Local energy conservation law

$$\partial_t s \cdot [\rho \partial_t^2 s = \nabla \cdot T]$$

$$\partial_t \left(\tfrac{1}{2} \rho s \cdot s\right) - \underbrace{\partial_t s_j \cdot \partial_i \left(c_{ijkl} \partial_k s_l\right)}_{*} = 0$$

$$* = - \partial_i \left(c_{ijkl} \partial_k s_l \, \partial_t s_j\right) + c_{ijkl} \partial_k s_l \, \partial_t (\partial_i s_j)$$

$$= - \partial_i \left(T_{ij} \partial_t s_j\right) + \partial_t \left(\tfrac{1}{2} c_{ijkl} \partial_i s_j \partial_k s_l\right)$$

Get $\quad \partial_t E + \nabla \cdot K = 0$

$$E = \tfrac{1}{2} \rho (\partial_t s)^2 + \tfrac{1}{2} \varepsilon : c : \varepsilon$$

$$K = - \partial_t s \cdot T$$

Generic conservation law for ϕ-stuff

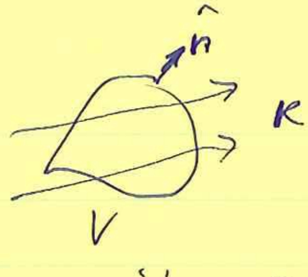
$$\partial_t \phi + \nabla \cdot K = k$$

ϕ, k order q

K order $q+1$

integrate over fixed volume V:

$$\underbrace{\frac{d}{dt} \int_V \phi \, dV}_{\text{rate of change of } \phi\text{-stuff}} = \underbrace{- \int_{\partial V} \hat{n} \cdot K \, dA}_{\text{flux}} + \underbrace{\int_V k \, dV}_{\text{creation of } \phi\text{-stuff}}$$



ϕ: density of ϕ-stuff
K: flux of ϕ-stuff (per unit area per sec)
k: creation rate of ϕ-stuff.

Momentum equation: $\partial_t(\rho s) + \nabla\cdot T = 0$

$\phi = \rho s$ momentum density (per unit vol.)

T = momentum flux

T_{ij} = flux of j component of momentum in i direction

Energy: ϕ-stuff = energy

$$\phi = E = \underbrace{\tfrac{1}{2}\rho(\partial_t s)^2}_{\text{ke density}} + \underbrace{\tfrac{1}{2}\,\varepsilon : \Gamma : \varepsilon}_{\text{pe density}}$$

Energy flux in a linear elastic medium

$$K = -\partial_t s \cdot T$$ (minus velocity dot stress)

Example: plane P wave

$$s_x = s_y = 0 \qquad s_z = A\sin(kz-\omega t)$$

$$T_{ij} = (\kappa - \tfrac{2}{3}\mu)\,\nabla\cdot s\,\delta_{ij} + 2\mu(\partial_i s_j + \partial_j s_i)$$

$T_{xx} = T_{yy} = (\kappa - \tfrac{2}{3}\mu)\partial_z s_z$, $T_{zz} = (\kappa + \tfrac{4}{3}\mu)\partial_z s_z$, all rest zero

$$K_z = -\omega A[-\cos(kz-\omega t)]\; k(\kappa + \tfrac{4}{3}\mu) A\cos(kz-\omega t)$$
$$= \omega k(\kappa+\tfrac{4}{3}\mu)A^2\cos^2(kz-\omega t) = \rho\omega^2\alpha A^2\cos^2(kz-\omega t)$$
$$\langle K_z\rangle = \tfrac{1}{2}\rho\omega^2\alpha A^2$$