Class # 10

Eulerian vs Lagrangian

planets $\mathbf{r}_i(t)$, $i = 1, 2, \dots, N$

$\mathbf{r}(\mathbf{x}, t)$, $\mathbf{r}(\mathbf{x}, 0) = \mathbf{x}$ initial posn = particle label

$$\mathbf{u}^L(\mathbf{x}, t) = \partial_t \mathbf{r}(\mathbf{x}, t)$$

Eulerian = weather bureau

$\mathbf{r}$ = fixed in space

$\mathbf{u}^E(\mathbf{r}, t)$

Equivalent, complete descriptions

$$\mathbf{u}^E(\mathbf{r}(\mathbf{x}, t), t) = \mathbf{u}^L(\mathbf{x}, t)$$

Any physical variable, e.g. ϕ = temp, pressure, stress

$$\phi^E(\mathbf{r}(\mathbf{x}, t), t) = \phi^L(\mathbf{x}, t)$$

Chain rule: $\partial_t \phi^L = \partial_t \phi^E + \mathbf{u}^E \cdot \nabla_r \phi^E$
$= D_t \phi^E$

$D_t = \partial_t + \mathbf{u}^E \cdot \nabla_r$ material derivative

Take it for granted that you know the exact Eulerian conservation laws:

$$\partial_t \rho^E + \nabla\cdot(\rho^E \mathbf{u}^E) = 0 \qquad \text{continuity}$$

$$D_t \rho^E + \rho^E \nabla\cdot \mathbf{u}^E = 0$$

gravity

$$\rho^E D_t \mathbf{u}^E + \nabla\cdot \mathbf{T}^E = \rho^E \mathbf{g}^E \qquad \text{momentum}$$

Cauchy (symmetric)

SNREI Earth

$a = 6371$ km

undeformed state $\rho^0(r)$

grav. pot $\phi^0(r)$

$$\mathbf{g}^0 = -\nabla\phi^0$$

Poisson's eqn

$$\nabla^2\phi^0 = 4\pi\rho^0 \qquad G = 6.67\times10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$$

$$\mathbf{g}^0(\mathbf{r}) = -g^0(r)\,\hat{\mathbf{r}} \quad , \quad g^0(r) = \partial_r\phi^0$$

$$\partial_r g^0 + 2r^{-1}g^0 = 4\pi G\rho^0$$

$$\frac{1}{r^2}\partial_r(r^2 g^0) = 4\pi G\rho^0$$

$$g^0(r) = 4\pi G r^{-2}\int_0^r \rho^0(r)\,r^2\,dr$$
$$= G M_{\text{encl}}(r)/r^2$$

Static initial stress <u>hydrostatic</u>

$$\mathbf{T}^0(\mathbf{r}) = -p^0(r)\,\mathbf{I}$$

Mech. equil. $\rho^0 \nabla \phi^0 + \nabla \cdot \mathbf{T}^0 = 0$

$$\rho^0 \nabla \phi^0 + \nabla p^0 = 0$$ asphericity $\Rightarrow$ non-hydrostatic

Take curl: $\nabla \rho^0 \times \nabla \phi^0 = 0$

mult. by $\nabla p^0 \times$ $\nabla p^0 \times \nabla \phi^0 = 0$ level surfaces all coincide (spheres)

$$\partial_r p^0 + \rho^0 g^0 = 0$$

$$p^0(r) = \int_r^a \rho^0(r) g^0(r)\, dr$$ vanishes at $r = a$

Fig. 8.2 DfT: $p^0_{\oplus}$ center $= 364$ GPa

Now consider small oscillations of this model

$$\mathbf{r}(\mathbf{x}, t) = \mathbf{x} + \mathbf{s}(\mathbf{x}, t)$$

$\uparrow$ displacement

density $\rho^0(r) + \rho^{E1}(\mathbf{r}, t)$

grav. pot. $\phi^0(r) + \phi^{E1}(\mathbf{r}, t)$

stress $-p^0(r)\mathbf{I} + \mathbf{T}^{E1}(\mathbf{r}, t)$

Continuity eqn $\partial_t(\rho^0 + \rho^{E1}) + \nabla \cdot [(\rho^0 + \rho^{E1})\mathbf{u}^E] = 0$

$$\partial_t \rho^{E1} + \nabla \cdot (\rho^0 \mathbf{u}^E) + \cancel{\nabla \cdot (\rho^{E1} \mathbf{u}^E)} = 0$$

ignore

$$\mathbf{u}^E = D_t \mathbf{r} = D_t \mathbf{s} \approx \partial_t \mathbf{s}$$

integrate $\partial_t [\rho^{E1} + \nabla \cdot (\rho^0 \mathbf{s})] = 0$

$$\boxed{\rho^{E1} = -\nabla\cdot(\rho^0 \mathbf{s})} \leftarrow \text{1st order relation}$$

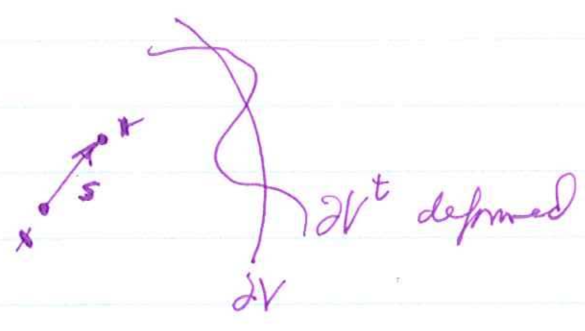
$$\rho^L(\mathbf{x},t) = \rho^E(\mathbf{r}(\mathbf{x},t),t)$$

$$\cancel{\rho^0} + \rho^{L1} = \cancel{\rho^0} + \mathbf{s}\cdot\nabla\rho^0 + \cdots + \rho^{E1} + \cdots$$

$$\boxed{\rho^{L1} = -\rho^0\,\nabla\cdot\mathbf{s}}$$

$$\rho^{L1} = \rho^{E1} + \underbrace{\mathbf{s}\cdot\nabla\rho^0}_{\text{advection term}}$$

Strictly speaking these eqns hold at $\mathbf{r}$ in deformed volume. But we regard as true at $\mathbf{x}$ in undeformed spherical volume. Then need to consider b.c. at boundaries



Change in grav pot due to redistr. of ⊕ mass

$$\nabla^2(\phi^0 + \phi^{E1}) = 4\pi G(\rho^0 + \rho^{E1})$$ exact at $\mathbf{r}$ in space

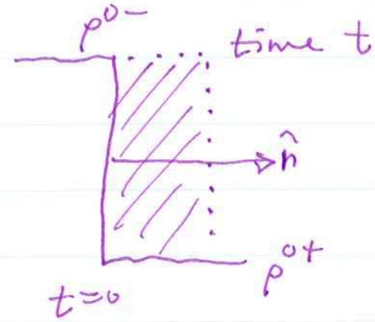
$$\boxed{\nabla^2\phi^{E1} = 4\pi G\rho^{E1}} \leftarrow$$ also interpret as valid at $\mathbf{x} \in$ undeformed

subject to b.c.

$[\phi^{E1}]_-^+$ on Σ

$$[\hat{n}\cdot\nabla\phi^{E1}]_-^+ = -4\pi G[\rho^0]_-^+(\hat{n}\cdot s)$$

relative to $t=0$ $\exists$ a surface mass layer



surface mass density $= \hat{n}\cdot s\,[\rho^{0-}-\rho^{0+}]$
>0 in this picture
(outward displ. of CMB)

$$\phi^{E1}(x,t) = -G\int_\oplus \frac{\rho^{E1}(x',t)}{\|x-x'\|}\,d^3x' + G\int_\Sigma \frac{\overset{\text{at } x'}{[\rho^0]_-^+(\hat{n}\cdot s)}}{\|x-x'\|}\,d^2x'$$

$$= -G\int_\oplus \frac{\rho^0(x')\,s(x',t)\cdot(x-x')}{\|x-x'\|^3}\,d^3x'$$

Pert. in gravity $g^{E1} = -\nabla_x\phi^{E1}$

$$g^{E1}(x,t) = G\int_\oplus \rho^0(x')\,s(x',t)\cdot\Pi(x-x')\,d^3x'$$

$$\Pi = \frac{I}{\|x-x'\|^3} - \frac{3(x-x')(x-x')}{\|x-x'\|^5}$$

Deformation <u>everywhere</u> affects ϕ^{E1} and g^{E1} since gravity is a long-range force.

Momentum eqn

$$(\underbrace{\rho^0+\cancel{\rho^{E1}}}_{\text{ignore}})\underbrace{(D_t u^E)}_{\approx \partial_t^2 s} = \nabla\cdot(-p^0 \mathbf{I} + \mathbf{T}^{E1}) - (\rho^0+\rho^{E1})\nabla(\phi^0+\phi^{E1})$$

$$\rho^0 \partial_t^2 s = \underbrace{\cancel{-\nabla p^0} + \nabla\cdot\mathbf{T}^{E1} - \cancel{\rho^0\nabla\phi^0}}_{\text{equil.}} - \rho^0\nabla\phi^{E1} - \rho^{E1}\nabla\phi^0$$

$$\rho^0 \partial_t^2 s = \nabla\cdot\mathbf{T}^{E1} - \rho^0\nabla\phi^{E1} - \rho^{E1}\nabla\phi^0$$

Now we just set $\mathbf{T}^{E1} = \mathbf{C} : \mathbf{m} = \mathbf{C} : \varepsilon$ right.

WRONG!

Elastic constants must pertain to a parcel of matter; it is the Lagrangian perturbation $\mathbf{T}^{L1}$ not $\mathbf{T}^{E1}$ that is related to the strain by Hooke's law

Read quotes from Rayleigh (1906) and Love (1911) from page 6.

$$\mathbf{T}^{L1} = \mathbf{T}^{E1} + s\cdot\nabla\mathbf{T}^0 = \mathbf{T}^{E1} - (s\cdot\nabla p^0)\mathbf{I}$$
$$= \mathbf{T}^{E1} + (\rho^0 s\cdot\nabla\phi^0)\mathbf{I}$$

Summarizing

$$\star \quad \rho^0 \partial_t^2 s = -\rho^0 \nabla\phi^{E1} - \rho^{E1}\nabla\phi^0 - \nabla(\rho^0 s\cdot\nabla\phi^0) + \nabla\cdot T^{L1}$$

$$\rho^{E1} = -\nabla\cdot(\rho^0 s)$$ mixed Eulerian–Lagrangian description

$$\nabla^2\phi^{E1} = 4\pi G\rho^{E1}$$

$$T^{L1} = (\kappa - \tfrac{2}{3}\mu)(\nabla\cdot s)\, I + 2\mu\varepsilon$$

$\star$ is an integro-differential eqn because $-\nabla\phi^{E1} = g^{E1}$ is an integral over whole $\oplus$.

In Ch. 8 we get rid of all subscripts

don't give these forms

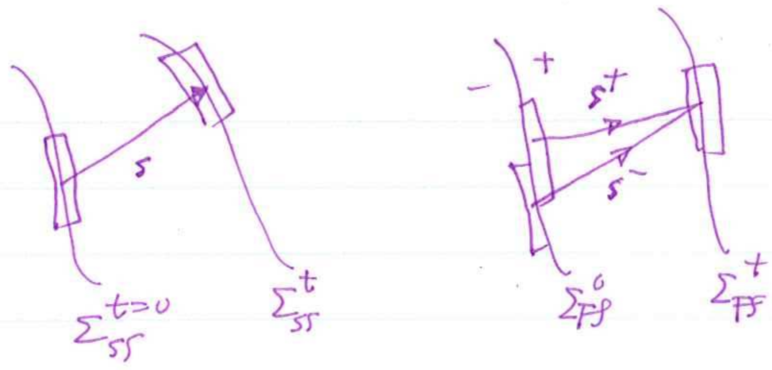
$$\rho\partial_t^2 s = -\rho\nabla\phi + \nabla\cdot(\rho s)\nabla\Phi - \nabla(\rho s\cdot\nabla\Phi) + \nabla\cdot T$$
$$\nabla^2\phi = -4\pi G\,\nabla\cdot(\rho s)$$
$$T = (\kappa - \tfrac{2}{3}\mu)(\nabla\cdot s)\, I + 2\mu\varepsilon$$

In a SNREI the momentum eqn is

$$\rho\partial_t^2 s = -\rho\nabla\phi - (4\pi G\rho^2 s_r)\hat{r} - \rho g\left[\nabla s_r - (\nabla\cdot s + \tfrac{2}{r}s_r)\hat{r}\right] + \nabla\cdot T = 0$$

Must be supplemented by b.c. on undeformed boundaries.

end class 10 here after 1.5 hours



Upshot: kinematic

$$[s]^+_- = 0 \quad \text{on} \quad \Sigma_{SS}$$

$$[\hat{n}\cdot s]^+_- = 0 \quad \text{on} \quad \Sigma_{FS}$$

grav.

$$[\phi^{E1}]^+_- = 0$$

$$[\hat{n}\cdot\nabla\phi^{E1} + 4\pi G\rho^0\hat{n}\cdot s]^+_- = 0 \quad \text{on } \Sigma$$

dynamic

$$[\hat{n}\cdot T^{L1}]^+_- = 0 \quad \text{on} \quad \Sigma_{SS}$$

$$[\hat{n}\cdot T^{L1}]^+_- = \hat{n}[\hat{n}\cdot T^{L1}\cdot\hat{n}]^+_- = 0 \quad \text{on } \Sigma_{FS}$$

$$\hat{n}\cdot T^{L1} = 0 \quad \text{on} \quad \partial V$$

Tables 3.3 and 3.4 pp. 103–104 D&T

Conservation of energy D&T Sect. 3.11.4

Take momentum eqn. Multiply by $\partial_t s$ and integrate over $\oplus$. Integrate by parts and apply the b.c.

No... wait... first talk about displacement versus displacement – potential points of view

Displacement : regard ⋆ as integro - differential

$-\nabla\phi^{E1} = g^{E1}$ given explicitly by

$$g^{E1} = G \int_{\oplus} \rho^0(x)\, s(x,t) \cdot \Pi(x - x')\, d^3x'$$

Solve only for s

Displacement - potential : treat ϕ^{E1} as an additional unknown. Then need and additional eqn $\nabla^2\phi^{E1} = 4\pi G\rho^{E1}$ plus associated b.c.

We very systematically do both — I will limit to displacement point of view in class

Now conservation of energy : DfT 3.11.4

$$\frac{d}{dt}\frac{1}{2}\int_{\oplus} [\underbrace{\rho^0\|\partial_t s\|^2}_{\text{kinetic}} + \underbrace{\varepsilon : C : \varepsilon}_{\text{elastic}} + \underbrace{\rho^0 s \cdot \nabla\phi^{E1}}_{\text{grav}} + \underbrace{\rho^0 s \cdot \nabla\nabla\phi^0 \cdot s + \rho^0 \nabla\phi^0 \cdot (s \cdot \nabla s - s\, \nabla \cdot s)}_{\text{grav}}]\, dV = 0$$

This the conventional interpretation though not strictly correct. Actual elastic energy given by DfT 3.288 where J given by DfT 3.263. Actual gravitational energy (assemble from dispersal at ∞ into two states) given by DfT 3.223

Both elastic & grav. energy have a first order as well as 2nd order terms.

Look for normal mode solutions

$$\underline{s}(\underline{x},t) = \underline{s}(\underline{x})\, e^{i\omega t}$$

$$\mathcal{H}\underline{s} = \omega^2 \underline{s}$$

$$\rho^0 \mathcal{H}\underline{s} = -\nabla\cdot\underline{\underline{T}}^{L1} + \nabla(\rho^0 \underline{s}\cdot\nabla\phi^0) + \rho^0\nabla\phi^{E1} + \rho^{E1}\nabla\phi^0$$

"together with the b.c.

Can show that $\mathcal{H}$ is Hermitian

$$\langle \underline{s}, \mathcal{H}\underline{s}' \rangle = \langle \mathcal{H}\underline{s}, \underline{s}' \rangle$$

Need to use that Σ is a level surface of ρ^0, ϕ^0, p^0.

Rayleigh's principle

$$\omega^2 = \frac{\mathcal{V}_e + \mathcal{V}_g}{\mathcal{T}} \quad \text{stationary}$$

$\mathcal{V}_e$, $\mathcal{V}_g$ given by D&T (4.168) – (4.169)

fractional elastic + grav. energies of a mode.

Cowling approximation