Small oscillations : DT Sect 7.1 , Goldstein 2nd ed., ch. 6

coordinates $q_1, \dots, q_N$
velocities $\dot{q}_1, \dots, \dot{q}_N$

kinetic energy quadratic in velocities

$$T = \frac{1}{2} T_{ij}(q_1, \dots, q_N)\, \dot{q}_i \dot{q}_j \quad , \quad T_{ij} = T_{ji}$$

also assume $T > 0$ if at least one $\dot{q}_i \neq 0$

potential energy $V = V(q_1, \dots, q_N)$
conservative; see p. 8.

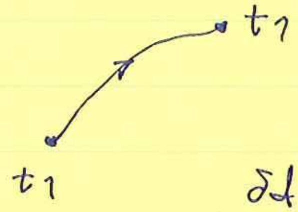
Lagrangian $L = T - V$

$L(q_1, \dots q_N ; \dot{q}_1, \dots \dot{q}_N)$

Hamilton's principle

$L(\mathbf{q}, \dot{\mathbf{q}})$
vectors $\mathbf{q}, \dot{\mathbf{q}}$

$$\text{action} \quad I = \int_{t_1}^{t_2} L \, dt$$



fixed endpoints $q_i(t_1)$, $q_i(t_2)$

$$\delta I = \delta \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}})\, dt = 0$$

$$\delta I = \int_{t_1}^{t_2} \Big[\frac{\partial L}{\partial \mathbf{q}} \cdot \delta \mathbf{q} + \underbrace{\frac{\partial L}{\partial \dot{\mathbf{q}}}}_{u} \cdot \underbrace{\delta \dot{\mathbf{q}}}_{dv} \Big] dt$$

$$= \cancel{\left[\frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \delta \mathbf{q} \right]_{t_1}^{t_2}} + \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) \right] \cdot \delta \mathbf{q} \, dt$$

zero, since endpoints fixed

$$\boxed{\text{Go to page 8; conservation of energy here}}$$

Euler–Lagrange : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = 0$

Second-order non linear

Now suppose $\mathbf{q} = \mathbf{0}$ is an equil. configuration

$$\frac{\partial V}{\partial q_i} = 0 \quad \text{at} \quad q_i = 0$$

Small oscillations

$$T = \frac{1}{2} T_{ij}(\mathbf{q}) \, \dot{q}_i \dot{q}_j$$

$$T_{ij}(\mathbf{q}) = T_{ij}(\mathbf{0}) + \cancel{\frac{\partial T_{ij}}{\partial q_k}(\mathbf{0}) \, q_k} + \cdots$$

$$= T_{ij}$$

$$V(\mathbf{q}) = V(\mathbf{0}) + \underbrace{\cancel{\frac{\partial V}{\partial q_i}}}_{\text{zero}}(\mathbf{0}) \, q_i + \frac{1}{2} \frac{\partial^2 V}{\partial q_i \partial q_j}(\mathbf{0}) \, q_i q_j$$

Define matrices $T = \left(T_{ij} \right)$, $V = \left(V_{ij} \right)$

both real symmetric $\qquad = \left(\left. \frac{\partial^2 V}{\partial q_i \partial q_j} \right|_{\mathbf{0}} \right)$

$$V = V_0 + \tfrac{1}{2}\, \underline{q}^T \mathbb{V} \underline{q}$$

$$T = \tfrac{1}{2}\, \dot{\underline{q}}^T \mathbb{T} \dot{\underline{q}}$$

$$L = T - V = \tfrac{1}{2}\left(\dot{\underline{q}}^T \mathbb{T} \dot{\underline{q}} - \underline{q}^T \mathbb{V} \underline{q} \right)$$

Euler–Lagrange $\quad \mathbb{T}\ddot{\underline{q}} + \mathbb{V}\underline{q} = 0$

$\mathbb{T}, \mathbb{V}$ real symmetric
$\mathbb{T}$ positive definite : $\dot{\underline{q}}^T \mathbb{T} \dot{\underline{q}} > 0$
for all $\dot{\underline{q}} \neq 0$

Thus $\mathbb{T}^{-1}$ exists

comment on FT sign convention

Normal mode solutions $\quad \underline{q}(t) = \underline{q}\, e^{i\omega t}$

$\mathbb{V}\underline{q} = \omega^2 \mathbb{T} \underline{q}$, generalized eigenvalue problem

$$\left(\mathbb{T}^{-1}\mathbb{V}\right)\underline{q} = \omega^2 \underline{q}$$

$\mathbb{H} = \mathbb{T}^{-1}\mathbb{V} \qquad \mathbb{H}\underline{q} = \omega^2 \underline{q}$, ordinary

<u>But</u> $\mathbb{H}$ is <u>not</u> symmetric : $\mathbb{H}^T = \mathbb{V}\mathbb{T}^{-1} \neq \mathbb{H}$.

Nice theorems don't apply, e.g. ω^2 may be complex. But there's an easy fix

Introduce new (kinetic energy) inner product

$$\langle \underline{q}, \underline{q}' \rangle = \underline{q}^T \mathbb{T} \underline{q} = \overline{}\, (\, \mathbb{T}\,)(\,)$$

Then H is Hermitian w.r.t. $\langle , \rangle$

$$\langle \underline{q}, H\underline{q}' \rangle = \underline{q}^T T (T^{-1} V \underline{q}')$$

$$= \underline{q}^T V \underline{q}' = \underline{q}'^T V \underline{q}$$

$$= \underline{q}'^T T (T^{-1} V \underline{q}) = \langle \underline{q}', H\underline{q} \rangle$$

Eigenvalues <u>are</u> real

$$H\underline{q} = \omega^2 \underline{q} \quad , \quad H\underline{q}' = \omega'^2 \underline{q}'$$

$$\langle \underline{q}, H\underline{q}' \rangle = \langle \underline{q}, \omega'^2 \underline{q}' \rangle = \omega'^2 \langle \underline{q}, \underline{q}' \rangle$$

$$\langle \underline{q}', H\underline{q} \rangle = \langle \underline{q}', \omega^2 \underline{q} \rangle = \omega^2 \langle \underline{q}', \underline{q} \rangle = \omega^2 \langle \underline{q}, \underline{q}' \rangle$$

$$(\omega^2 - \omega'^2) \langle \underline{q}, \underline{q}' \rangle = 0$$

$\langle \underline{q}, \underline{q}' \rangle = 0$ if $\omega^2 \neq \omega'^2$ orthogonality

May be repeated roots; can choose $\underline{q}$ orthogonal in degenerate eigenspace

normalization $\langle \underline{q}, \underline{q} \rangle = 1$ or $\underline{q}^T T \underline{q} = 1$

eigenvector matrix

$$Q = \begin{pmatrix} \vdots & & \vdots \\ \underline{q}_1 & & \underline{q}_N \\ \vdots & & \vdots \end{pmatrix}$$

$$Q^T T Q = I$$

$$\begin{pmatrix} \text{---} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} & & \\ & T & \\ & & \end{pmatrix}\begin{pmatrix} | & & \\ | & & \\ & & \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$VQ = TQ\Omega^2 , \qquad \Omega^2 = \begin{pmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_N^2 \end{pmatrix}$$

why on right?

$$Q^T V Q = \Omega^2 , \qquad Q^T T Q = I$$

simultaneous diagonalization by congruent transformation Q

$$VQ = TQ\Omega^2$$
$$(T^{-1}V)Q = Q\Omega^2$$

similarity transform

$$Q^{-1}(T^{-1}V)Q = Q^{-1}HQ = \Omega^2$$

Note that $Q^{-1} \neq Q^T$, Q <u>not</u> an orthogonal transformation

Initial value problem:
find $\mathbf{q}(t)$ given $\mathbf{q}(0)$ and $\dot{\mathbf{q}}(0)$

normalized eigenvectors $e_1, \dots e_N$
normal modes

general solution a sum of $\mathbf{e}_i e^{\pm i\omega_i t}$

$$\underline{q}(t) = \sum_{n=1}^{N} \left[A_n \cos\omega_n t + \omega_n^{-1} B_n \sin\omega_n t \right] e_n$$

$$\underline{q}_0 = \underline{q}(0) = \sum_{n=1}^{N} A_n e_n$$

$$\dot{\underline{q}}_0 = \dot{\underline{q}}(0) = \sum_{n=1}^{N} B_n e_n$$

$$A_n = \langle e_n, \underline{q}_0 \rangle \quad , \quad B_n = \langle e_n, \dot{\underline{q}}_0 \rangle \omega_n^{-1}$$

$$\underline{q}(t) = \sum_{n=1}^{N} e_n \left[\langle e_n, \underline{q}_0 \rangle \cos\omega_n t + \omega_n^{-1} \langle e_n, \dot{\underline{q}}_0 \rangle \sin\omega_n t \right]$$

$$= \sum_{n=1}^{N} C_n e_n \cos(\omega_n t + \phi_n)$$

$$C_n \cos\phi_n = \langle e_n, \underline{q}_0 \rangle$$

$$C_n \sin\phi_n = -\omega_n^{-1} \langle e_n, \dot{\underline{q}}_0 \rangle$$

Shapes of normal mode oscillations are completely determined; amplitudes and phases are determined by the i.c.

Green matrix

$$T\ddot{G} + VG = I\,\delta(t)$$

or $$T\ddot{G} + VG = 0 \qquad G(0) = 0, \; \dot{G}(0) = T^{-1}$$

$$G(t) = Q\cos(\Omega t) A + Q \sin(\Omega t) B$$

$$QA = 0 \qquad Q\Omega B = T^{-1}$$

multiply on left by $Q^T T$

$$A = 0 \qquad B = \Omega^{-1} Q^T$$

sin Ωt is diagonal so $G = G^T$ source-receiver reciprocity

$$G(t) = Q\Omega^{-1}\sin(\Omega t)\,Q^T H(t)$$

Response to arbitrary forcing: convolve with Green matrix

$$T\ddot{\underline{q}} + V\underline{q} = f(t)$$

$$\underline{q}(t) = \int_{-\infty}^{t} G(t-t')\,f(t')\,dt'$$

$$\dot{\underline{q}}(t) = \cancel{G(0)f(t)} + \int_{-\infty}^{t} \dot{G}(t-t')\,f(t')\,dt'$$

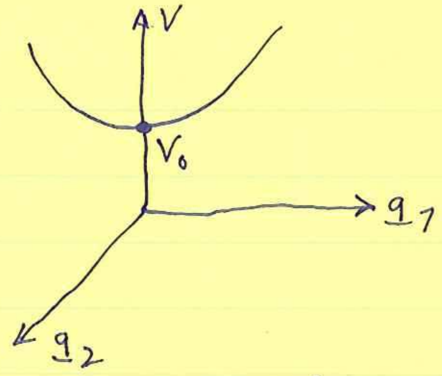
$$\ddot{\underline{q}}(t) = \dot{G}(0)\,f(t) + \int_{-\infty}^{t} \ddot{G}(t-t')\,f(t')\,dt'$$

$$T\ddot{\underline{q}} + V\underline{q} = TT^{-1}f(t) + \int_{-\infty}^{t} \cancel{[T\ddot{G} + VG]}(t-t')\,f(t')\,dt'$$

$$= f(t), \text{ check}$$

Linear stability analysis: ω^2 is real but may have either sign

If all $\omega_n^2 > 0$ then $V = V_0 + \frac{1}{2}\underline{q}^T V \underline{q}$ is a local <u>minimum</u>:

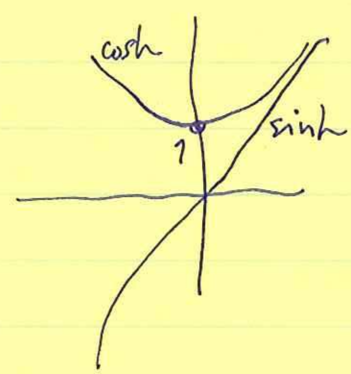


Suppose some squared eigenfrequency, $\omega_1^2 < 0 \Rightarrow \omega_1 = ip_1$

$\cos \omega_1 t = \cosh p_1 t$

$\omega_1^{-1} \sin \omega_1 t = p_1^{-1} \sinh p_1 t$

exponential instability



Plot of $V(\underline{q})$ a saddle

Conservation of energy: general Lagrangian $L(\underline{q}, \dot{\underline{q}}, t)$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$\dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \dot{q}_i \frac{\partial L}{\partial q_i} = 0$$

$$\frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} - \dot{q}_i \frac{\partial L}{\partial q_i} = 0$$

$$\frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) + \frac{\partial L}{\partial t} = 0$$

In our case $T = \frac{1}{2} \dot{q}_i T_{ij}(\underline{q}) \dot{q}_j + V(\underline{q})$

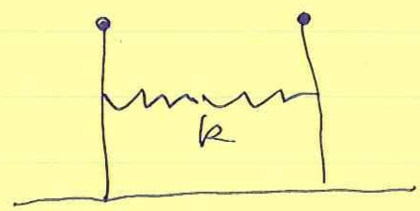
$$\frac{\partial L}{\partial t} = 0$$

also $\quad \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = 2T$

$$\frac{d}{dt}(T+V) = 0 \qquad \text{conservation of energy}$$

Now do double pendulum example
pages 12-16 1982 notes

Inverted pendulum



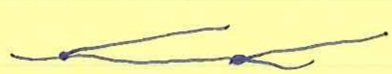
$$g \to -g$$

$$\omega_1 = \sqrt{-\frac{g}{\ell}} \qquad \text{unstable}$$

$$\omega_2 = \sqrt{-\frac{g}{\ell} + \frac{2k}{m}} \qquad \text{can be stable}$$

if spring is
strong enough

Hard to avoid



collapse

made it to here
end of day 1 — 2 hours

Day 2 review

$$\underline{q}(t) = \begin{pmatrix} q_1(t) \\ \vdots \\ q_N(t) \end{pmatrix} \qquad \underline{\dot{q}}(t) = \begin{pmatrix} \dot{q}_1(t) \\ \vdots \\ \dot{q}_N(t) \end{pmatrix}$$

$$T\underline{\ddot{q}} + V\underline{q} = 0 \quad , \quad T, V \text{ symmetric}$$
T pos. def

$$\frac{d}{dt}(T+V) = 0$$

$$\underset{ke}{T = \tfrac{1}{2}\underline{\dot{q}}^T T \underline{\dot{q}}} \,, \quad \underset{pe}{V = \tfrac{1}{2}\underline{q}^T V \underline{q}}$$

normal mode solutions: $\underline{q}(t) = \underline{q}\, e^{i\omega t}$
¿ why?

$$V\underline{q} = \omega^2 T \underline{q} \,, \quad \text{generalized eigenvalue problem}$$

ke inner product $\langle \underline{q}, \underline{q}' \rangle = \underline{q}^T T \underline{q}'$

Then $\langle \underline{q}, H\underline{q}' \rangle = \langle H\underline{q}, \underline{q}' \rangle$, $H = T^{-1}V$

simultaneous diagonalization by a congruent transformation

$$Q = \begin{pmatrix} | & & | \\ \underline{q}_1 & & \underline{q}_N \\ | & & | \end{pmatrix}$$

$$Q^T T Q = I \quad , \quad Q^T V Q = \Omega^2$$

$$\Omega^2 = \begin{pmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_N^2 \end{pmatrix}$$

orthonormality w.r.t. $\langle , \rangle$

Also

$$Q^{-1} H Q = Q^{-1}(T^{-1} V) Q = \Omega^2$$

similarity transform

One question: how do we know Q is invertible?

Alternative approach — closer to optimal numerical technique

$$T = R^2 \leftarrow \text{square root of } T \quad , \quad R^T = R$$

$$V\underline{q} = \omega^2 R^2 \underline{q} = \omega^2 R (R\underline{q})$$

$$R^{-1} V (R^{-1} R) \underline{q} = (R^{-1} V R^{-1})(R\underline{q}) = \omega^2 (R\underline{q})$$

Let $y = R\underline{q}$

$$\underbrace{(R^{-1} V R^{-1})}_{\text{symmetric}} y = \omega^2 y$$

now an ordinary eigenvalue problem

$$Y = \begin{pmatrix} | & & | \\ y_1 & \cdots & y_N \\ | & & | \end{pmatrix} \qquad \text{Then} \qquad Y = RQ$$

$$G(t) = \sum_{n=1}^{N} \omega_n^{-1} (q_n q_n^T) \sin \omega_n t$$

Green tensor

$$G(t) = Q\, \Omega^{-1} \sin \Omega t\, Q\, H(t)$$

source-receiver reciprocity:

$$G^T(t) = G(t)$$

$Y^T Y = I$

$\Rightarrow Y^{-1} = Y^T$

orthogonal

$Y^T (R^{-1} V R^{-1}) Y = \Omega^2$

$Y^T Y = Q^T R^T R Q = Q^T (R^2) Q$

$= Q^T T Q = I$ re orthonormality

$Q^T R^T R^{-1} V R^{-1} R Q$

$= Q^T V Q = \Omega^2$

but also $Y = RQ = Q = \underbrace{R^{-1}}_{\text{both invertible}} Y$

$\Rightarrow Q^{-1} = Y^{-1} R \quad \exists$

$Y^{-1} (R^{-1} V R^{-1}) Y$

$= Q^{-1} R^{-1} (R^{-1} V R^{-1}) R Q$

$= Q^{-1} (R^{-2} V) Q$

$= Q^{-1} (T^{-1} V) Q = Q^{-1} H Q = \Omega^2$

Numerically — use a cholesky decomposition $T = L L^T$ instead.

Rayleigh's principle D&T p. 113

$$H\underline{q} = \omega^2 \underline{q} \quad , \quad H = T^{-1} V$$

$$\omega^2 = \frac{\langle \underline{q}, H\underline{q} \rangle}{\langle \underline{q}, \underline{q} \rangle} = \frac{\underline{q}^T V \underline{q}}{\underline{q}^T T \underline{q}}$$ Rayleigh quotient

potential energy over kinetic energy

eigenvalue ω^2 is stationary for arbitrary $\delta \underline{q}$ iff $\underline{q}$ is an eigenvector with associated eigenvalue ω^2

Write schematically as $\omega^2 = \frac{\mathcal{V}}{\mathcal{T}}$

$$\delta \omega^2 = \delta\left(\frac{\mathcal{V}}{\mathcal{T}}\right) = \frac{\delta \mathcal{V}}{\mathcal{T}} - \frac{\mathcal{V}}{\mathcal{T}^2} \delta \mathcal{T}$$

$$= \frac{\delta \mathcal{V} - \omega^2 \delta \mathcal{T}}{\mathcal{T}} = \frac{1}{\mathcal{T}} \delta(\mathcal{V} - \omega^2 \mathcal{T})$$

* later next page

In normal mode context convenient to define a freq domain action

$$\mathcal{A} = \tfrac{1}{2}(\omega^2 \mathcal{T} - \mathcal{V}) = \tfrac{1}{2}\omega^2 (\underline{q}^T T \underline{q}) - \tfrac{1}{2} \underline{q}^T V \underline{q}$$

$$\delta \omega^2 = - \frac{\delta \mathcal{A}}{2\langle \underline{q}, \underline{q} \rangle} = \; = - \frac{\delta \mathcal{A}}{2 \underline{q}^T T \underline{q}}$$

ω^2 is stationary if $\mathcal{A}$ is and vice-versa

$$\delta(\mathcal{V} - \omega^2 \mathcal{T}) = \delta(\underline{q}^T V \underline{q} - \omega^2 \underline{q}^T T \underline{q})$$

$$= 2\,\delta \underline{q}^T (V\underline{q} - \omega^2 T \underline{q}) = 0 \quad \text{iff}$$

$$V\underline{q} - \omega^2 T \underline{q}$$

* from previous page here

Yet a third version: $\mathcal{V} = \underline{q}^T V \underline{q}$
is stationary subject to the normalization
constraint $\mathcal{T} = \underline{q}^T T \underline{q} = 1$

stationarity of ω^2 physically appealing

We know 2 things about $\mathcal{L}$:
1. stationary
2. equal to zero $\mathcal{L} = \frac{1}{2}(\omega^2 \mathcal{T} - \mathcal{V}) = 0$ at stationary point

Now suppose that $\mathbf{T}$ and $\mathbf{V}$ depend
on parameters $\mathbf{p}$, e.g. k, ℓ, m
for the pendula

$$\mathcal{L} = \mathcal{L}(\omega, \mathbf{q}, \mathbf{p}) = 0$$

Now suppose

$$\mathbf{p} \to \mathbf{p} + \delta\mathbf{p}, \quad \omega \to \omega + \delta\omega, \quad \mathbf{q} \to \mathbf{q} + \delta\mathbf{q}$$

$$\mathcal{L}(\omega, \underline{q}, \underline{p}) = \mathcal{L}(\omega+\delta\omega, \underline{q}+\delta\underline{q}, \underline{p}+\delta\underline{p}) = 0$$

Take the <u>total</u> variation w.r.t. all arguments

$$\delta_{total}\, \tfrac{1}{2}\left(\omega^2 \underline{q}^T \mathbf{T} \underline{q} - \underline{q}^T \mathbf{V} \underline{q}\right)$$

$$= \tfrac{1}{2}\delta\omega^2\, \underline{q}^T \mathbf{T} \underline{q} + \cancel{\delta\underline{q}^T\left(\omega^2 \mathbf{T}\underline{q} - \mathbf{V}\underline{q}\right)}$$

$$+ \tfrac{1}{2}\left(\omega^2 \underline{q}^T \delta\mathbf{T}\, \underline{q} - \underline{q}^T \delta\mathbf{V}\, \underline{q}\right) = 0$$

$$\delta\omega^2 = \frac{\underline{q}^T \delta\mathbf{V}\, \underline{q} - \omega^2 \underline{q}^T \delta\mathbf{T}\, \underline{q}}{\underline{q}^T \mathbf{T} \underline{q}}$$

Double pendulum example – page 28 of 1982 notes.