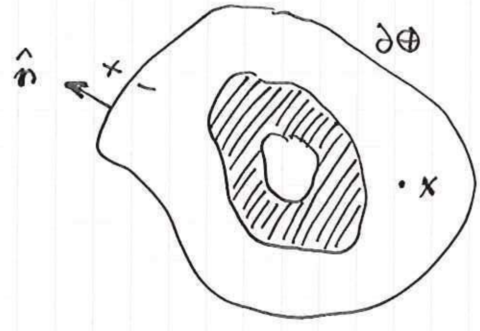
Free Oscillations and Surface Waves

Consider an Earth model with a solid mantle & crust, a fluid outer core and a solid inner core



$\oplus_S$: solid regions

$\oplus_F$: fluid regions

$\oplus = \oplus_S \cup \oplus_F$

$\Sigma = \Sigma_{SS} + \Sigma_{FS} + \partial\oplus$

solid–solid e.g. Moho (Σ_{SS}) ; fluid–solid e.g. CMB (Σ_{FS})

We ignore the Earth's self-gravitation, even though it is quantitatively important for the low-frequency normal modes.

Elastodynamic equation of motion: $\rho\, \partial_t^2 \mathbf{s} = \nabla \cdot \mathbf{T}$

Hooke's law: $\mathbf{T} = \mathbf{C} : \boldsymbol{\varepsilon}$ or $T_{ij} = c_{ijkl}\, \varepsilon_{kl}$

where $\boldsymbol{\varepsilon} = \frac{1}{2}[\nabla \mathbf{s} + (\nabla \mathbf{s})^T]$ or $\varepsilon_{ij} = \frac{1}{2}(\partial_i s_j + \partial_j s_i)$

is the elastic strain.

The tensor c_{ijkl} is the elastic tensor:

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$$

$\mathbf{T}^T = \mathbf{T}$; $\boldsymbol{\varepsilon}^T = \boldsymbol{\varepsilon}$; hyperelastic material

A general anisotropic material (triclinic xtal) has 21 independent elastic components.

Isotropic material: $c_{ijkl} = (\kappa - \frac{2}{3}\mu)\,\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

$$\mathbf{T} = (\kappa - \tfrac{2}{3}\mu)(\nabla\cdot\mathbf{s})\,\mathbf{I} + 2\mu\boldsymbol{\varepsilon} \quad \text{, or}$$

$$\mathbf{T} = \kappa(\nabla\cdot\mathbf{s})\,\mathbf{I} + 2\mu\mathbf{d} \quad \text{where}$$

$\delta = \varepsilon - \frac{1}{3}(\mathrm{tr}\,\varepsilon)\mathbb{I} = \varepsilon - \frac{1}{3}(\nabla\cdot s)\mathbb{I}$ is the deviatoric strain ($\mathrm{tr}\,\delta = 0$)

$\mathbb{T} = \kappa(\nabla\cdot s)\mathbb{I} + 2\mu\delta$

isotropic stress — deviatoric stress

κ – bulk modulus or incompressibility
μ – shear modulus or rigidity

Fluid region: $\mu = 0$ (no rigidity)

$$\mathbb{T} = \kappa(\nabla\cdot s)\mathbb{I} \quad \text{in } \oplus_F$$

Boundary conditions:

kinematic: $[s]^{+}_{-} = 0$ on Σ_{SS} (welded)

$[\hat{n}\cdot s]^{+}_{-} = 0$ on Σ_{FS} (tangential slip allowed)

dynamic: $[\hat{n}\cdot\mathbb{T}]^{+}_{-} = 0$ traction continuous on both Σ_{SS} & Σ_{FS}

$\hat{n}\cdot\mathbb{T} = 0$ on $\partial\oplus$: free outer surface

Conservation of energy:

Consider $\displaystyle\int_{\oplus} \partial_t s \cdot (\rho\,\partial_t^2 s - \nabla\cdot\mathbb{T})\,dV$

First term is $\displaystyle\frac{d}{dt}\int_{\oplus} \frac{1}{2}\rho\,|\partial_t s|^2\,dV$

Second term is $-\int_{\oplus} \partial_t s_j \, \partial_i T_{ij} \, dV$

$$= \int_{\Sigma} [\, \partial_t s_j \cancel{n_i T_{ij}} \,]_-^+ \, d\Sigma + \int_{\oplus} T_{ij} \, \partial_t (\partial_i s_j) \, dV$$

$\hookrightarrow$ vanishes by b.c.

why: if clause in Gauss' theorem

(note — must apply Gauss' theorem separately to each sub-region)

$$= \int_{\oplus} c_{ijkl} \, \partial_k s_l \, \partial_t (\partial_i s_j) \, dV$$

$$= \frac{d}{dt} \int_{\oplus} \tfrac{1}{2} c_{ijkl} \, \partial_i s_j \, \partial_k s_l \, dV$$

since $c_{ijkl} = c_{klij}$

In summary:

$$\frac{d}{dt} \int_{\oplus} \left[\tfrac{1}{2} \rho |\partial_t s|^2 + \tfrac{1}{2} c_{ijkl} \, \partial_i s_j \, \partial_k s_l \right] dV = 0$$

$\frac{1}{2} \rho |\partial_t s|^2$: kinetic energy density

$\frac{1}{2} c_{ijkl} \, \varepsilon_{ij} \, \varepsilon_{kl}$: stored elastic potential energy density

The sum of the kinetic + potential energy is <u>conserved</u> : $\frac{d}{dt}(T + V) = 0$.

In an isotropic material, the elastic energy density is $\underbrace{\tfrac{1}{2} \kappa (\nabla \cdot s)^2}_{\text{compressional energy density}} + \underbrace{\mu \, \delta : \delta}_{\text{shear energy density}}$

stability: $\kappa > 0$, $\mu > 0$

more generally $\mathbf{c}$ is positive definite

every deformation requires work

A hyperelastic material ($c_{ijkl} = c_{klij}$) is one having an elastic energy density.

Hamilton's principle

The equations of motion and boundary conditions can be derived from a variational principle. Define the action integral

$$I = \int_{t_1}^{t_2} \int_{\oplus} L \, dV \, dt \qquad L(s, \partial_t s, \nabla s)$$

not really

where $L = \frac{1}{2}\rho|\partial_t s|^2 - \frac{1}{2} C_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl}$ is the Lagrangian density

kinetic − potential

We regard I as a functional of the dynamical path $s(x,t)$ between times t_1 and t_2.
Consider the variation of this functional for fixed end-points $\delta s(x,t_1) = \delta s(x,t_2) = 0$.
A variation δs is admissible if $[\delta s]^{\pm} = 0$ on Σ_{SS} and $[\hat{n}\cdot\delta s]^{\pm} = 0$ on Σ_{FS}.

$$\delta I = \int_{t_1}^{t_2} \int_{\oplus} [\, \delta s \cdot \partial_s L + \partial_t(\delta s) \cdot \partial_{\partial_t s} L + \nabla(\delta s) \cdot \partial_{\nabla s} L \,] \, dV \, dt$$

$$= \int_{t_1}^{t_2} \int_{\oplus} \delta s \cdot \left[\partial_s L - \partial_t(\partial_{\partial_t s} L) - \nabla\cdot(\partial_{\nabla s} L) \right] dV \, dt$$

$$+ \int_{\oplus} \cancel{\left[\delta s \cdot \partial_{\partial_t s} L \right]_{t_1}^{t_2}} \, dV$$

zero since ends of path are fixed

$$- \int_{t_1}^{t_2} \int_{\Sigma} [\, \delta s \cdot (\hat{n} \cdot \partial_{\nabla s} L) \,]^{\pm} \, d\Sigma \, dt$$

δI vanishes for an arbitrary <u>admissible</u> δs if and only if

$$\partial_s L - \partial_t(\partial_{\partial_t s} L) - \nabla\cdot(\partial_{\nabla s} L) = 0 \quad \text{in } \oplus$$

Euler – Lagrange eqn

$$[\hat{n}\cdot(\partial_{\nabla s} L)]^{\pm} \quad \text{on } \Sigma$$

$$\partial_s L = 0$$

$$\partial_{\partial_t s} L = \rho\,\partial_t s \quad (\text{momentum density})$$

$$\partial_{\nabla s} L = -\mathbf{T} \quad (-\text{ stress})$$

Euler – Lagrange equation : $\rho\,\partial_t^2 s = \nabla\cdot\mathbf{T}$

Boundary condition : $[\hat{n}\cdot\mathbf{T}]^{\pm} = 0$

The <u>underlined</u> term is $-\int_{t_1}^{t_2}\int_{\Sigma} [\delta s\cdot(\hat{n}\cdot\mathbf{T})]^{\pm}\, d\Sigma\, dt$

end here 6th class

on Σ_{SS} : $[\delta s]^{\pm} = 0$

on Σ_{FS} : $[\hat{n}\cdot\delta s]^{\pm} = 0$ but $\hat{n}\cdot\mathbf{T} = \hat{n}(\hat{n}\cdot\mathbf{T})$

<u>dynamical</u> <u>b.c.</u> <u>arise</u> <u>naturally</u> no shear stress on Σ_{FS}.

<u>Normal mode solutions</u> : oscillatory in time

Ch. IV of D&T.

$$s(x,t) = s(x)\begin{Bmatrix}\sin\omega t\\ \cos\omega t\end{Bmatrix}$$

why? because $\rho(x)$ and $C(x)$ ind. of time t

today Sect. 4.1

$s(x)$: real eigenfunction

ω : real eigenfrequency

$$-\rho\omega^2 s = \nabla\cdot\mathbf{T}$$

dependence on ω^2 $\Rightarrow$ two eigenfrequencies $\pm\omega$ associated with every s.

Write in abstract operator notation :

$\mathcal{H}s = \omega^2 s$ where $\mathcal{H}s = -\frac{1}{\rho}\nabla\cdot(\mathbf{C}:\nabla s)$

or $\mathcal{H}s_j = -\frac{1}{\rho}\partial_i(C_{ijkl}\,\partial_k s_l)$

$\mathcal{H}$ stands for the differential operator together with the b.c.

eigenvalue

Eigenvalue problem : $\mathcal{H}s = \omega^2 s$

Define the inner product of two functions s and s':

$$\langle s, s'\rangle = \int_{\oplus} \rho\, s\cdot s'\, dV$$

density weighting

The operator $\mathcal{H}$ is Hermitian or self-adjoint with respect to $\langle\cdot,\cdot\rangle$, i.e.

$$\langle s, \mathcal{H}s'\rangle = \langle \mathcal{H}s, s'\rangle = \langle s', \mathcal{H}s\rangle$$

Proof:

$$\langle s, \mathcal{H}s'\rangle = -\int_{\oplus} s\cdot(\nabla\cdot \mathbf{T}')\, dV$$

$$= \int_{\oplus} C_{ijkl}\,\partial_i s_j\,\partial_k s'_l\, dV + \int_{\Sigma} [\hat{n}\cdot\mathbf{T}'\cdot s]_-^+\, d\Sigma$$

equal since $C_{ijkl} = C_{klij}$

$$\langle s', \mathcal{H}s\rangle = -\int_{\oplus} s'\cdot(\nabla\cdot\mathbf{T})\, dV$$

$$= \int_{\oplus} C_{ijkl}\,\partial_i s'_j\,\partial_k s_l\, dV + \int_{\Sigma} [\hat{n}\cdot\mathbf{T}\cdot s']_-^+\, d\Sigma$$

The surface integrals vanish by virtue of the b.c. (this is why $\mathcal{H}$ "includes the b.c.")

It is noteworthy that the manipulations required to show that H is Hermitian are the same as those used to show $\frac{d}{dt}(T+V)=0$. Illustrates a general principle — that physical systems governed by Hermitian operators are energy-conserving.

Consider now the inner product of $Hs=\omega^2 s$ with s' and the inner product of $Hs'=\omega'^2 s'$ with s:

$$\omega^2\langle s',s\rangle = \langle s', Hs\rangle$$
$$\omega'^2\langle s,s'\rangle = \langle s, Hs'\rangle$$

Subtracting and using the Hermiticity we find

$$(\omega^2-\omega'^2)\langle s,s'\rangle = 0 \quad \text{or}$$

$$\langle s,s'\rangle = 0 \quad \text{if} \quad \omega^2 \neq \omega'^2$$

Eigenfunctions associated with discrete eigenfrequencies $\omega\neq\omega'$ are <u>orthogonal</u> in the sense $\langle s,s'\rangle=0$. Because of this every eigensolution ω, s is referred to as a normal mode.

<u>Rayleigh's principle</u>:

Every Hermitian eigenvalue problem of the form $Hs=\omega^2 s$ is associated with a variational principle known as Rayleigh's principle.

Consider the Rayleigh quotient

$$\omega^2 = \frac{\langle s, \mathcal{H}s\rangle}{\langle s, s\rangle}$$

Regard right side as a functional which assigns a scalar $\omega^2(s)$ to every possible displacement s. Then Rayleigh's principle asserts that this functional is stationary for every admissible δs iff s is an eigenfunction with associated squared eigenfrequency ω^2. To verify this,

$$\delta\omega^2 = \frac{\langle \delta s, \mathcal{H}s\rangle + \langle s, \mathcal{H}\delta s\rangle - \omega^2\langle \delta s, s\rangle - \omega^2\langle s, \delta s\rangle}{\langle s, s\rangle}$$

$$= \frac{2\langle \delta s, \mathcal{H}s - \omega^2 s\rangle}{\langle s, s\rangle}$$

Evidently $\delta\omega^2 = 0$ for an arbitrary δs iff $\mathcal{H}s = \omega^2 s$.

We may alternatively consider the quantity

$$\mathcal{L} = \tfrac{1}{2}\omega^2\langle s, s\rangle - \tfrac{1}{2}\langle s, \mathcal{H}s\rangle$$

rather than ω^2 to be the stationary functional.

An alternative notation – define the kinetic and potential energy quadratic functionals

$$\mathcal{T} = \int_{\oplus} \rho\, s\cdot s\, dV$$

$$\mathcal{V} = \int_{\oplus} C_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl}\, dV.$$

Then $\quad \omega^2 = \dfrac{\mathcal{V}}{\mathcal{T}} = \dfrac{\text{potential energy}}{\text{kinetic energy}}$

$$\mathcal{L} = \tfrac{1}{2}\left(\omega^2 \mathcal{T} - \mathcal{V}\right) = \text{kinetic energy} - \text{potential energy}$$

Fleshing out the above schematic proof:

$$\delta \mathcal{L} = \int_{\oplus} \delta \mathbf{s} \cdot \left[\omega^2 \rho \mathbf{s} + \nabla \cdot \mathbf{T} \right] dV + \int_{\Sigma} \left[\delta \mathbf{s} \cdot (\hat{\mathbf{n}} \cdot \mathbf{T}) \right]_{-}^{+} d\Sigma$$

Clearly $\delta\mathcal{L}$ vanishes for an arbitrary $\delta \mathbf{s}$ iff

$$-\rho \omega^2 \mathbf{s} = \nabla \cdot \mathbf{T} \quad \text{in } \oplus$$

$$\left[\hat{\mathbf{n}} \cdot \mathbf{T} \right]_{-}^{+} = 0 \quad \text{on } \Sigma$$

The stationary value of $\mathcal{L}$ at the eigensolutions $\omega^2, \mathbf{s}$ is $\mathcal{L} = 0$. Physically, $\omega^2 \mathcal{T}$ and $\mathcal{V}$ are twice the average k.e. and p.e. during a cycle of free oscillation $\mathbf{s}(\mathbf{x}) \cos \omega t$ or $\mathbf{s}(\mathbf{x}) \sin \omega t$.

Trivial modes:

Every Earth model has a 6-dimensional space of trivial rigid-body modes

$$\omega^2 = 0, \quad \mathbf{s}(\mathbf{x}) = \mathbf{X} + \mathbf{Q} \cdot \mathbf{x}$$

3 translations $\quad$ 3 rotations

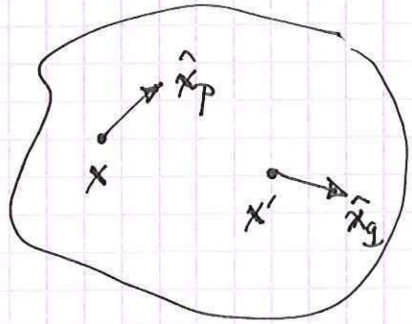
In addition there is an ∞-dimensional family of trivial geostrophic ($\omega^2 = 0$) modes confined to the fluid core.

Go to page 14.2: do Rayleigh's principle before this.

Green's tensor

The response to an earthquake, meteorite impact, nuclear explosion, etc. can be conveniently expressed in terms of the Green's tensor or impulse response:

Definition: $G_{pq}(x, x'; t)$ is the $\hat{x}_p$ component of the response at x, t to a unit impulsive force in the $\hat{x}_q$ direction at $x', 0$.



In other words: $\rho(\partial_t^2 G + \mathcal{H} G) = \mathbb{I}\,\delta(x - x')\,\delta(t)$

Equivalently, we can solve the homogeneous eqn

$$\rho(\partial_t^2 G + \mathcal{H} G) = 0 \quad , \quad t \geq 0$$

subject to the initial conditions

$$G(x, x'; 0) = 0$$

$$\partial_t G(x, x'; 0) = \rho^{-1}\,\mathbb{I}\,\delta(x - x')$$

Label the eigensolutions $\pm\omega_k$, s_k with an index k and normalize such that

$$\langle s_k, s_{k'} \rangle = \int_\oplus \rho\, s_k \cdot s_{k'}\, dV = \delta_{kk'}$$

Look for a solution that is a linear combination of free oscillations (assume the s_k are <u>complete</u>), for $t \geq 0$:

$$G(x, x'; t) = \sum_k s_k(x) \left[a_k(x') \cos\omega_k t + b_k(x') \sin\omega_k t \right] H(t)$$

This satisfies $\rho(\partial_t^2 G + \mathcal{H} G) = 0$.
The b.c. are satisfied if

$$\sum_k s_k a_k = 0 \quad ; \quad \sum_k \omega_k s_k b_k = \rho^{-1} \mathbf{I}\, \delta(x - x')$$

Take inner product with s_k and use orthonormality:

$$a_k = 0 \quad ; \quad b_k = \omega_k^{-1} s_k(x')$$

Thus the normal-mode Green's tensor is

$$G(x, x'; t) = \sum_k \omega_k^{-1} s_k(x) s_k(x') \sin\omega_k t \, H(t)$$

Every mode begins oscillating with the same phase, like $\sin\omega_k t$.
Note that

$$G(x, x'; t) = G^T(x', x; t)$$

$G_{pq}(x, x'; t) = G_{qp}(x', x; t)$

This is the principle of source-receiver <u>reciprocity</u>. Note that the <u>orientations</u> of source and receiver as well as their <u>locations</u> must be interchanged.

Response to a transient force:

The response to a general applied body force $\mathbf{f}$ in $\oplus$ and surface force $\mathbf{t}$ on $\partial\oplus$ can be found by convolution with the impulse response:

$$s(x,t) = \int_{-\infty}^{t} \int_{\oplus} G(x,x';t-t') \cdot f(x',t')\, dV'\, dt'$$

$$+ \int_{-\infty}^{t} \int_{\partial\oplus} G(x,x';t-t') \cdot t(x',t')\, d\Sigma'\, dt'$$

Superposition plus causality (upper limit is t). Lower limit can be any time before f and t begin to act (entire past history).

Substituting the mode-sum Green's tensor gives

$$s(x,t) = \sum_k \omega_k^{-1} s_k(x) \int_{-\infty}^{t} A_k(t') \underbrace{\sin\omega_k(t-t')}_{\text{impulse response}}\, dt'$$

where

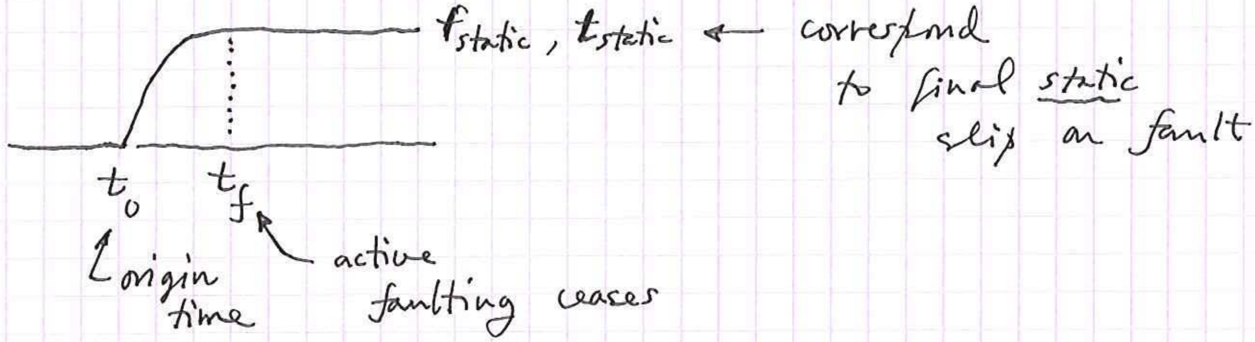
$$A_k(t) = \int_{\oplus} f(x,t)\cdot s_k(x)\, dV + \int_{\partial\oplus} t(x,t)\cdot s_k(x)\, d\Sigma$$

projection onto the mode

Integrating by parts with respect to time, can also write in form

$$s(x,t) = \sum_k \omega_k^{-2} s_k(x) \int_{-\infty}^{t} \partial_{t'} A_k(t') \underbrace{\left[1 - \cos\omega_k (t-t') \right] dt'}_{\text{unit step response}}$$

As we shall see later, the equivalent forces to an earthquake have the character



The responce to such a transient force, for times $t \geq t_f$, is

$$s(x,t) = \sum_k \omega_k^{-2} \left(a_k^f - a_k \cos\omega_k t - b_k \sin\omega_k t \right) s_k(x) \quad , \quad t \geq t_f$$

$$a_k^{stat} = \int_{\oplus} f_{stat} \cdot s_k \, dV + \int_{\partial\oplus} t_{stat} \cdot s_k \, d\Sigma$$

$$a_k = \int_{t_0}^{t_f} \int_{\oplus} \partial_t f \cdot s_k \cos\omega_k t \, dV \, dt + \int_{t_0}^{t_f} \partial_t t \cdot s_k \cos\omega_k t \, d\Sigma \, dt$$

$$b_k = \int_{t_0}^{t_f} \int_{\oplus} \partial_t f \cdot s_k \sin\omega_k t \, dV \, dt + \int_{t_0}^{t_f} \partial_t t \cdot s_k \sin\omega_k t \, d\Sigma \, dt$$

integrals only over active faulting interval

Anelasticity causes the oscillations $\cos\omega_k t$, $\sin\omega_k t$ to decay with time:

$$\lim_{t\to\infty} s(x,t) \equiv s_{stat}(x) = \sum_k \omega_k^{-2}\, a_k^{stat}\, s_k(x)$$

This represents the permanent static deformation of the Earth produced by the faulting.

Modern seismic instruments are in essence accelerometers — the acceleration is

$$a(x,t) = \sum_k \underbrace{(a_k \cos\omega_k t + b_k \sin\omega_k t)}_{\text{free oscillations}}\, s_k(x)\,, \quad t \geq t_f$$

$s_k(x)$ is the geographic <u>shape</u> of the k^{th} oscillation

Every detail of every observed seismogram is of this form (with attenuation properly accounted for).

Rayleigh–Ritz method:

Let [redacted] $\mathbf{e}_k$ be a set of functions in $\oplus$, smooth everywhere except on Σ_{FS}, where they satisfy $[\hat{n}\cdot\mathbf{e}_k]=0$. For now the $\mathbf{e}_k$ can be considered arbitrary — later we will take them to be the eigenfunctions of a starting Earth model.

Write eigenfunction s as an expansion

$$s=\sum_k a_k \mathbf{e}_k$$

Substitute into the action $\mathcal{I}=\frac{1}{2}(\omega^2\mathcal{T}-\mathcal{V})$

Get new form for action:

$$\mathcal{I}=\frac{1}{2}\mathbf{a}^T\cdot(\omega^2\mathbf{T}-\mathbf{V})\cdot\mathbf{a}$$

where

$$\mathbf{a}=\begin{pmatrix}\vdots\\ a_k\\ \vdots\end{pmatrix},\quad \mathbf{T}=\begin{pmatrix} & \vdots & \\ \cdots & T_{kk'} & \cdots\\ & \vdots & \end{pmatrix},\quad \mathbf{V}=\begin{pmatrix} & \vdots & \\ \cdots & V_{kk'} & \cdots\\ & \vdots & \end{pmatrix}$$

$$T_{kk'}=\int_\oplus \rho\,\mathbf{e}_k\cdot\mathbf{e}_{k'}\,dV \qquad V_{kk'}=\int_\oplus \boldsymbol{\varepsilon}_k:\mathbf{C}:\boldsymbol{\varepsilon}_{k'}\,dV$$

$$=\int_\oplus [\kappa(\nabla\cdot\mathbf{e}_k)(\nabla\cdot\mathbf{e}_{k'})+2\mu\,\mathbf{d}_k:\mathbf{d}_{k'}]\,dV$$

$\mathbf{T}$ and $\mathbf{V}$ are symmetric matrices: $\mathbf{T}^T=\mathbf{T}$, $\mathbf{V}^T=\mathbf{V}$

The variation of the action with respect to $\mathbf{a}$ is

$$\delta\mathcal{I}=\frac{1}{2}\delta\mathbf{a}^T\cdot(\omega^2\mathbf{T}-\mathbf{V})\cdot\mathbf{a}+\frac{1}{2}\mathbf{a}^T\cdot(\omega^2\mathbf{T}-\mathbf{V})\cdot\delta\mathbf{a}$$

$$=\delta\mathbf{a}^T\cdot(\omega^2\mathbf{T}-\mathbf{V})\cdot\mathbf{a}=0 \quad \text{iff} \quad \mathbf{V}\cdot\mathbf{a}=\omega^2\mathbf{T}\cdot\mathbf{a}$$

$\mathbf{T}$ is symmetric and positive definite $\mathbf{T}>0$ so it can be inverted: $(\mathbf{T}^{-1}\cdot\mathbf{V})\cdot\mathbf{a}=\omega^2\mathbf{a}$, an ordinary matrix eigenvalue problem. No need to solve ODE's — only need to do matrix algebra.

Rayleigh's principle can also be used to calculate the change in an eigenfrequency ω due to a change in the Earth model:

$$\rho \to \rho + \delta\rho \;, \quad \kappa \to \kappa + \delta\kappa \;, \quad \mu \to \mu + \delta\mu$$

as a result $\omega \to \omega + \delta\omega$, $s + \delta s$

↑ no need to calculate

We regard the Lagrangian density L as a functional not only of s and ∇s, but also of ω and $\oplus \equiv \{\rho, \kappa, \mu\}$. Write the action as

$$\mathcal{A} = \int_\oplus L(s, \nabla s, \omega, \oplus)\, dV \quad \text{where}$$

$$L = \tfrac{1}{2}\left[\omega^2 \rho\, s\cdot s - \kappa(\nabla\cdot s)^2 - 2\mu\, \delta : \delta\right]$$

do this in terms of C_{ijkl}

We know 2 things about $\mathcal{A}$:

(1) it is stationary

(2) it's value at the stationary points (eigensolutions) is zero

Consider $\quad \mathcal{A} = \int_\oplus L(s, \nabla s, \omega^2, \oplus)\, dV = 0$

Take the <u>total</u> variation w.r.t. $s, \nabla s, \omega^2, \oplus$

$$\int_\oplus \left[\delta s \cdot \partial_s L + \nabla s : \partial_{\nabla s} L + \delta\omega^2\, \partial_{\omega^2} L + \delta\oplus\, \partial_\oplus L\right] dV = 0$$

$$\int_\oplus \delta s \cdot \underbrace{\left[\partial_s L - \nabla\cdot(\partial_{\nabla s} L)\right]}_{\rho\omega^2 s + \nabla\cdot T = 0} dV - \int_\Sigma \left[\delta s \cdot \underbrace{(\hat{n}\cdot\partial_{\nabla s} L)}_{-\hat{n}\cdot T = 0}\right]_-^+ d\Sigma$$

$$+ \int_\oplus \left[\delta\omega^2\, \partial_{\omega^2} L + \delta\oplus\, \partial_\oplus L\right] dV = 0$$

$$\delta\omega^2 \int_\oplus \partial_\omega^2 L \, dV = -\int_\oplus \delta\Phi \, \partial_\Phi L$$

write in terms of δC_{ijkl}

$$= -\int_\oplus [\, \delta\rho \, \partial_\rho L + \delta\kappa \, \partial_\kappa L + \delta\mu \, \partial_\mu L \,] \, dV$$

$$\partial_\omega^2 L = \tfrac{1}{2} \rho \, s \cdot s$$

If we adopt the normalization $\int_\oplus \rho s \cdot s \, dV = 1$:

$$\delta\omega^2 = \int_\oplus [\, \delta\kappa \, (\nabla \cdot s)^2 + 2 \delta\mu \left(\mathbf{d} : \mathbf{d} \right) - \delta\rho \, \omega^2 s \cdot s \,] \, dV$$

$\boldsymbol{\varepsilon} : \delta C : \boldsymbol{\varepsilon}$

$$\delta\omega^2 = \int_\oplus [\, \delta\kappa \, (\nabla \cdot s)^2 + 2\mu \, (\mathbf{d} : \mathbf{d}) - \delta\rho \, \omega^2 s \cdot s \,] \, dV$$

This is the basis for SNREI Earth model inversion. One measures the observed frequencies of vibration after an earthquake and calculates the residuals $\delta\omega = \omega_{obs} - \omega_{model}$ relative to some starting or initial model. The above result can then be used to adjust the model to provide a best fit to the data.

Quasi-degenerate perturbation theory

The above only works if the initial eigenfrequencies are well isolated in the spectrum. More generally we can seek to find all the zeroth-order eigenfunctions s and first-order

eigenfrequencies $\omega_0 + \delta\omega$ in the vicinity of some fiducial or reference frequency ω_0.
Use the Rayleigh-Ritz form of the action but now choose the basis vectors $\mathbf{e}_k$ to be the [redacted] eigenfunctions of the initial model: $\mathbf{e}_k = \mathbf{s}_k$. The kinetic and potential energy matrices now take the form

$$\mathbf{T} = \mathbf{I} + \delta\mathbf{T}$$
$$\mathbf{V} = \mathbf{\Omega}^2 + \delta\mathbf{V} \qquad \text{where} \qquad \mathbf{\Omega} = \begin{pmatrix} \ddots & & \\ & \omega_k & \\ & & \ddots \end{pmatrix}$$

Reason: $\quad T_{kk'} = \int_{\oplus} (\rho + \delta\rho)\, \mathbf{s}_k \cdot \mathbf{s}_{k'}\, dV \,, \quad$ etc.

Substitute in $\mathbf{V}\cdot\mathbf{a} = \omega^2\, \mathbf{T}\cdot\mathbf{a}$ and neglect second-order terms

$$(\mathbf{\Omega}^2 + \delta\mathbf{V})\cdot\mathbf{a} = (\omega_0^2 + 2\omega_0\,\delta\omega)(\mathbf{I} + \delta\mathbf{T})\cdot\mathbf{a}$$

$$(\mathbf{\Omega}^2 - \omega_0^2\,\mathbf{I} + \delta\mathbf{V} - \omega_0^2\,\delta\mathbf{T})\cdot\mathbf{a} = 2\omega_0\,\delta\omega\,\mathbf{a}$$

$$\mathbf{H}\cdot\mathbf{a} = \delta\omega\,\mathbf{a} \leftarrow \text{ an } N\times N \text{ algebraic eigenvalue problem}$$

$$\mathbf{H} = \frac{1}{2\omega_0}(\mathbf{\Omega}^2 - \omega_0^2\,\mathbf{I} + \delta\mathbf{V} - \omega_0^2\,\delta\mathbf{T})$$

$$\mathbf{H} = \underbrace{\frac{1}{2\omega_0}\begin{pmatrix} \ddots & & \\ & \omega_k^2 - \omega_0^2 & \\ & & \ddots \end{pmatrix}}_{\text{diagonal}} + \underbrace{\frac{1}{2\omega_0}\begin{pmatrix} & \vdots & \\ \cdots & \delta V_{kk'} - \omega_0^2\,\delta T_{kk'} & \cdots \\ & \vdots & \end{pmatrix}}_{\text{symmetric}}$$

The zeroth-order eigenfunctions of the perturbed model are $\mathbf{s} = \sum_k a_k \mathbf{s}_k$ and the associated first-order eigenfrequencies are $\omega_0 + \delta\omega$

begin class #8
Ch. 5 D&T

Seismic source representation:

The simplest and most general approach to this problem uses the concept of the stress glut, introduced by Backus & Mulcahy in 1976.

We have used two equations

$\rho \partial_t^2 s = \nabla \cdot \mathbb{T}$: Newton's second law

b.c. $\hat{n} \cdot \mathbb{T} = 0$ on $\partial\oplus$

$\mathbb{T} = \mathbf{C} : \varepsilon$: Hooke's "law" (linearized)

The first is a bona fide law of physics — the second is not. If both "laws" were always valid there would be no earthquakes — the equations are homogeneous.

Every indigenous source, which does not involve forces exerted by other bodies (e.g. a meteorite strike) are the result of a localized, transient failure of Hooke's law (including slip on faults, phase changes, etc.)

We regard Newton's law and the b.c. as true for the true physical stress $\mathbb{T}_{true}$:

$\rho \partial_t^2 s = \nabla \cdot \mathbb{T}_{true}$ in $\oplus$

$\hat{n} \cdot \mathbb{T}_{true} = 0$ on $\partial\oplus$

But Hooke's law only defines the Hooke stress

$\mathbb{T}_{Hooke} = \mathbf{C} : \varepsilon$

Define the Backus–Mulcahy stress glut by

$$\mathbf{T}_{glut} \equiv \mathbf{T}_{Hooke} - \mathbf{T}_{true} \qquad \text{D\&T call the glut } \mathbf{S}$$

Rewrite eqn & b.c. in form

$$\rho \partial_t^2 \mathbf{s} = \nabla \cdot \mathbf{T}_{Hooke} - \nabla \cdot \mathbf{T}_{glut} \quad \text{in } \oplus$$

$$\hat{\mathbf{n}} \cdot \mathbf{T}_{Hooke} = \hat{\mathbf{n}} \cdot \mathbf{T}_{glut} \quad \text{on } \partial\oplus$$

Define the <u>equivalent</u> body and surface forces

$$\mathbf{f} \equiv -\nabla \cdot \mathbf{T}_{glut} \quad \text{in } \oplus$$

$$\mathbf{t} \equiv \hat{\mathbf{n}} \cdot \mathbf{T}_{glut} \quad \text{on } \partial\oplus$$

Then have a <u>non-homogeneous</u> problem

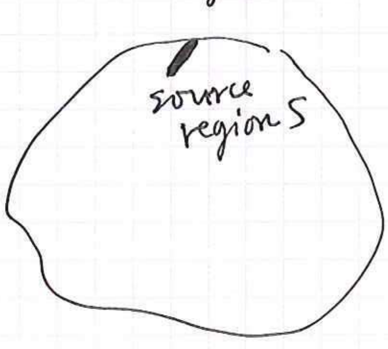
$$\left.\begin{aligned} \rho \partial_t^2 \mathbf{s} &= \nabla \cdot \mathbf{T} + \mathbf{f} \quad \text{in } \oplus \\ \mathbf{T} &= \mathbf{C} : \boldsymbol{\varepsilon} \quad \text{in } \oplus \\ \hat{\mathbf{n}} \cdot \mathbf{T} &= \mathbf{t} \quad \text{on } \partial\oplus \end{aligned}\right\}$$

where $\mathbf{T}$ denotes $\mathbf{T}_{Hooke}$ – the linear model stress

$\mathbf{f}$ and $\mathbf{t}$ act as sources that can excite the free oscillations of the Earth.

In general the breakdown of Hooke's law will be confined to some source region S

$\mathbf{T}_{glut} = \mathbf{0}$ in $\oplus - S$ and on $\partial S - \partial S \cap \partial\oplus$



$$\mathbf{f} = 0 \quad \text{in } \oplus - S$$

$$\mathbf{t} = 0 \quad \text{on } \partial S - \underbrace{\partial S \cap \partial\oplus}_{\text{fault may intersect surface (as shown)}}$$

this eqn not right
$\mathbf{t}$ only defined on $\partial\oplus$

The total force exerted on the Earth by $\boldsymbol{f}$ and $\boldsymbol{t}$ is

$$\boldsymbol{F}_{total} = \int_{\oplus} \boldsymbol{f}\, dV + \int_{\partial\oplus} \boldsymbol{t}\, d\Sigma$$

$$= -\int_{S} \boldsymbol{\nabla}\cdot\boldsymbol{T}_{glut}\, dV + \int_{\partial S \cap \partial\oplus} \hat{\boldsymbol{n}}\cdot\boldsymbol{T}_{glut}\, d\Sigma$$

$$= -\int_{\partial S - \partial S \cap \partial\oplus} \hat{\boldsymbol{n}}\cdot\boldsymbol{T}_{glut}\, d\Sigma = 0$$

The total torque likewise vanishes:

$$\boldsymbol{N}_{total} = \int_{\oplus} \boldsymbol{x}\times\boldsymbol{f}\, dV + \int_{\partial\oplus} \boldsymbol{x}\times\boldsymbol{t}\, d\Sigma$$

$$= -\int_{S} \boldsymbol{x}\times(\boldsymbol{\nabla}\cdot\boldsymbol{T}_{glut})\, dV + \int_{\partial S \cap \partial\oplus} \boldsymbol{x}\times(\hat{\boldsymbol{n}}\cdot\boldsymbol{T}_{glut})\, d\Sigma$$

The ith component is

$$-\int_{S} \varepsilon_{ijk}\, x_j\, \partial_l T_{lk}^{glut}\, dV + \int_{\partial S \cap \partial\oplus} \varepsilon_{ijk}\, x_j\, n_l T_{lk}^{glut}\, d\Sigma$$

$\varepsilon_{ilk} T_{lk}^{glut}$

$$= \int_{S} \underbrace{\varepsilon_{ijk} T_{jk}^{glut}}_{\substack{\text{vanishes} \\ \text{since} \\ T_{jk}^{glut} = T_{kj}^{glut}}} dV - \int_{\partial S - \partial S \cap \partial\oplus} \varepsilon_{ijk}\, x_j\, n_l T_{lk}^{glut}\, d\Sigma = 0$$

An indigenous source exerts no net force or torque on the Earth — the trivial modes are not excited as a result.

Recall the acceleration response to a transient applied $\mathbf{f}$ and $\mathbf{t}$:

$$\mathbf{a}\,\cancel{\mathbf{s}}(\mathbf{x},t) = \sum_k \left(a_k \cos\omega_k t + b_k \sin\omega_k t\right) \mathbf{s}_k(\mathbf{x})$$

What are the excitation amplitudes for a stress-glut source?

$$\begin{Bmatrix} a_k \\ b_k \end{Bmatrix} = \int_{t_0}^{t_f} \underbrace{\left[\int_{\oplus} \partial_t \mathbf{f} \cdot \mathbf{s}_k \, dV + \int_{\partial\oplus} \partial_t \mathbf{t} \cdot \mathbf{s}_k \, d\Sigma \right]} \begin{Bmatrix} \cos\omega_k t \\ \sin\omega_k t \end{Bmatrix} dt$$

$$-\int_{S} \nabla \cdot \partial_t \mathbf{T}_{\text{glut}} \cdot \mathbf{s}_k \, dV + \int_{\partial S \cap \partial\oplus} \hat{\mathbf{n}} \cdot \partial_t \mathbf{T}_{\text{glut}} \cdot \mathbf{s}_k \, d\Sigma$$

$$= \int_S \partial_t \mathbf{T}_{\text{glut}} : \nabla \mathbf{s}_k \, dV - \int_{\partial S - \partial S \cap \partial\oplus} \hat{\mathbf{n}} \cdot \partial_t \mathbf{T}_{\text{glut}} \cdot \mathbf{s}_k \, d\Sigma$$

$$= \int_S \partial_t \mathbf{T}_{\text{glut}} : \boldsymbol{\varepsilon}_k \, dV$$

integral only over source volume

strain associated with k^{th} eigenfunction

$$\boldsymbol{\varepsilon}_k = \tfrac{1}{2}\left[\nabla \mathbf{s}_k + (\nabla \mathbf{s}_k)^T\right]$$

In summary:

$$a_k = \int_{t_0}^{t_f} \int_S \partial_t \mathbf{T}_{\text{glut}} : \boldsymbol{\varepsilon}_k \, dV \cos\omega_k t \, dt$$

$$b_k = \int_{t_0}^{t_f} \int_S \partial_t \mathbf{T}_{\text{glut}} : \boldsymbol{\varepsilon}_k \, dV \sin\omega_k t \, dt$$

glut-rate

Moment tensor D&T Sect. 5.4.1

skip this for now
go to page 20

In the limit of long wavelengths ($\gg$ source dimensions) and long periods ($\gg$ source duration) we can approximate

$$\varepsilon_k(x)\begin{Bmatrix}\cos\omega_k t\\ \sin\omega_k t\end{Bmatrix} \approx \varepsilon_k(x_s)\begin{Bmatrix}\cos\omega_k t_s\\ \sin\omega_k t_s\end{Bmatrix}$$

Then x_s, t_s are the epicentral location in this point-source approximation

$$a_k = \mathbf{M} : \varepsilon_k(x_s)\ \cos\omega_k t_s$$
$$b_k = \mathbf{M} : \varepsilon_k(x_s)\ \sin\omega_k t_s$$

where
$$\mathbf{M} = \int_{t_0}^{t_f}\int_S \partial_t \mathbb{T}_{glut}\, dV\, dt$$

$\mathbf{M} = \mathbf{M}^T$ is the moment tensor — the integrated glut-rate, or equivalently,

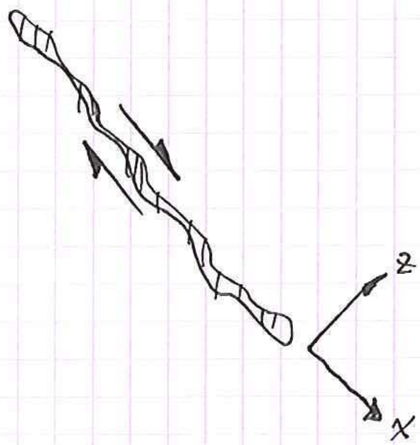
$$\mathbf{M} = \int_S \mathbb{T}_{glut,\,stat}\, dV \ ,$$ the integrated final static stress glut

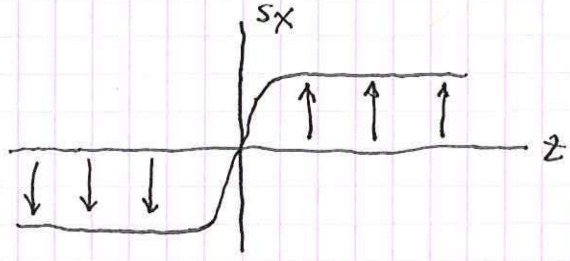
The acceleration response to such a moment tensor source is — using $\cos\omega_k t\cos\omega_k t_s + \sin\omega_k t\sin\omega_k t_s = \cos\omega_k(t-t_s)$:

$$a(x,t) = \sum_k \underbrace{\mathbf{M}:\varepsilon_k(x_s)}_{\text{amplitude of oscillation}}\ \underbrace{s_k(x)}_{\text{shape of oscillation}}\ \underbrace{\cos\omega_k(t-t_s)}_{\text{begins oscillating at } t=t_s}\ ,\quad t \geq t_f$$

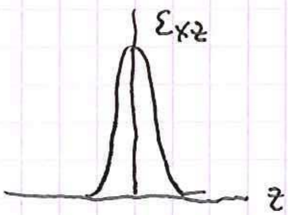
Earthquake fault source   D&T Sect 5.2

Whence the terminology stress "glut"?
Consider a narrow fault zone such as the San Andreas filled with gouge. The x component of displacement s_x looks like this:

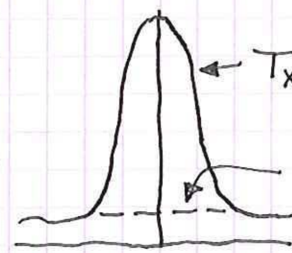




The strain $\varepsilon_{xz} = \frac{1}{2}(\partial_z s_x)$ looks like this:



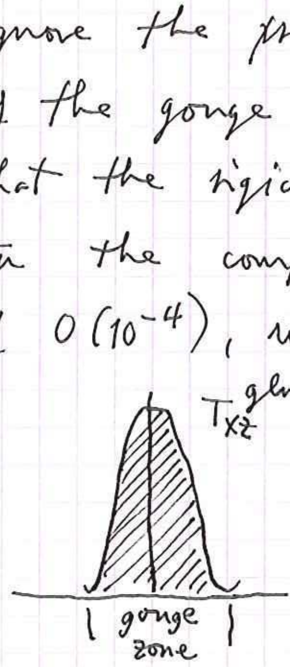
The Hooke stress and true stress look like this:



$T_{xz}^{Hooke} = \mu(\partial_z s_x) + T_{xy}^{initial}$

$T_{xz}^{true} \approx T_{xz}^{initial}$, by continuity

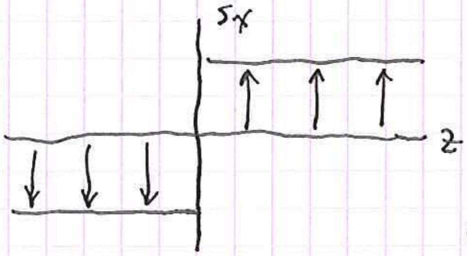
To calculate T_{xz}^{Hooke} we ignore the presence of the gouge and assume that the rigidity $\mu = \mu_{const} \approx$ constant.
In the competent rock ε_{xz} is very small, of $O(10^{-4})$, whereas in the gouge it is large, $\gg 1$.



$T_{xz}^{glut} = T_{xz}^{Hooke} - T_{xz}^{true}$

There is an excess or glut of Hooke or model stress in the ~~[illegible]~~ gouge zone, where Hooke's laws fails.

In the limit of a ~~very~~ very narrow fault zone:



$$T^{glut}_{xz} = T^{glut}_{zx} = \mu \Delta s \, \delta(z - z_{fault})$$

where $\Delta s = [s_x]^+_-$ is the total slip on the fault and $\delta(z - z_{fault})$ is a Dirac delta function.

Distribution theory:

To generalize the above simplified analysis it is useful to review some elementary notions from the theory of distributions or generalized functions (L. Schwartz).

A distribution is a continuous linear functional on a space of test functions ϕ. The test functions are assumed to be smooth in Φ and vanishing on $\partial\Phi$. We denote the scalar that the distribution f assigns to the test function ϕ by:

$$\langle f, \phi \rangle \rightarrow \text{real scalar}$$

continuous in this slot

Every ordinary function in Φ can be regarded as a distribution provided we define

$$\langle f, \phi \rangle = \int_{\Phi} f \phi \, dV$$

When it is important to distinguish functions and distributions we write the distribution associated with a function f in the form $\mathcal{D}f$.

By analogy we also frequently write

$$\langle f, \phi \rangle = \int_\Phi f\phi \, dV$$

for a more general distribution, the "integration" being purely symbolic. All distributions of the form $\mathcal{D}f$ are said to be <u>regular</u>; all others are <u>singular</u>.

If f is an ordinary differentiable function (regular distribution) we can write

$$\int_\Phi (\nabla f)\phi \, dV = -\int_\Phi f \nabla\phi \, dV \qquad \text{(since the } \phi\text{'s vanish on } \partial\Phi\text{)}$$

More generally we <u>define</u> the gradient of a singular distribution f by:

$$\langle \nabla f, \phi \rangle \equiv -\langle f, \nabla\phi \rangle$$

If the test functions ϕ are smooth enough then every distribution can be differentiated any number of times in this sense.

The most familiar example of a singular distribution is the <u>Dirac</u> <u>delta</u> <u>distribution</u> (or "function") defined by

$$\langle \delta_\xi, \phi \rangle = \int_\Phi \delta_\xi \phi \, dV = \phi(\xi)$$ ← no ordinary function has this property — just selects $\phi(\xi)$ at a point ξ.

in book $\langle \delta_0, \phi \rangle = \phi(x_0)$

A more common notation for this:

$$\int_\Phi \delta(x-\xi)\,\phi(x)\,d^3x = \phi(\xi)$$

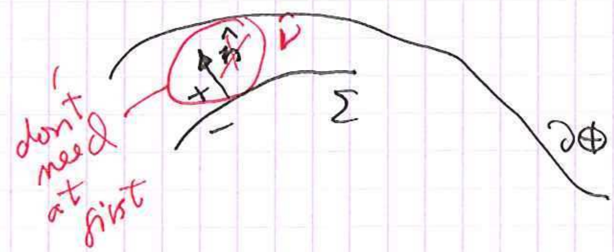
The gradient $\nabla\delta_\xi$ is defined in accordance with the general definition by

$$\langle \nabla\delta_\xi, \phi \rangle = \int_\Phi \nabla_x \delta(x-\xi)\,\phi(x)\,d^3x$$

$$= -\int_\Phi \delta(x-\xi)\,\nabla_x\phi(x)\,d^3x = -\langle \delta_\xi, \nabla\phi \rangle.$$

$$= -\nabla\phi(\xi)$$

A useful singular distribution in the present context is defined as follows:
Let Σ be a smooth 2-d surface in Φ and let w a smooth ordinary function on Σ.



Define the distribution $w\delta_\Sigma$ by

$$\langle w\delta_\Sigma, \phi \rangle = \int_\Sigma w\phi \, d\Sigma$$

A more suggestive notation:

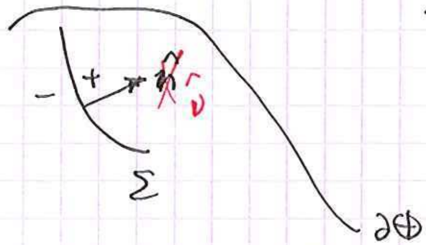
$$w\delta_\Sigma = \int_\Sigma w(\xi)\,\delta(x-\xi)\,d^2\xi$$

Then:

$$\langle w\delta_\Sigma, \phi \rangle = \int_\Phi \int_\Sigma w(\xi)\,\delta(x-\xi)\,\phi(x)\,d^2\xi\,d^3x = \int_\Sigma w(\xi)\phi(\xi)\,d^2\xi$$

We can regard $w\delta_\Sigma$ as a weighted "distribution" of Dirac deltas on Σ just as $\sum_k w_k \delta(x-\xi_k)$ is a weighted discrete "distribution".
The "value" of $w\delta_\Sigma$ (this concept can be made rigorous) is zero everywhere except on Σ just as the "value" of $\sum_k w_k \delta(x-\xi_k)$ is zero everywhere except at the points ξ_k.

We now make a calculation — suppose that f is an ordinary function that is smooth everywhere in Φ except on a surface Σ where it exhibits a jump discontinuity $[f]^{\pm}$



What is ∇f? It does not exist as an ordinary function everywhere in Φ, notably on Σ. But we can calculate $\nabla(\mathcal{D}f)$.

$$\langle \nabla(\mathcal{D}f), \phi \rangle \equiv -\langle \mathcal{D}f, \nabla\phi \rangle$$

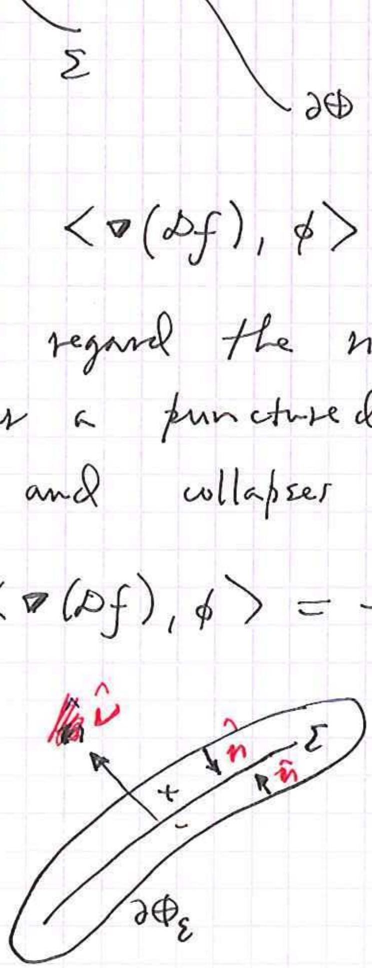
We regard the right side as an integral over a punctured volume that envelops Σ and collapses onto it in the limit $\varepsilon \to 0$

$$\langle \nabla(\mathcal{D}f), \phi \rangle = -\lim_{\varepsilon\to 0} \int_{\Phi-\Phi_\varepsilon} f \,\nabla\phi \, dV$$

now an ordinary smooth function in this domain

why do we do this — because we want to be able to integrate by parts

better to call this $\hat{\nu}$



We can apply Gauss' theorem <u>before</u> taking the limit:

$$-\int_{\oplus-\oplus_\varepsilon} f\,\nabla\phi\, dV = -\int_{\partial\oplus_\varepsilon} \hat{n} f\phi\, d\Sigma + \int_{\oplus-\oplus_\varepsilon} \nabla f\,\phi\, dV$$

unit <u>inward</u> normal to $\partial\oplus_\varepsilon$

Now take limit $\varepsilon \to 0$

$$= \int_\Sigma \hat{n}\,[f]^\pm\,\phi\, d\Sigma + \int_\oplus \nabla f\,\phi\, dV$$

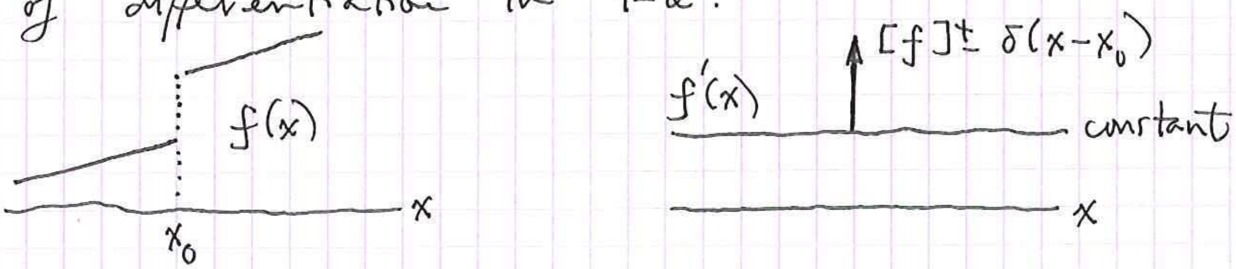
In conclusion:

$$\langle \nabla(\mathcal{D}f), \phi\rangle = \langle \mathcal{D}(\nabla f) + \hat{n}[f]^\pm \delta_\Sigma, \phi\rangle$$

Since these two distributions assign the same scalar to every test function ϕ, they must be equal:

$$\nabla(\mathcal{D}f) = \mathcal{D}(\nabla f) + \hat{n}[f]^\pm \delta_\Sigma$$

Loosely speaking, $\nabla(\mathcal{D}f)$ consists of the "ordinary" gradient ∇f, which is well defined everywhere except on Σ, plus a delta function contribution on Σ arising from the discontinuity $[f]^\pm$. Generalization of differentiation in 1-d.



<u>Ideal fault</u>: a surface Σ in $\oplus$ across which there is a <u>tangential</u> slip discontinuity $\Delta s = [s]^\pm$ satisfying $\hat{n}\cdot\Delta s = 0$ (no opening up or interpenetration).

The Hooke stress $\mathbb{T}_{Hooke} = \mathbb{C} : \varepsilon = \mathbb{C} : \nabla s$ does not exist everywhere, notably on Σ, as an ordinary function so we consider the associated distribution:

$$\mathbb{T}_{Hooke} = \mathbb{C} : \nabla(\mathcal{D}s)$$

$$T_{ij}^{Hooke} = C_{ijkl}\, \partial_k (\mathcal{D} s_l)$$ — a singular distribution

The true physical stress is, on the other hand, well defined everywhere. The associated distribution is regular:

discontinuous but non-singular

$$\mathcal{D}(\mathbb{T}_{true}) = \mathbb{C} : \mathcal{D}(\nabla s)$$

$$\mathcal{D}(T_{ij}^{true}) = C_{ijkl}\, \mathcal{D}(\partial_k s_l)$$

The stress glut is defined as a singular distribution by

$$\mathbb{T}_{glut} = \mathbb{T}_{Hooke} - \mathcal{D}(\mathbb{T}_{true})$$

$$= \mathbb{C} : \left[\nabla(\mathcal{D}s) - \mathcal{D}(\nabla s) \right]$$

$$= \mathbb{C} : \overset{\nu}{\cancel{\hat{n}}}\, \Delta s\, \delta_\Sigma$$

long name — wrong units

Define the ~~stress glut density~~ on Σ:

$$\mathbf{m} = \mathbb{C} : \overset{\nu}{\cancel{\hat{n}}} \Delta s$$

$$m_{ij} = C_{ijkl}\, \overset{\nu_k}{\cancel{n_k}}\, \Delta s_l$$

units of $\mathbf{m}$: $\frac{\text{force} \times \text{distance}}{\text{area}} = \frac{m\ell t^{-2} \times \ell}{\ell^2} = \frac{m}{t^2}$

Then $\mathbb{T}_{glut} = \mathbf{m}\, \delta_\Sigma$ — confined to fault surface as expected.

end here Thurs Mar 2

Quantity $\mathbf{m}$ also called <u>moment tensor density</u>.

The product of a discontinuous function times a Dirac delta is not defined so we require that

$$[C]^{+}_{-} = 0$$ — elastic parameters continuous across the fault Σ.

Ampuero & Dahlen BSSA (2005)

The equivalent body and surface force densities for an ideal fault are

$$f = -\mathbf{m} \cdot \nabla \delta_\Sigma$$

$$t = (\hat{n} \cdot \mathbf{m})\, \delta_\Sigma$$

normal to $\partial\oplus$, not Σ

Note that this is completely general, for dynamic, time-dependent faulting in an anisotropic Earth.

skip

The moment tensor of an ideal fault is

$$\mathbf{M} = \int_S \mathbf{T}_{\text{glut, stat}}\, dV = \int_\oplus \mathbf{T}_{\text{glut, stat}}\, dV$$

$$= \int_\oplus \mathbf{m}\, \delta_\Sigma\, dV$$

moment tensor / unit area

$$\mathbf{M} = \int_\Sigma \mathbf{m}\, d\Sigma$$

If the Earth is isotropic:

$$C_{ijkl} = \underbrace{\left(\kappa - \tfrac{2}{3}\mu\right) \delta_{ij}\, \delta_{kl}}_{\text{no contribution since } n_k \Delta s_k = 0} + \mu(\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk})$$

$$\mathbf{m} = \mu \Delta s\, (\underset{\hat{\nu}}{\hat{n}}\hat{e} + \hat{e}\underset{\hat{\nu}}{\hat{n}}) \quad \text{where} \quad \Delta \mathbf{s} = \Delta s\, \hat{e}$$

slip direction on fault

skip

The moment tensor in this case is

$$\mathbf{M} = \int_\Sigma \mu \Delta s\, (\underset{\hat{\nu}}{\hat{n}}\hat{e} + \hat{e}\underset{\hat{\nu}}{\hat{n}})\, d\Sigma$$

If the fault surface is planar & the slip is uni-directional ($\hat{n}$ & $\hat{e}$ constant)

$$\mathbf{M} = M_0 (\hat{n}\hat{e} + \hat{e}\hat{n})$$

where $M_0 = \int_\Sigma \mu \Delta s \, d\Sigma$, the scalar seismic moment first defined by Aki

slip

Note the fault plane – auxiliary plane ambiguity: cannot distinguish $\hat{n}$ from $\hat{e}$

For the largest events $M_0 \sim 10^{30}$ dyne-cm (10^{23} N-m)

e.g. 1964 Alaskan 'quake

$\mu \sim 3 \times 10^{11}$ dyne/cm^2

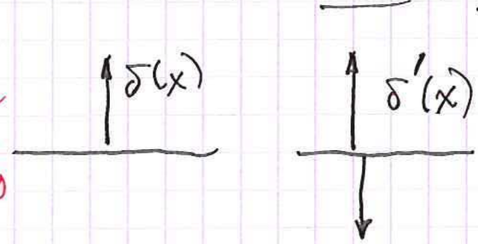
$\overline{\Delta s} = 10$ m

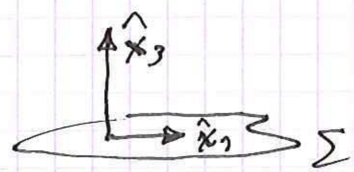
$A = 1000$ km $\times$ 250 km

$M_0 \sim 7.5 \times 10^{29}$ dyne-cm (Kanamori)

The equivalent point-source body force is a classical double couple: $\mathbf{f} = -\mathbf{M} \cdot \nabla \delta(\mathbf{x} - \mathbf{x}_s)$

do this for moment density





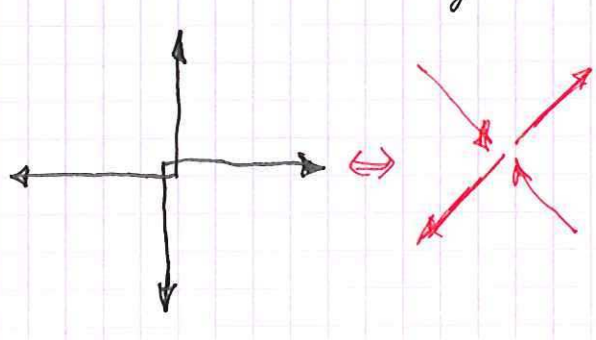
$\hat{n} = \hat{x}_3$

$\hat{e} = \hat{x}_1$

Then $M_{13} = M_{31} = M_0$

All other $M_{ij} = 0$

$$f_1 = -M_0 \delta(x_1)\delta(x_2)\delta'(x_3)$$
$$f_2 = -M_0 \delta'(x_1)\delta(x_2)\delta(x_3)$$



Note – obviously no net force or torque

More generally f is a "distribution" of double couples over the fault surface Σ — this is an exact dynamical result for all frequencies & wavelengths.

<u>Harvard CMT project</u>: solves for $\mathbf{M}$ as well as an updated location $x_s + \Delta x$, $t_s + \Delta t$. In this case the acceleration response is

$$a(x,t) = \sum_k \mathbf{M} : \varepsilon_k(x_s)\, s_k(x) \cos\omega_k(t - t_s)$$

$$+ \sum_k \Delta x\, \mathbf{M} : \nabla\varepsilon_k(x_s)\, s_k(x) \cos\omega_k(t - t_s)$$

$$+ \sum_k \omega_k \Delta t : \varepsilon_k(x_s)\, s_k(x) \sin\omega_k(t - t_s)$$

<u>Linear</u> inverse problem for $\mathbf{M}$, Δx, Δt

Can be shown that the <u>centroid</u> shift ~~[illegible]~~ is given by:

$$\Delta x = \frac{1}{M_0} \int_\Sigma (x - x_s)\, \mu\, \Delta s_{stat}\, d\Sigma$$

$$\Delta t = \frac{1}{M_0} \int_{t_0}^{t_f} \int_\Sigma (t - t_s)\, \mu\, \partial_t \Delta s\, d\Sigma\, dt$$

Centroid of source in space-time: may differ from rupture initiation point.

To compare with near-field geodetic observations and for other reasons often seek best-fitting double couple source. The constraints

trace $\mathbf{M} = 0$

det $\mathbf{M}$

guarantee this.

The first is linear & easily imposed – the second is not. Customary to impose trace $\mathbf{M} = 0$ and find the best-fitting deviatoric tensor $\mathbf{M}'$ by fitting seismograms. Then find best-fitting $\mathbf{M}_{dc}$ by

$$(\mathbf{M}_{dc} - \mathbf{M}') : (\mathbf{M}_{dc} - \mathbf{M}') = \text{minimum}$$

Solution found by diagonalization of $\mathbf{M}'$:

$$\mathbf{M}' = \begin{pmatrix} M_{maj} & & \\ & -M_{maj} - M_{min} & \\ & & M_{min} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} M_{maj} & & \\ & -M_{maj} & \\ & & 0 \end{pmatrix}}_{\substack{\text{major double} \\ \text{couple} \\ \text{(best-fitting)}}} + \underbrace{\begin{pmatrix} 0 & & \\ & -M_{min} & \\ & & M_{min} \end{pmatrix}}_{\substack{\text{minor double} \\ \text{couple}}}$$

where $|M_{maj}| \geq |M_{min}|$

A measure of the amount by which $\mathbf{M}'$ deviates from a double couple is given by

$$\varepsilon = \frac{M_{max} + M_{min}}{\max(|M_{max}|, |M_{min}|)}$$

Then $\varepsilon = 0$ corresponds to a double couple and $-\frac{1}{2} \leq \varepsilon \leq \frac{1}{2}$ in general.

In the Harvard catalogue, 4% of the mechanisms are significantly non-double-couple $|\varepsilon| \geq \frac{1}{3}$. The most likely cause is <u>curvature</u> of the <u>fault plane</u>. Ekstrom (1994) shows that several can be associated with <u>volcanic</u> <u>ring faults</u>.