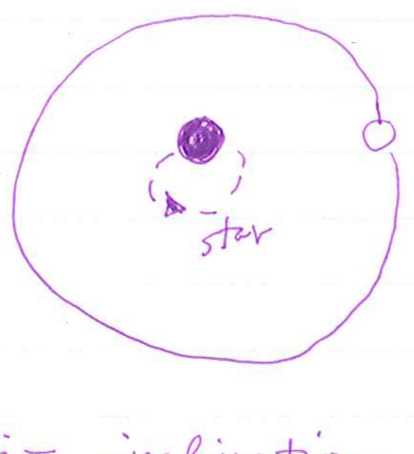
1 AU $= 500$ light seconds

Pluto ≈ 40 AU $\approx 2 \cdot 10^4$ light seconds $\approx 10^{-3}$ light years

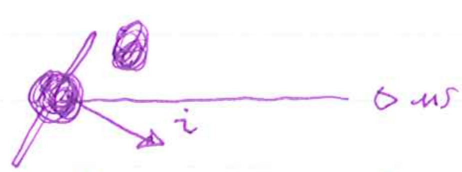
if solar system $\simeq 1$ m (on projection screen) nearest star is 4 km away Market Fair Mall — or dam on Lake Carnegie

Resolution of Doppler shift method is $v \sim 3$ m/sec $\Rightarrow$ $\Delta f_0 / f_0 \approx 10^{-8}$

cannot resolve extrasolar planets — look for Doppler shift of spectral lines in ★



if plane of orbit is $\perp$ line of sight — no signal



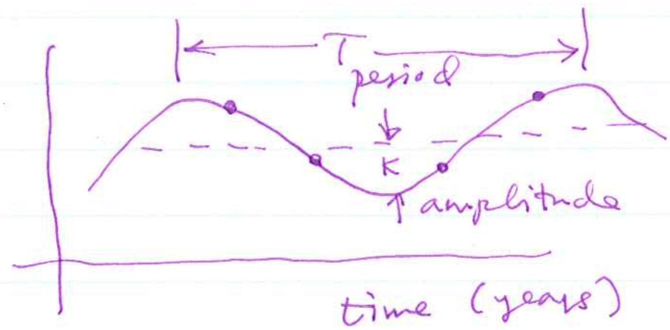
$i =$ inclination of orbital plane to line of sight

observable quantity is $v_\star \sin i$ —
line of sight velocity

Astronomers call this $K = v_\star \sin i$

Can also observe period of motion

Doppler velocity



$$\frac{4\pi^2 R^3}{T^2} = GM_\star$$

$$v_p = \sqrt{\frac{GM_\star}{R}} \qquad \text{i.e.} \quad \frac{v_p^2}{R} = \frac{GM_\star}{R^2}$$

Cons of momentum: $M_p v_p = M_\star v_\star$

$$v_\star = \left(\frac{M_p}{M_\star}\right) v_p \ll v_p$$

$$K = v_\star \sin i$$

$$= \frac{M_p \sin i}{M_\star} v_p = \frac{M_p \sin i}{M_\star} \sqrt{\frac{GM_\star}{R}}$$

$$\boxed{K = \sqrt{\frac{G}{M_\star}} \,(M_p \sin i)\, R^{-1/2}}$$ — the detection threshold

$$R^{-1/2} = \left(\frac{G M_\star T^2}{4\pi^2} \right)^{-1/6}$$

$$\boxed{K = \left(\frac{2\pi G}{T} \right)^{1/3} \frac{(M_p \sin i)}{M_\star^{2/3}}}$$

Can solve for $M_p \sin i$ as fcn of K, T.

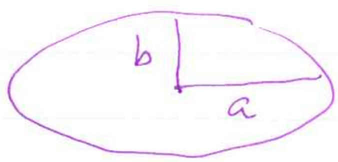
Easiest to find "hot Jupiters"

Before first detection — in 1995 — presumption was that all solar systems would look like ours.

More precise result in GN notes

$$K = \left(\frac{2\pi G}{T} \right)^{1/3} \frac{M_p \sin i}{(M_\star + M_p)^{2/3}} \frac{1}{\sqrt{1-e^2}}$$

$e =$ eccentricity



$b = a\sqrt{1-e^2}$

HD 209458 Transit

- *Download preprint accepted by Astrophysical Journal Letters, 23 November, 1999 (310Kb PS file or 162Kb PDF file).*

- *Download ASCII file of HD 209458 photometric data.*

Planetary Transit Across Star HD 209458 Detected by STARE Project Astronomers

At the end of August, 1999, STARE Project Astronomers Dr. Tim Brown and David Charbonneau started observing the star HD 209458 on the advisement of Dr. David Latham of the Harvard-Smithsonian Center for Astrophysics. Latham inferred from his radial-velocity analysis of the motion of the star (refer to Exoplanet Search Methods for more information) that a planetary companion was present and possibly orbiting in an edge-on orientation. When an orbit is aligned this way, the planet will pass between its star and the Earth (transit) once each orbit, causing a slight dimming of the star's light as in the figure below (see STARE Search Method for a more detailed explanation). After analyzing data collected by the STARE telescope, such a dimming was discovered occurring on the nights of September 9th and 16th. The times of these transits coincided exactly with the expected times predicted by the radial-velocity analysis. A partial transit in November was observed by G. Henry (Tennessee State University). Further details on this observation are available at the TSU Automated Astronomy Group website.

Light Curve of a Star During Planetary Transit

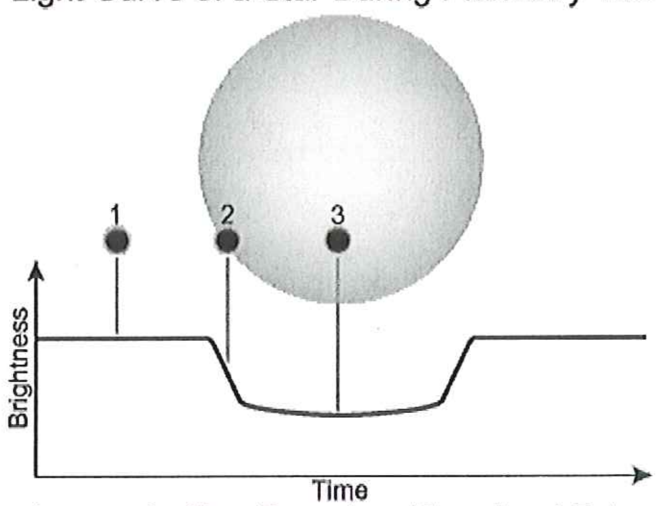
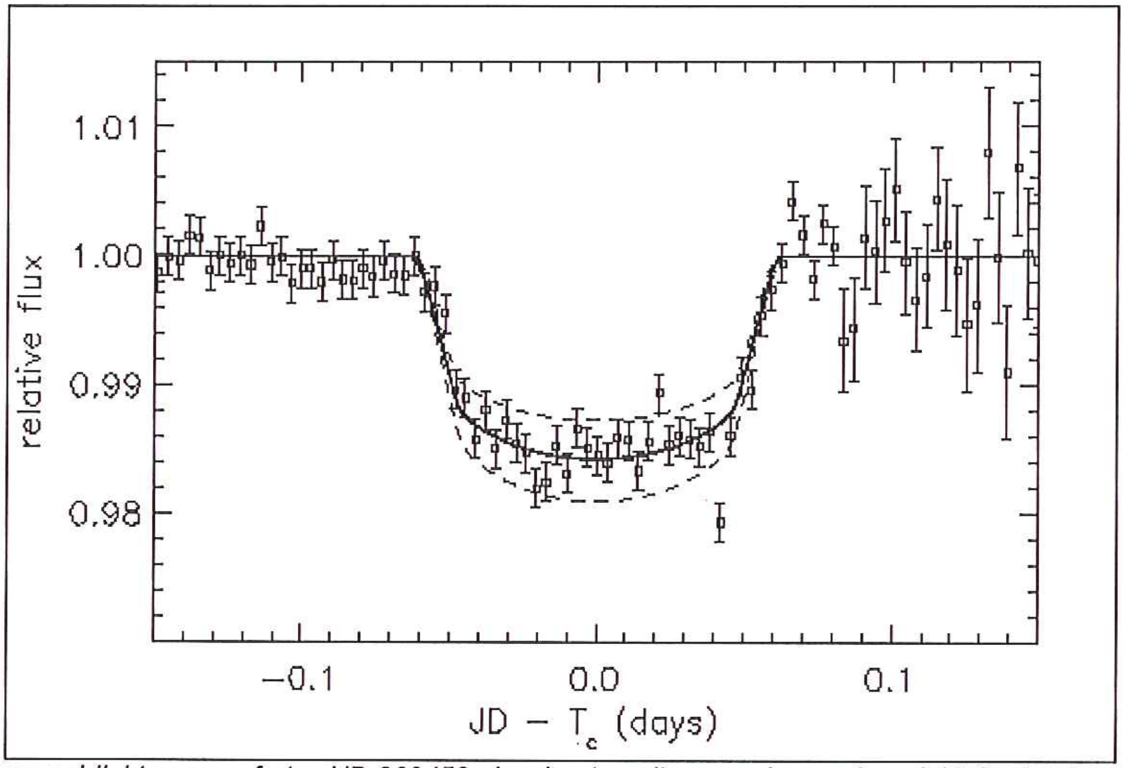


Figure based on one by Hans Deeg, from 'Transits of Extrasolar Planets'

When the properties of the parent star are known (such as radius, color, and limb darkening), subsequent analysis of the transit light curve (below) can be used to determine many properties of the planet not discernible by radial-velocity measurements alone. In the case of HD 203458, the radius of the planet is estimated at 1.27 times that of Jupiter (or around 91,000 km). This measurement, coupled with the mass determined by both the radial velocity and transit parameters (approximately 63% that of Jupiter, or 1.2 x 10^27 kg), allows for the calculation of surface gravity (at 9.7 m/s^2, just slightly lower than Earth's) and density (approximately 0.38 g/cm^3, so low that if there was an ocean large enough, the planet would float!)



Superposed lightcurves of star HD 209458 showing transits occurring on 9 and 16 September, 1999

This first planetary transit discovery is very exciting in that it gives astronomers a wealth of new information about the properties and formation of other solar systems (such as the make-up of extra-solar planetary atmospheres). Please revisit the STARE web page in the future for updates on this and any other transit discoveries.

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Radius : 1.27 x Jupiter

Mass : 0.63 x Jupiter

Density : 0.38 x water