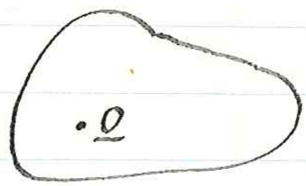
The inertia tensor

Say we have a body e.g. the $\oplus$



$\rho(\underline{r})$ given $\underline{r} \in$ volume V

$$M = \int_{\oplus} \rho(\underline{r})\, d^3\underline{r}$$

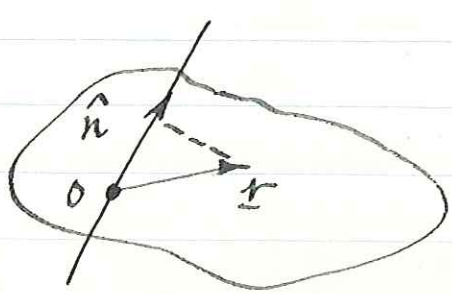
define center of mass (c.o.m.)

$$\langle \underline{r} \rangle = \frac{1}{M} \int_{\cancel{V}\,\oplus} \rho(\underline{r})\, \underline{r}\, \cancel{dV}\, d^3\underline{r}$$

From now on we'll always take origin at c.o.m.

$$\boxed{\underline{0} \equiv \langle \underline{r} \rangle}$$

define moment of inertia about an axis $\hat{n}$ thru c.o.m.



$$I(\hat{n}) = \int_{\cancel{V}\,\oplus} \overset{d^3\underline{r}}{dV}\, \rho(\underline{r}) \underbrace{\left[r^2 - (\hat{n}\cdot\underline{r})^2 \right]}_{\perp \text{ distance to axis}}$$

mass $M = \int_{\oplus} \rho(\underline{r})\, d^3\underline{r}$

e.g. homogeneous sphere radius $\cancel{a}$ R

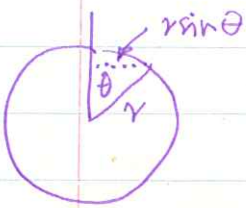
$$\rho(\underline{r}) = \rho$$

$$\text{mass} \quad M = \tfrac{4}{3}\pi\rho \cancel{a^3}\, R^3$$

Clearly $I(\hat{n})$ same for all axes $\hat{n}$

$$I(\hat{n}) = 2\pi \int_0^{\cancel{a}\,R} \int_0^{\pi} (r\sin\theta)^2 \,\rho\, r^2 \sin\theta \, dr\, d\theta$$

$$= 2\pi\rho \frac{\cancel{a^5}\,R^5}{5} \int_0^{\pi} \sin^3\theta \, d\theta$$



$$= \frac{8\pi}{15}\rho \cancel{a^5}\,R^5 = \frac{2}{5}\left(\tfrac{4}{3}\pi\rho \cancel{a^3}\,R^3\right) \cancel{a^2}\,R^2$$

$$\boxed{I(\hat{n}) = \tfrac{2}{5} M \cancel{a^2}\,R^2}$$

well-known result

Do we have to compute $I(\hat{n})$ separately for each $\hat{n}$? No.

skip to page 6

Useful extension: concept of inertia tensor.

$$\boxed{\underline{\underline{C}} = \int_V dV\, \rho(\underline{r}) \left[(\underline{r}\cdot\underline{r})\, \underline{\underline{I}} - \underline{r}\,\underline{r} \right]}$$

where $\underline{\underline{I}}$ is the so-called identity tensor

Mathematical aside:

The dot product of two vectors is a scalar.

A tensor of order two is a mathematical object whose dot product with a vector gives another vector. We write

$$\underline{u} = \underline{\underline{T}} \cdot \underline{v}$$

Note in general $\underline{\underline{T}} \cdot \underline{v} \neq \underline{v} \cdot \underline{\underline{T}}$. If $\underline{\underline{T}} \cdot \underline{v} = \underline{v} \cdot \underline{\underline{T}}$, $\underline{\underline{T}}$ called a symmetric tensor.

Identity tensor $\underline{\underline{I}} \cdot \underline{v} = \underline{v} \cdot \underline{\underline{I}} = \underline{v}$. Reason for name.

Components of a tensor:

vector $\underline{v} = v_i \hat{x}_i$

$v_i = \hat{x}_i \cdot \underline{v}$ 3 comps.

tensor $\underline{\underline{T}} = T_{ij} \hat{x}_i \hat{x}_j$

$T_{ij} = \hat{x}_i \cdot \underline{\underline{T}} \cdot \hat{x}_j$ 9 comps.

Can write v_i as a column matrix

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

and T_{ij} as 3×3 matrix

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

Then in terms of components $\underline{u} = \underline{\underline{T}} \cdot \underline{v}$ becomes

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \text{matrix mult.}$$

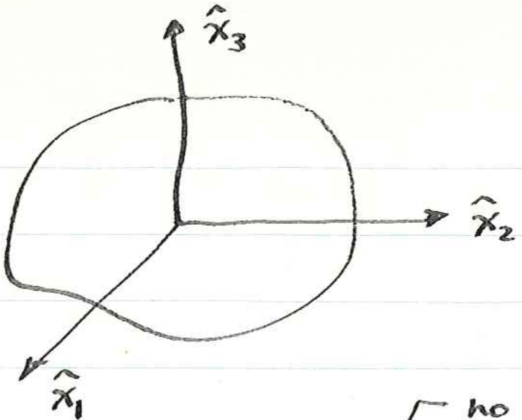
Answer to our question: if we know $\underline{\underline{C}}$ easy to find $I(\hat{n})$ for any $\hat{n}$ since

$$\boxed{I(\hat{n}) = \hat{n} \cdot \underline{\underline{C}} \cdot \hat{n}}$$

proof: $$\hat{n} \cdot \underline{\underline{C}} \cdot \hat{n} = \int_V dV \, \rho(\underline{r}) \left[(\underline{r} \cdot \underline{r})(\hat{n} \cdot \underline{\underline{I}} \cdot \hat{n}) - (\hat{n} \cdot \underline{r})(\underline{r} \cdot \hat{n}) \right]$$

$$= \int_V dV \, \rho(\underline{r}) \left[r^2 - (\hat{n} \cdot \underline{r})^2 \right]$$

What are components of $\underline{\underline{C}}$?



no sum

for $i=j$ $$C_{ii} = \int_V \rho(\underline{r}) \left[r^2 - (\hat{x}_i \cdot \underline{r})^2 \right] dV$$

moment of inertia about $\hat{x}_i$ axis

for $i \neq j$ $$C_{ij} = -\int_V \rho(\underline{r})\, x_i x_j \, dV$$

so-called product of inertia

$\underline{\underline{C}}$ is symmetric $C_{ij} = C_{ji}$

Important theorem in mechanics (or in linear algebra): <u>principal</u> <u>axis</u> <u>theorem</u>.

$\boxed{\exists \text{ a Cartesian axis system } \{\hat{x}_1, \hat{x}_2, \hat{x}_3\} \text{ in which products of inertia vanish}}$

There may be more than one but there is for any body no matter how misshapen at least one. Furthermore, the moments of inertia about the principal axes are <u>extrema</u>.

These axes are called the <u>principal axes</u> of inertia. The components of $\underline{\underline{C}}$ take the form

$$\underline{\underline{C}} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

Conventional to order $A \leq B \leq C$.
Least, intermediate and greatest moments of inertia. If $\hat{n} = n_i \hat{x}_i$ in principal axis system, then

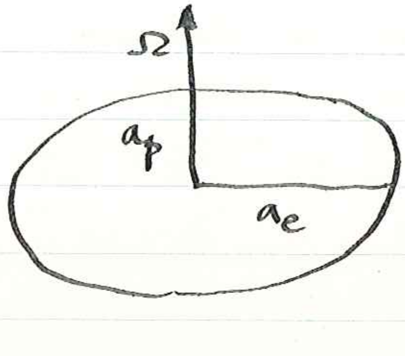
$$I(\hat{n}) = A n_1^2 + B n_2^2 + C n_3^2 .$$

Examples:

1. spherical body $\rho(\underline{r}) = \rho(r)$ only: <u>any</u> $\hat{x}_1, \hat{x}_2, \hat{x}_3$ is a principal axis system, and $A = B = C$. (degenerate case)

skip to here

2. the Earth: equatorial bulge well-known



$a_e \approx 6378$ km
$a_p \approx 6356$ km
20 km difference

Clear that $\hat{x}_3$ axis of greatest principal inertia aligned along Ω rotation axis.

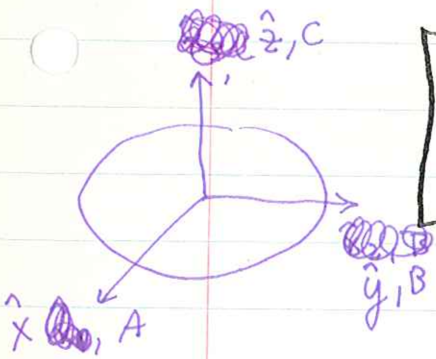
Actually $\hat{x}_3 \parallel \langle \underline{\Omega} \rangle$ since $\exists$ Chandler + annual wobbles, etc.

$n = n_x \hat{x} + n_y \hat{y} + n_z \hat{z}$, $n_x^2 + n_y^2 + n_z^2 = 1$

$I(\hat{n}) = A n_x^2 + B n_y^2 + C n_z^2$

$C_\oplus$ = mom. of inertia about rotation axis, A and B about equatorial axes.

Furthermore $C-A$ and $C-B \ll B-A$. or $B \approx A$.



$$\boxed{C - \tfrac{1}{2}(A+B) \text{ a measure of size of bulge}}$$

$B-A$ a measure of deviations from rotational symmetry.

As we shall see $B - A \approx 10^{-3}(C-A)$

e.g. Turcotte & Schubert

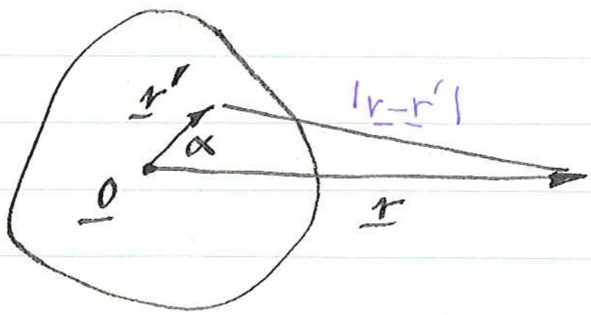
Simplified discussions often take $A = B$.

If decide to skip discussion of inertia tensor, define $C \equiv$ greatest principal (possible) moment of inertia $\equiv$ mean rotation axis for $\oplus$; $A \equiv$ least principal (possible) moment of inertia. Then a theorem that A is equatorial; define $B \equiv \perp$ both C, A.

The external gravitational potential of Earth

$$V(\underline{r}) = -G \int_{V_\oplus} \frac{\rho(\underline{r}')\, d\cancel{V'}\, d^3\underline{r}'}{|\underline{r} - \underline{r}'|}$$

Now take $\underline{0}$ at c.o.m. of $\oplus$.



use law of cosines

$$\frac{1}{|\underline{r}-\underline{r}'|} = \frac{1}{r}\left[1 - 2\frac{r'}{r}\cos\alpha + \left(\frac{r'}{r}\right)^2\right]^{-1/2}$$

valid for $r > r'$ — for $r \gg r'$ this has form $\frac{1}{r}(1-\varepsilon)^{-1/2} = \frac{1}{r}\left(1 + \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 + \cdots\right)$

Now expand by binomial theorem, get

$$\boxed{\frac{1}{|\underline{r}-\underline{r}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^{l} P_l(\cos\alpha)}$$

$$P_0(\cos\alpha) = 1$$
$$P_1(\cos\alpha) = \cos\alpha$$
$$P_2(\cos\alpha) = \frac{1}{2}(3\cos^2\alpha - 1)$$
$$\vdots$$

so-called Legendre polys.

Substitute in for $V(\underline{r})$

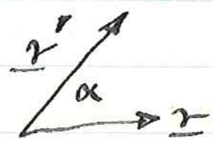
$$V(\underline{r}) = -\frac{G}{r}\Big[\int_V \rho(\underline{r}')\, dV'$$

$$+ \frac{1}{r}\int_V r' \cos\alpha \; \rho(\underline{r}')\, dV'$$

$$+ \text{etc.} \Big]$$

But $\int_V \rho(\underline{r}')\, dV' = M$, the $\oplus$ <u>mass</u>

and $\cos\alpha = \hat{r}\cdot\hat{r}'$

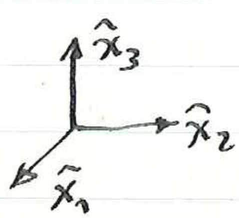


so

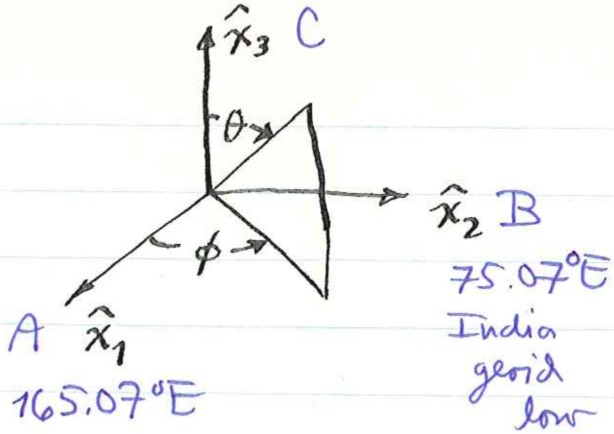
$$r'\cos\alpha = \hat{r}\cdot\underline{r}'$$

$$\int_V r'\cos\alpha\; \rho(\underline{r}')\, dV' = \hat{r}\cdot\underbrace{\int_V \underline{r}'\rho(\underline{r}')\, dV'}_{=\ \underline{0}\ \text{since c.o.m.}}$$

Furthermore let



be <u>principal</u> <u>axes</u>, i.e. A, B, C respectively



θ, ϕ spherical coords w.r.t. principal axes.

Then can be shown that next term (involving $P_2(\cos\alpha)$) takes form

measure of equatorial bulge

$$V(r,\theta,\phi) = -\frac{GM}{r}\left[1 + \frac{C - \frac{1}{2}(A+B)}{Ma^2}\left(\frac{a}{r}\right)^2\left(\frac{1}{2} - \frac{3}{2}\cos^2\theta\right) + \frac{B-A}{Ma^2}\left(\frac{a}{r}\right)^2 \frac{3}{4}\sin^2\theta\cos 2\phi + O(a/r)^3\right]$$

deviation from axial symmetry

terms of order $(a/r)^3$

do not erase need later on page 4

Guust writes in slightly different form: $V = -\frac{GM}{r} - \frac{G}{2r^3}(A+B+C-3I)$

This called <u>MacCullagh's formula</u>. It assumes <u>O</u> at c.o.m. and $\hat{x}_1, \hat{x}_2, \hat{x}_3$ <u>principal axes</u>. Stacey eqn. 3.11 writes it in arbitrary axes. Garland 11.2.7 includes centrifugal term. Gives $V(\underline{r})$ in terms of M, A, B, C

a is size of body, could e.g. be $a_e \equiv$ mean equatorial radius.

Approx. improves as $r \gg a$ very far away.

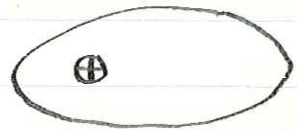
Dominant term is $-\frac{GM}{r}$. Very far away any body looks like a point mass — makes physical sense.
A little closer can see $(a/r)^2$ effects of equatorial bulge (note ind. of ϕ) as well as $B-A$ deviations from axial symmetry.

Now we know something else about question: given $V(\underline{r})$ what can we learn about $\oplus$?

Best way to determine $V(\underline{r})$ orbits of <u>artificial satellites</u>.

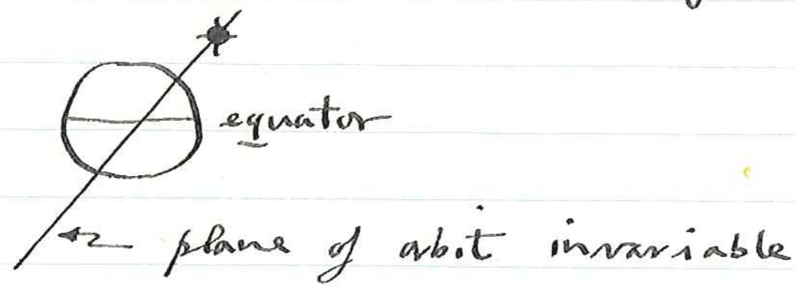
Recall <u>Kepler's laws</u> of motion. If $\oplus$ were a sphere $A = B = C$ and
$V(\underline{r}) = -GM/r$ exactly.

Orbit an ellipse $\oplus$ at focus.

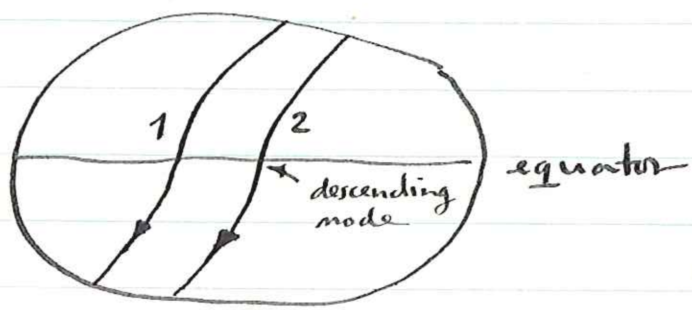


Since $\oplus$ <u>almost</u> spherical orbit almost an ellipse but one whose parameters change slowly with time.

Suppose for simplicity $A = B$. Consider an inclined orbit. If spherical



But with bulge



Constant pull of bulge leads to an eastward motion of (regression) the descending node. Plane of orbit thus precesses around axis of symmetry. One can measure this rate of motion by tracking satellites. The rate is $\propto$ size of equatorial bulge $\dfrac{C - \frac{1}{2}(A+B)}{Ma^2}$.

~~Three~~ Four ways of tracking:

1. camera tracking (earliest: measures angular posn w.r.t. fixed ✶)
2. Doppler tracking (measure line of sight velocity, integrate to find position)
3. laser ranging (newest, since '67,

4. GPS satellite

Liu & Chao GJI 106, 699–702

A axis 165.07°E longitude
B axis 75.07°E longitude (geoid low S of India)

measure radial distance directly, travel time of nanosecond laser pulse, 2-way time, corner cube on satellite). GPS also measures distance.

$$\frac{C - \frac{1}{2}(A+B)}{Ma^2}$$ can be measured with great accuracy, can monitor slow changes for a long time.

Big effort, military reasons, many satellites + observing stations.

Gaposchkin JGR 1974

$$J_2 = \frac{C - \frac{1}{2}(A+B)}{Ma_e^2} = 1.082637 \cdot 10^{-3}$$

$$\frac{B-A}{Ma_e^2} = 7.2 \cdot 10^{-6}$$ much smaller

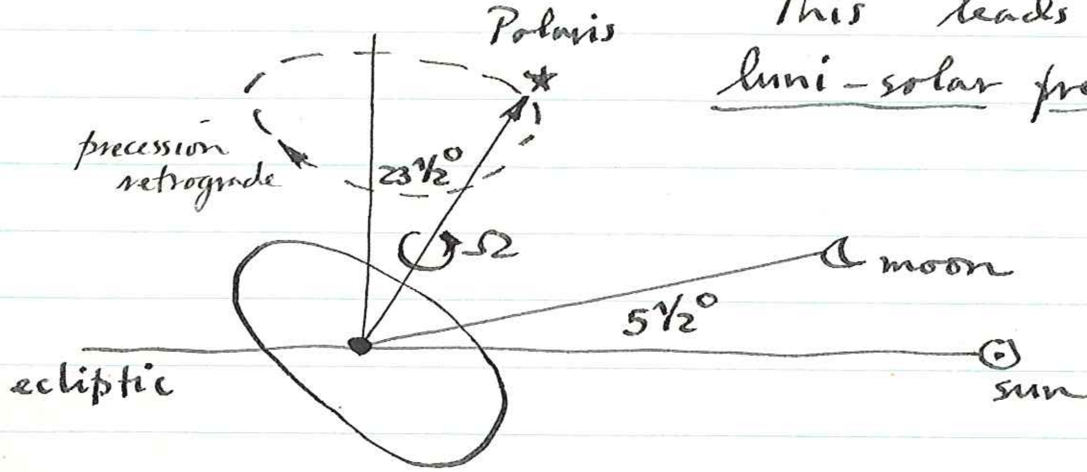
a_e = mean equatorial radius

Note: $V(\underline{r})$ does not allow A, B or C to be measured, only differences. How is C determined?

A two-pronged attack.

<u>Precession</u> <u>of</u> <u>the</u> <u>equinoxes</u>

Bulge does not coincide with ecliptic or with plane of lunar orbit.

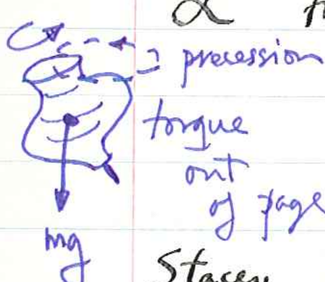


This leads to <u>luni-solar precession</u>

Torque (out of page) $\propto$ bulge i.e.

$$\propto \frac{C - \frac{1}{2}(A+B)}{Ma^2}$$

Like a gyroscope, response of $\oplus$ to torque is to <u>precess</u>. Rate of precession $\propto$ torque $\div$ angular momentum or



$$\text{rate} \propto \frac{C - \frac{1}{2}(A+B)}{C\Omega}$$

Stacey 3.2 gives the derivation, also Garland Appendix B.

$$\boxed{\omega_\odot = -\frac{3}{2}\frac{G}{\Omega}\left[\frac{C - \frac{1}{2}(A+B)}{C}\right]\frac{M_\odot}{R^3_{\odot\oplus}}\cos\theta \qquad \theta = 23\tfrac{1}{2}^\circ}$$

Similarly for contribution of moon ☾.
Can observe precession astronomically.
Very slow rate $= 50.37''/\text{year}$
or period $= 25{,}800$ years.

About half due to ☉, half to ☾.
Enables $H = \dfrac{C - \frac{1}{2}(A+B)}{C}$ precession constant
to be measured.

$$\boxed{\frac{C - \frac{1}{2}(A+B)}{C} = \frac{1}{305.4\cancel{377}}}$$

Now combine 2 measurements

$$\frac{C}{Ma^2} = \frac{C - \frac{1}{2}(A+B)}{Ma^2} \Bigg/ \frac{C - \frac{1}{2}(A+B)}{C}$$

$$= (1.083 \cdot 10^{-3})(305.4)$$

$$= .3308\cancel{0}$$

$$\boxed{C = 0.3308\cancel{0}\, Ma_e^2}$$

Recall $C = 0.4\, Ma^2$ for constant density sphere. Another indication that $\rho(r) \nearrow$ as one goes down.

Two reasons for $0.33089 < 0.4$:

1. ρ of silicate mantle $\uparrow$ due to pressures in $\oplus$
2. Fe core in center.

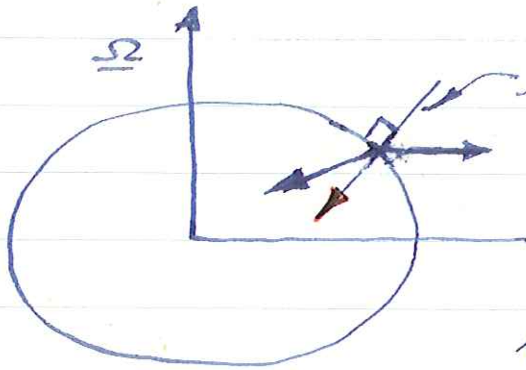
In contrast C/Ma^2 for moon (det. similarly from lunar orbiters + observed <u>libration</u> in potential of $\oplus$)

$$\boxed{(C/Ma^2)_{☾} = 0.392}$$

Moon a much more homogeneous body, no core, smaller, pressures in mantle less, radius $a \cong 1700$ km.

Centrifugal potential and the geoid

Terrestial gravity measurements are affected by ⊕'s rotation.



We take $\underline{\Omega} = \Omega \hat{x}_3 = \Omega \hat{z}$

$\Omega = |\underline{\Omega}| = 2\pi$ rad / day

By defn, centrifugal force is (on a unit test mass)

$$\underline{f}\,(\text{per unit mass}) = \Omega^2 \left[r^2 - (\hat{z} \cdot \underline{r})^2 \right]^{1/2} \hat{r}$$

$$= -\Omega^2 \left[\hat{z} \times (\hat{z} \times \underline{r}) \right]$$

$$= \Omega^2 (\underline{x} + \underline{y}) = \Omega^2 (x^2 + y^2)^{1/2} \hat{r}$$

($\hat{r}$ ≡ unit vector in cyl. coords)

This force is conservative or derivable from a potential.

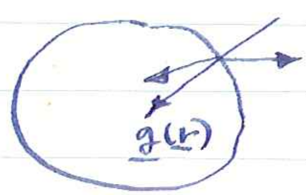
$\underline{f}(\underline{r}) = -\nabla \psi(\underline{r})$ where

$$\boxed{\begin{aligned}\Psi(\underline{r}) &= -\tfrac{1}{2}\Omega^2(x^2+y^2)\\ &= -\tfrac{1}{2}\Omega^2 r^2 \sin^2\theta\\ &= -\tfrac{1}{3}\Omega^2 r^2\left[1-P_2(\cos\theta)\right]\end{aligned}}$$

Total force on a unit test mass in ⊕ frame derivable from so-called geopotential

$$\mathcal{U}(\underline{r}) = V(\underline{r}) + \Psi(\underline{r})$$

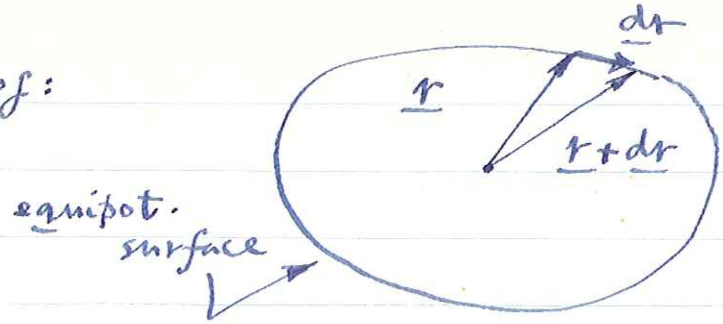
Then
$$\begin{aligned}\underline{g}(\underline{r}) &= -\nabla\mathcal{U}(\underline{r})\\ &= -\nabla\left[V(\underline{r})+\Psi(\underline{r})\right]\end{aligned}$$
called the local gravity vector.



It points in the direction a plumbob hangs.

Surfaces $\mathcal{U}(\underline{r})$ = constant called equipotential surfaces (equi ≡ same)
Their physical significance: they are everywhere ⊥ to local gravity vector.

Proof:



This is defn of grad

$$U(\underline{r}+\underline{dr}) = U(\underline{r}) + \underline{dr}\cdot\nabla U(\underline{r}) + \cdots$$
$$= U(\underline{r}) - \underline{dr}\cdot\underline{g}(\underline{r}) + \cdots$$

Take $\underline{dr}$ along equipotential so that $U(\underline{r}+\underline{dr}) = U(\underline{r})$. Then $\underline{dr}\cdot\underline{g}(\underline{r}) = 0$.

q.e.d.

If not for action of wind and luni-solar tidal forces surface of ocean would be an equipot. surface.

Define: geoid is that equipot surface which coincides with mean $\$$ where $\exists$ oceans

To visualize think of continents cut by a network of fine canals. Geoid important to surveyors. Elevations measured w.r.t. geoid. Denver is one mile above water level in Denver – LA – NYC canal.

Surface of $\oplus$ very bumpy: max elevation above geoid 8.8 km, max ocean depths ~ 11 km.

Surface of geoid <u>much</u> smoother. To a very close approx (about ± 100 m) it is an ellipsoid of revolution. Because of this it is expedient to determine the <u>best-fitting ellipsoid</u> to the geoid.

We have $\mathcal{U}(\underline{r}) = V(\underline{r}) + \psi(\underline{r})$

$$= -\frac{GM}{r}\left[1 + \frac{C - \frac{1}{2}(A+B)}{Ma^2}\left(\frac{a}{r}\right)^2\left(\frac{1}{2} - \frac{3}{2}\cos^2\theta\right) + \frac{B-A}{Ma^2}\left(\frac{a}{r}\right)^2 \frac{3}{4}\sin^2\theta \cos 2\phi\right]$$

$$- \frac{1}{2}\Omega^2 r^2 \sin^2\theta$$

It's quicker to write this as * on p. 5 first.
Note, however, that it already sets $B \equiv A$,

We'll consistently neglect small <u>quantities</u>. To this order of approx we take $A = B$ ($A \neq B$ causes a "bump" of order 100 m away from best-fitting ellipsoid).

Customary to denote

$$\longrightarrow J_2 \equiv \frac{C - \frac{1}{2}(A+B)}{Ma^2} \sim 10^{-3}$$

quadrupole moment of $\oplus$

$$m \equiv \Omega^2 a^3 / GM = \frac{a\Omega^2}{GM/a^2}$$

$$\approx \frac{\text{cent. force at equator}}{\text{grav. force at equator}}$$

$$m \approx 1/290$$

$$\mathcal{U}(\underline{r}) \approx -\frac{GM}{r}\left[1 + J_2\left(\frac{a}{r}\right)^2\left(\frac{1}{2} - \frac{3}{2}\cos^2\theta\right) + \frac{1}{2}m\left(\frac{r}{a}\right)^3 \sin^2\theta\right] \quad \star$$

Now we ask: what is shape of geoid of a body with $\mathcal{U}(\underline{r}) = \star$ <u>exactly</u>?

Geoid defined by $\mathcal{U}(\underline{r}) = \mathcal{U}_0$, const.

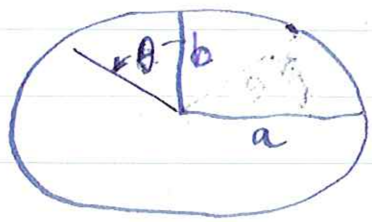
Solve for r. Since J_2, m small

$$r \approx -\frac{GM}{\mathcal{U}_0}\left[1 + J_2\left(\frac{1}{2} - \frac{3}{2}\cos^2\theta\right) + \frac{1}{2}m\sin^2\theta\right]$$

$$r \approx \underbrace{-\frac{GM}{U_0}\left[1+\tfrac{1}{2}J_2+\tfrac{1}{2}m\right]}_{\text{call this } a}\left[1-\underbrace{\left(\tfrac{3}{2}J_2+\tfrac{1}{2}m\right)}_{\text{call this } \varepsilon}\cos^2\theta\right]$$

$$\boxed{\text{get} \quad r = a\,(1-\varepsilon\cos^2\theta)}$$

This eqn of an ellipsoid of revolution



$$\boxed{\varepsilon = \frac{a-b}{a} = \text{flattening}}$$

semi-major axis a, flattening or ellipticity ε

check $\theta = 0$, $r = a(1-\varepsilon) = b$; $\theta = \pi/2$, $r = a$

The flattening ε is given by

measure gravity — infer shape of ⊕ — no surveying — why — intimate connection — plumbbob ⊥ to geoid

$$\boxed{\varepsilon = \frac{3}{2}J_2 + \frac{1}{2}m}$$ ← call this $\varepsilon_{\text{best-fitting}}$

This gives the flattening or shape of best-fitting ellipsoid in terms of measurable quantities :

J_2 : satellites, regression of nodes
$m = \Omega^2 a^3 / GM$
GM : Kepler's 3rd law
Ω : * and clocks
a : surveying à la Eratosthenes

Second order theory gives

$$\boxed{J_2 = \frac{2}{3}\varepsilon - \frac{1}{2}m - \frac{1}{3}\varepsilon^2 + \frac{2}{21}\varepsilon m + O(\varepsilon^3)} \quad *$$

Gaposchkin JGR 1974

$$GM = 3.986013 \cdot 10^{20} \text{ cm}^3/\text{sec}^2$$
$$a = 6.378140 \cdot 10^6 \text{ m}$$
$$\Omega = 7.292115 \cdot 10^{-5} \text{ s}^{-1}$$
$$J_2 = 1.082637 \cdot 10^{-3}$$

Plugging into * gives

$$\boxed{\varepsilon_{\text{best-fitting}} = 1/298.25}$$

This the ellipticity of best-fitting ellipsoid of revolution. Actual geoid has bumps of order ± 100 m w.r.t. this.

Figure 4.1 of Stacey shows best map

Garland Fig. 11.7 is the Gaposchkin 1974 map.

of geoid in 1971. This a contour map of geoid undulations w.r.t. best-fitting ellipsoid. This (and hydrostatic ellipsoid we'll discuss later) most common repr. of geoid. Only way to depict such a smooth surface as differences w.r.t. an even smoother one.

Now consider two questions:

1. how are such maps obtained?
2. how does one interpret such a map?

Answer to first question. We have consistently neglected terms of $O(a/r)^4$. Deviations from reference ellipsoid depend on these terms. Procedure: expand these terms in so-called spherical harmonics.

$$\boxed{V(\underline{r}) = -\frac{GM}{r}\left[1 + \sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\left(\frac{a}{r}\right)^{\ell} P_\ell^m(\cos\theta)\left(c_\ell^m \cos m\phi + s_\ell^m \sin m\phi\right)\right]}$$

GEM 8 (Wagner et al. JGR 82, no. 5 (1977)) has coefficients up thru $l=m=28$ (not complete, however)

c_l^m and s_l^m expansion coefficients

$P_l^m(\cos\theta)\begin{Bmatrix}\cos m\phi\\ \sin m\phi\end{Bmatrix}$ spherical harmonics: these are known fcns of θ, ϕ; very important in geophysics; Appendix A of Stacey has a discussion; knowledge of s.h. not required for this course. Two facts:

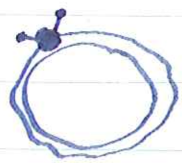
~~[illegible]~~ External grav. pot. of any mass distribution can be so expanded. Note $l=2 \Rightarrow$ MacCullagh terms, also $l=1$ absent if O at c.o.m.

Also: larger the index l or m the more wiggly is $P_l^m \cos m\phi$ or $P_l^m \sin m\phi$.

In our expansion a term of wiggliness l falls off as $(a/r)^{l+1}$. The bumpiness or resolution with which we can see bumps in geoid depends on wiggliness of $V(r,\theta,\phi)$; on how many coefficients c_l^m, s_l^m can measure.

Wiggliest terms have little effect on high satellites because of $(a/r)^{l+1}$ falloff.

Moral: use low satellites but then corrections for air drag in atmosphere required and these are difficult.

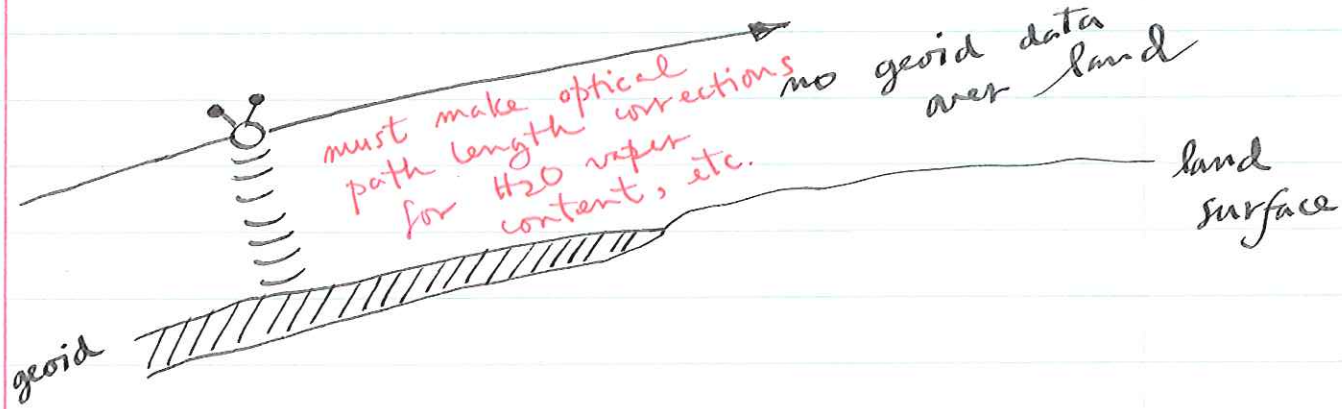


This an inherent limitation. Very hard to resolve fine scale features of geoid by tracking of satellites.

To get better resolution over oceans radar altimetry was developed; we shall discuss its results later.

If want geoid over land as well must use expansion method + tracking, hard to get good results above $l \sim 10$. To overcome this inherent limitation method used is to combine satellite data with terrestrial gravity data: measurements of gravitational acceleration on ⊕ surface.

Latest and best method of determining geoid over oceans only is radar altimetry



Measure 2-way travel time of a reflected radar pulse. Must know satellite position w.r.t. c.o.m. of ⊕ to with ~ 10 cm to determine geoid to within 10 cm. Gravimetric + tracking data adequate for this purpose. Also can see short-wavelength details easily, since ripples in satellite orbit tend to be longer wavelength. Two satellites GEOS-3 and SEASAT. Latter flown in 1978, failed after 105 days but collected lots of data before that. Map of coverage for 18 days shown in Fig. 1 of Marsh + Martin, JGR (Green), 87, C5, 3269–80 (1982).

Narrow lows over trenches are most notable.

How does one interpret such maps? what do they tell us?
In general a positive bump on either map ⇒ some kind of mass excess below and a negative bump a mass deficiency.
But problem of determining what the mass anomalies is as non-unique as in spherical case we considered.
For a given ~~[illegible]~~ bump anomaly could be small and shallow or larger and deeper. Geoid low over Hudson's Bay associated with deglaciation; highs over western S. America
We simply do not yet know the cause of and many bumps, e.g. that off India. New Guinea
Other bumps can be correlated with due to geological features, e.g. we shall subducted discuss the geoid over mid-ocean slabs. ridges, a weak signal, in some detail.

To interpret properly should employ a physically significant reference ellipsoid. Best-fitting has no physical significance.

The hydrostatic ellipsoid does. Physically it is the shape the ⊕ would have if it were a fluid in hydrostatic equilibrium. All "bumps" both surficial (mtns) and internal would be smoothed out.

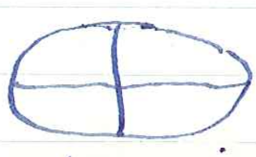
Entails 2 conditions:

1. hydro – material has no long-term shear strength – able to flow and smooth out bumps
2. static – no mantle convection – a convecting fluid could have (time-dependent in gen'l) internal "bumps".

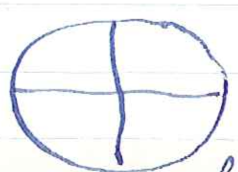
Answer: hydrostatic shape of $\oplus$ with same $\rho(r)$ stratification but smoothed out is an ellipsoid of revolution with $\varepsilon = 1/299.67$ instead of best-fitting $\varepsilon = 1/298.256$.

Direct physical significance of this figure enables us to begin interpretation. Gaposchkin maps of geoid and free air gravity w.r.t. this figure.

Note $\varepsilon_{hydro} < \varepsilon$

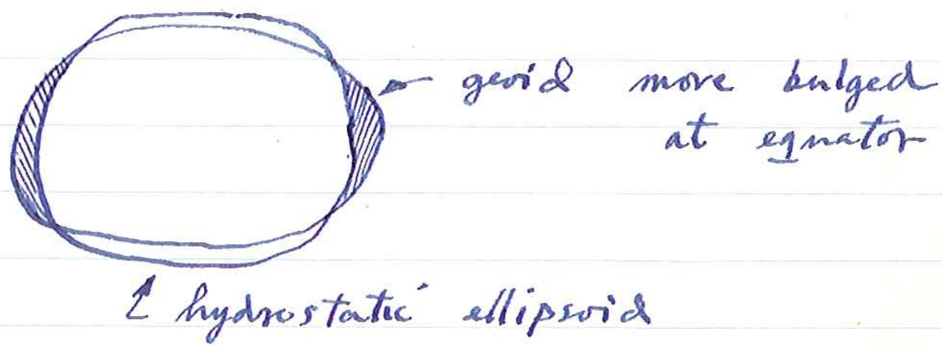


best-fitting



hydrostatic

Geoid map w.r.t. hydrostatic ellipsoid has low values at both poles because reference surface is less flat than b.f. ellipsoid. Positive band about equator and low at both poles. This difference between hydrostatic and b.f. ellipsoid is main broad scale feature of geoid



Two very different explanations have been advanced, one of which is now favored by most people.

First however let's discuss the hydrostatic ellipsoid in more detail.

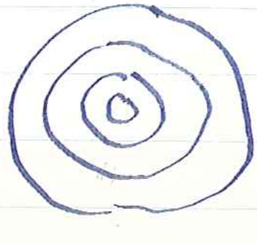
The hydrostatic ellipsoid

Deviations away from this figure have physical significance. They imply a violation of either assumption 1. or 2.

1. hydro : ⊕ on long time scales behaves as a fluid
2. static : no convection

Flow in a hydrostatic fluid would act to smooth out any lateral density differences. Constant density surfaces must in a hydrostatic fluid coincide with equipotentials. This can be proved from cont. mech. version of $F = ma$ but we won't here. It's a strongly intuitive result – reason ocean is "flat" i.e. geoid.

We ask question : what is hydrostatic figure of ⊕. If not rotating and fluid clear that must be spherically symmetric, density $\rho(r)$ only.



Now allow to rotate, rate Ω.
What will be new shape of constant density surfaces, esp. outer surface?
Answer depends on $\rho(r)$.

Theory ancient: shape of rotating fluid bodies. Goes back to Newton's Principia (1687).
He solved problem for special case of constant density fluid $\rho(r) = \rho$.
His solution remarkable for its insight + simplicity, long before J. Bernoulli invented concept of hydrostatic pressure. He recognized that:

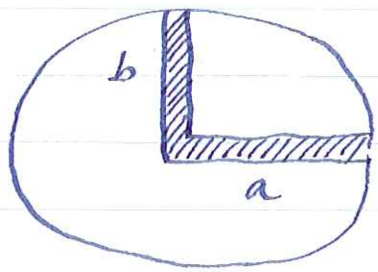
effect of Ω is to bulge the equator because of centrifugal force.
cause of bulge $\propto m$

$$m = (\text{cent. force} / \text{grav. force})_{\text{equator}}$$
$$= \Omega^2 a / (GM/a^2) = \Omega^2 a^3 / GM$$

effect is measured by ellipticity of surface

$$\varepsilon = \frac{\text{eq. rad.} - \text{polar rad.}}{\text{eq. rad.}}$$
$$= \frac{a-b}{a}$$

He imagined 2 holes to center of ⊕ filled with H_2O. Weight of two columns must be equal.



Eq. column is longer but wt. is diluted by centr. force

Centrifugal force (outward) $= \Omega^2 r \propto r$ varies linearly with dist. r from center.

Also inside a homog. sphere (this where assumption $\rho(r) = \rho$ comes in) gravity force $g(r) \propto r$ also. No attraction from mass outside, that inside attracts like pt. mass.

$$g(r) = \frac{GM(r)}{r^2} = \frac{4/3\,\pi G \rho r^3}{r^2} \propto r$$

Newton knew this as he had invented integral calculus.

Both $g(r)$ and centr. force $\propto r$. Hence dilution factor or percentage dilution remains constant in the column and equal to its surface value m.

this is grav. force only; dilution factor

$$\text{wt. of eq. column} = \underbrace{\tfrac{1}{2}}_{\text{average since } g \propto r} a\, g_{eq} \,(1-m)$$

$$\text{wt. of polar column} = \tfrac{1}{2}\, b\, g_{pole}$$

grav. only also

$$(1-m) = \frac{b}{a}\,\frac{g_{pole}}{g_{eq}}$$

Newton had also solved for the purely grav. attraction of an oblate body.

What is $g(\theta)$ on surface of const. density ellipsoid?

$\rho = $ const. A problem in integral calculus. If $b = a$, $g(\theta) = $ const.

Newton found that $\dfrac{g_{pole}}{g_{eq}} = 1 + \tfrac{1}{5}\varepsilon + O(\varepsilon^2)$

since further from c.o.m.

Furthermore ε is here the ellipticity of the body, not of an equipotential

<u>Note</u>: this not same as so-called Clairaut's theorem $g_{pole}/g_{eq} = 1 - \varepsilon + \tfrac{5}{2} m + O(\varepsilon^2)$ since there g <u>includes</u> centr. force. Above result for grav. force <u>only</u>.

$$(1-m) = \frac{b}{a}\,\frac{g_{pole}}{g_{eq}} = (1-\varepsilon)(1+\tfrac{1}{5}\varepsilon) + O(\varepsilon^2)$$
$$= 1 - \tfrac{4}{5}\varepsilon + O(\varepsilon^2)$$

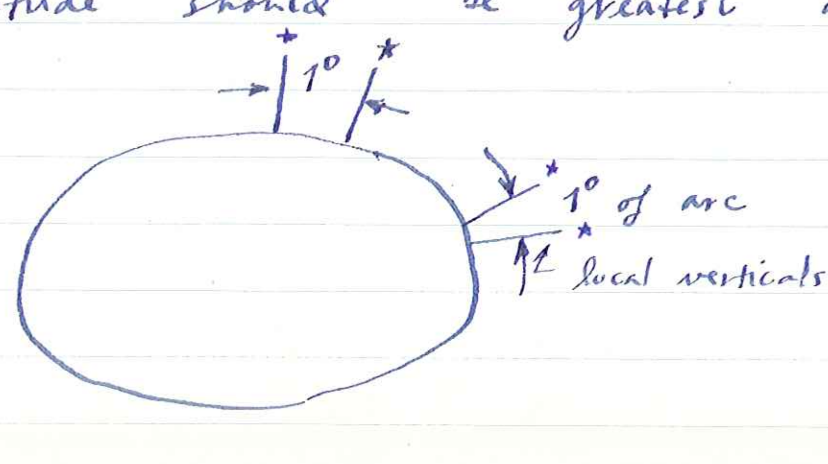
$$\boxed{\varepsilon = \frac{5}{4} m \quad , \quad m = \Omega^2 a^3 / GM}$$

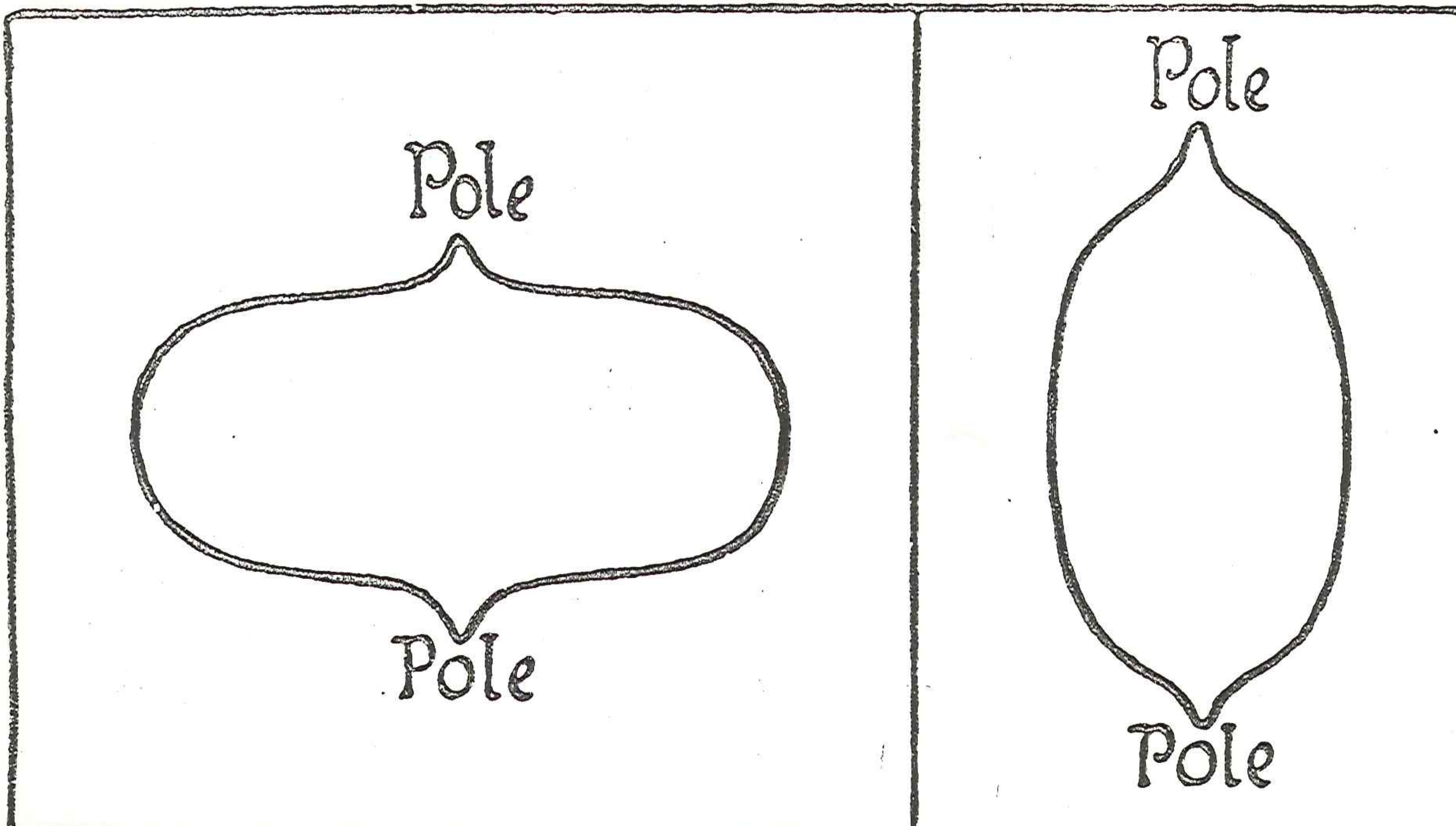
flattening of a homog. fluid sphere due to its rotation.

Known in Newton's time that $m \sim 1/290$.
Thus predicted $\varepsilon = \frac{5}{4} m \sim 1/230$.
Read underline quotes from Book III, Proposition XIX, Problem III of Principia.

This prediction of Newton in conflict with astronomical measurements of J. D. and Jacques Cassini (father + son) of l'Observatoire de Paris. Their measurements showed ⊕ to be flattened at equator not at poles.

Newton ⇒ arc length of one degree of latitude should be greatest at poles





NEWTON | CASSINI

Cassinis compared two short ~~[illegible]~~ arcs in N and S France. Toujours chauvinistes, les Français. Not sufficiently accurate and wrong. Led to long controversy between French + English schools of science. Satirized by Swift in <u>Gulliver's Travels</u>.

Not resolved until 1736, French Academy dispatched 2 expeditions, 1 to Lapland and 1 to Peru.
Expedition to Lapland led by famous scientific explorer <u>Maupertuis</u>. Two years of hardship in remote tundra, finally confirmed Newton's prediction.

Upon his return ~~[illegible]~~ he was dubbed by Voltaire "aplatisseur du monde et de Cassini". Later he lambasted Maupertuis too "Vous avez confirmé dan ces lieux pleins d'ennui ce que Newton connu sans sortir de chez lui".

Two rival expeditions produced 2 values 1/179 and 1/266; mean = 1/230!

Later measurements showed however actually nearer 1/300 than 1/230.

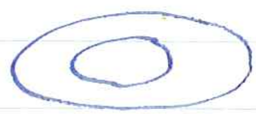
In 1743 Clairaut, French mathematician pointed out this could be a consequence of non-homogeneity of $\oplus$, $\rho(r) \neq$ const.

For homog. case, ellipticity of interior equipot. surfaces is the same for all. All have $\varepsilon = \frac{5}{4} m$.

$\rho(r) = \rho$

This because the "dilution factor" is constant.

For an inhomog. sphere this no longer true.



Ellipticity then $\varepsilon(r)$ a function of r. Recall const. density and equipot. surfaces coincide.

Clairaut showed that $\varepsilon(r)$ satisfied a 2$^{\underline{d}}$ order ODE (~~[illegible]~~ Clairaut's eqn)

$$\rho \left(\varepsilon'' - \frac{6\varepsilon}{r^2} \right) + \frac{6\bar{\rho}}{r} \left(\varepsilon' + \frac{\varepsilon}{r} \right) = 0$$

$\bar{\rho}(r) \equiv$ mean density inside radius r

2^d order ODE needs 2 b.c.:

$$\varepsilon(0) \text{ finite}$$

$$a\varepsilon'(a) = 5m - 2\varepsilon(a)$$

Nowadays simple matter to solve <u>numerically</u> for $\rho(r)$ of ⊕.

For many years after Clairaut, and long before seismology (which has enabled us to find $\rho(r)$ fairly well) it was the hope that Clairaut's eqn + surveying to find $\varepsilon(a)$ accurately could somehow be used to place some strong constraints on $\rho(r)$. It was not clear how, just a goal.

142 yrs. later Radau (1885), German, dashed these hopes by a clever transformation of variables. He found an excellent approximate (not exact) soln to Clairaut's eqn, namely

$$\boxed{\varepsilon_{hydro} = \varepsilon(a) = \frac{10\,m}{4 + 25\left(1 - \frac{3}{2}\frac{C}{Ma^2}\right)^2}}$$

Note $\varepsilon_{hydro} \propto m$ (the <u>cause</u> of bulge) depends on $\rho(r)$ only thru C/Ma^2. All $\rho(r)$ with same C/Ma^2 have same ε_{hydro}.

Check: in case $\rho(r) = \rho$ (Newton's case)

$$C/Ma^2 = \frac{2}{5}$$

$$\varepsilon_{hydro} = \frac{10m}{4 + 25\left(1 - \frac{3}{5}\right)^2} = \frac{5}{4} m.$$

If $\rho(r) \nearrow$ as we go down $C/Ma^2 < \frac{2}{5}$ and $\varepsilon_{hydro} < \varepsilon_{homog} = 1/230$

Recall C/Ma^2 of $\oplus$ known from J_2 from satellites and H from precession

$$(C/Ma^2)_\oplus = 0.3308$$

This gives $\varepsilon_{hydro} = 1/299.8$

Radau is an <u>approximation</u>. If more accuracy desired should integrate Clairaut. When this done for model 1066A of Gilbert + Dziewonski find that Radau is excellent

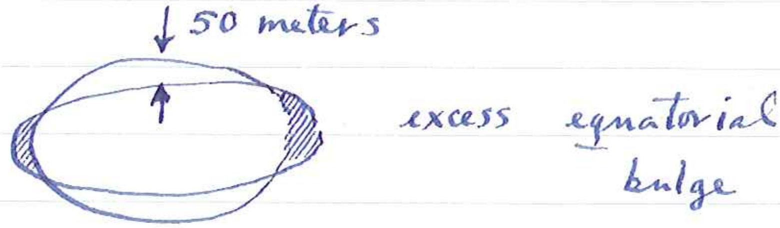
$$\boxed{\varepsilon_{hydro}^{1066A} = 1/299.8}$$

1066A constrained to have right C/Ma^2

Recall best-fitting $\varepsilon = 1/298.258$.

$$\varepsilon_{hydro} < \varepsilon_{b.f.}$$

Amounts to about a 50 m difference at poles



Can see this from Gaposchkin maps.

Other "bumps" on geoid smaller horizontal scales but of same amplitude as excess eq. bulge. Thus as measured by its amplitude there is nothing special about it.

It is however special in terms of its relationship to the rotation axis. Two reasons for or explanations of this excess equatorial bulge have been advanced.

The first ~ 1960 soon after its discovery (it was the first "bump" to be measured) the fossil bulge hypothesis of Munk and MacDonald.

Due to the action of tidal friction (to be discussed later, or see Section 4.5 of Stacey) the ⊕'s rotation rate is known to be slowing down

The rate of change of the l.o.d. is 2.7 ms / century determined from astronomical observations and comparison with atomic Cesium clocks.

Thus in past the equilibrium figure of the ⊕ would have been more oblate. They suggested that the excess bulge was a "fossil" bulge, a remnant of the faster rotation in the past.

How long ago was ε_{hydro} = current ε?
Since $\varepsilon \propto m \propto \Omega^2$

$$\delta\varepsilon / \varepsilon = 2\, \delta\Omega / \Omega$$

$$\delta\varepsilon / \varepsilon = \frac{1/298.256 - 1/299.8}{1/298.256} \sim 5 \cdot 10^{-3}$$

$$\delta\Omega / \Omega \sim 2.6 \cdot 10^{-3}$$

Current deceleration rate from Lambeck Phil. Trans. (1977) is $\dot{\Omega} = -7.22 \cdot 10^{-22}$ rad / sec^2

$$\delta t = \frac{\delta \Omega}{7.22 \cdot 10^{-22}} = 2.6 \cdot 10^{14} \text{ secs}$$

$$\boxed{\varepsilon_{b.f.} \text{ corresponds to } \varepsilon_{hydro} \text{ 8 m.y. ago}}$$

The lag in response of ⊕ to its changing rotation rate could be a consequence of the viscosity of the mantle.

If the viscosity ν were very low like H_2O and if Ω changed slowly then we would always have $\varepsilon = \varepsilon_{hydro}$.

If on the other hand $\nu = \infty$ and Ω started decreasing then ε would simply remain fixed at its initial value.

For a uniform ⊕ (ρ = const, ν = const) with no elastic rigidity (a so-called Newtonian fluid ⊕) the time scale for decay by a fraction e^{-1} of an old bulge to a new rotation rate can be shown to be

$$\boxed{\tau = \frac{19}{2}\frac{\nu}{\rho g a}}$$

if we deform a self-gravitating viscous sphere into an ellipsoid of ellipticity ε_0 and let go it flows back into a sphere like $e^{-t/\tau}$.

τ from fossil bulge hypothesis is $8 \cdot 10^6$ yr. This would imply

$$\nu \sim \frac{2}{19}(5.5)(980)(6.37 \cdot 10^8)(8 \cdot 10^6)(3.15 \cdot 10^7)$$

or $\nu \sim 9 \cdot 10^{25}$

$$\boxed{\text{Fossil bulge hypothesis implies mean } \nu \text{ of mantle} \sim 10^{26} \text{ poise.}}$$

In comparison ν of H_2O is 10^{-2} poise, honey is $\sim 10^3$, road tar is $\sim 10^6$ ~~[illegible]~~ except on hot days and glass is $\sim 10^{12}$, which is why telescope mirrors are rotated annually. — not true!

Post-glacial rebound studies give ν of asthenosphere in range:

$$10^{20} < \nu < 10^{21} \text{ poise}$$

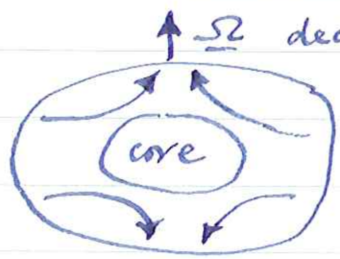
ice load melts

18 k.y.b.p.

Rule of thumb: depth of compensating flow $\approx$ lateral extent. Thus $10^{20} < \nu < 10^{21}$ is for upper 1000 km or so of mantle.

For decay of fossil bulge compensating flow is <u>mantle - wide</u>

Ω decreasing causes flow from equator to poles.



Implication of fossil bulge hypothesis: ν of lower mantle $\sim 10^5 - 10^6$ times that of upper mantle; effectively rigid. Convection, which must be driving plate motions, would be confined to upper mantle.

<u>Second hypothesis</u> advanced 1967 by Goldreich and Toomre. They argue that since size of $\varepsilon - \varepsilon_h$ is no bigger than other "bumps" there is no need to seek a special explanation. The <u>origin</u> of $\varepsilon - \varepsilon_h$ is same as that of other bumps, density anomalies assoc. with mantle convection and tectonics.

What makes $\varepsilon - \varepsilon_h$ special is its location around equator. This, they say, a natural consequence of polar wander.

Consider a simple analogy: spinning rigid sphere with bugs crawling on surface. They showed that Ω axis will always align itself along greatest principal axis of inertia of bugs.

Ω wanders w.r.t. the sphere but remains fixed in space (toward same ✱) by conservation of angular momentum.

Relevance for ⊕? The "bugs" represent mass anomalies (convection cells in mantle, lithospheric slabs, etc.). If it were not for equatorial bulge $\varepsilon_h \cong 1/300$ polar wander would clearly occur.

But the "bugs" or "bumps" on geoid are small ($\pm$ 50 – 100 m) compared to bulge ($a_{eq} - a_{pole}$ = 20 km)

EOS

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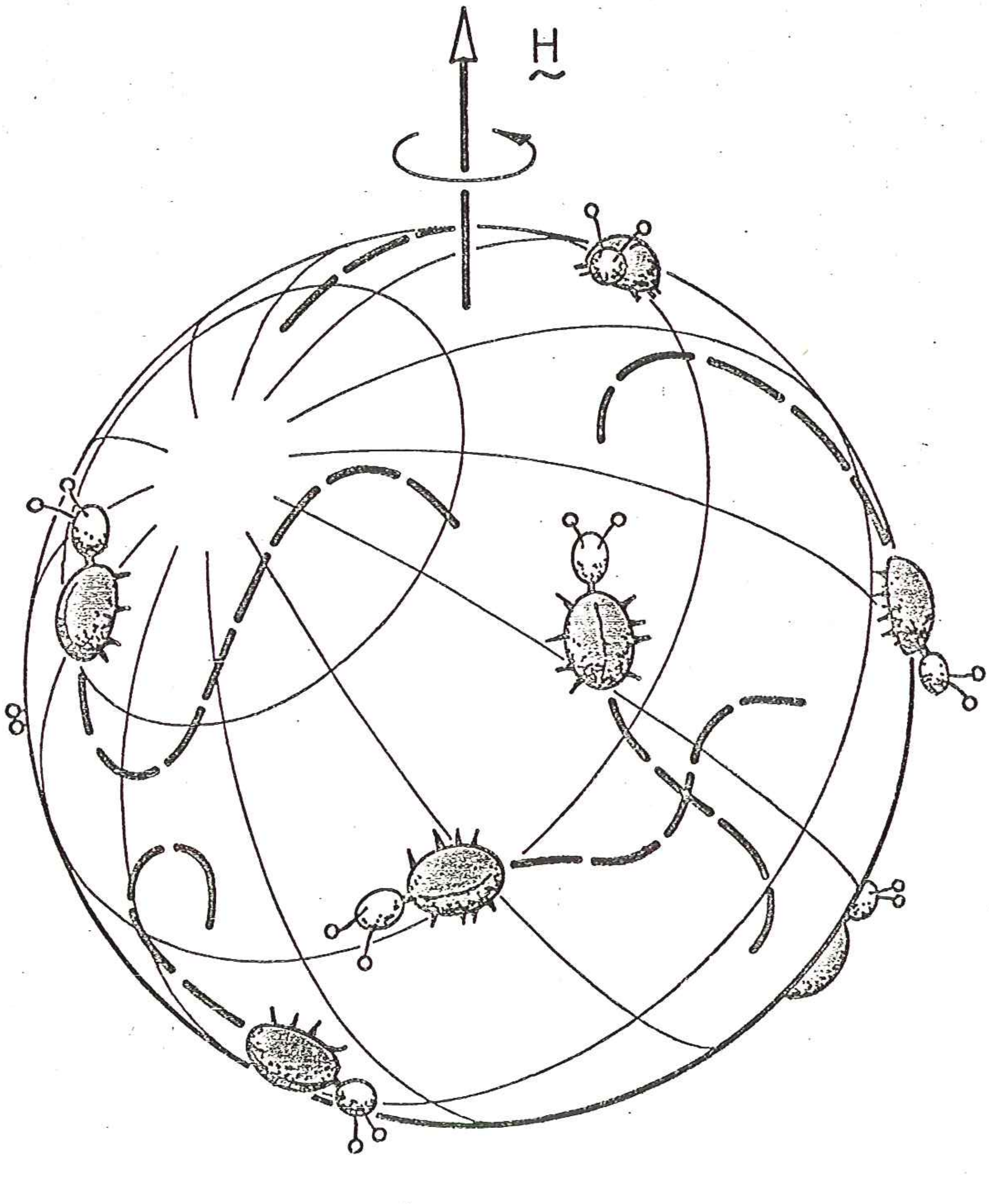
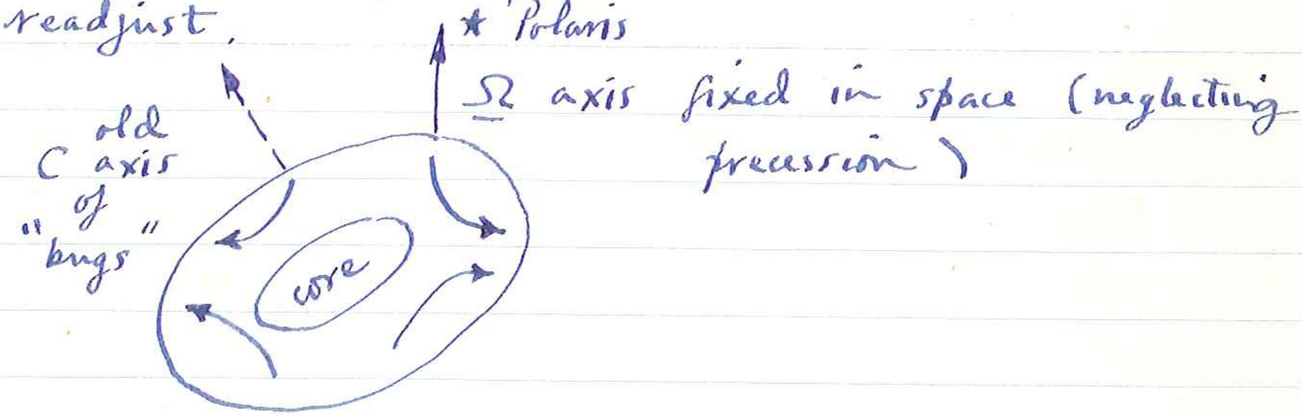


If the bulge were permanent very little polar wander could occur, but if mantle viscous bulge could continually readjust to keep Ω aligned along greatest principal axis of bugs. That is what we observe today (the excess bulge)

For this to work mantle as a whole must be sufficiently inviscid to allow hydrostatic bulge to readjust.



This in turn depends on how fast "bugs" are crawling or changing their mass.

Currently observed rate of polar wander from hot spot reconstruction of Jason Morgan combined with paleomag is about 1° per m.y. in the general direction of Greenland or Labrador.