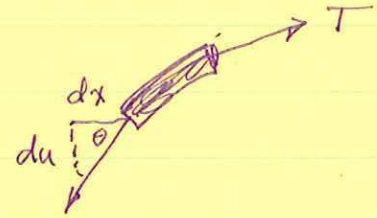
Violin string: tension T, density $\rho(x)$

$u(x,t)$ only in y direction



$$\tan\theta \approx \sin\theta = \frac{\partial u}{\partial x}$$

restoring force $T\left[\tan\theta(x+dx) - \tan\theta(x)\right]$

$$\approx T\left[\frac{\partial u}{\partial x}(x+dx) - \frac{\partial u}{\partial x}(x)\right]$$

Newton's law: $F = ma$

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T\left[\frac{\frac{\partial u}{\partial x}(x+dx) - \frac{\partial u}{\partial x}(x)}{dx}\right]$$

$$\rho(x)\,\partial_t^2 u = T\,\partial_x^2 u$$

fixed ends $u(0,t) = u(L,t) = 0$

Multiply by velocity $\partial_t u$ and $\int_0^L dx$

$$\text{lhs} = \int_0^L \rho\,\partial_t^2 u\,\partial_t u\,dx = \frac{d}{dt}\frac{1}{2}\int_0^L \rho(\partial_t u)^2\,dx$$

rhs by parts

$$T\int_0^L \partial_x^2 u \, \partial_t u \, dx = T\left[\cancelto{\text{zero}}{\partial_x u \, \partial_t u}\right]_0^L$$

$$- T\int_0^L \partial_x u \, \frac{\partial}{\partial t}(\partial_x u)\, dx$$

$$= -\frac{d}{dt}\frac{1}{2}\int_0^L T(\partial_x u)^2 \, dx$$

$$\frac{d}{dt}\int_0^L \left[\frac{1}{2}\rho(\partial_t u)^2 + \frac{1}{2}T(\partial_x u)^2\right] dx = 0$$

conservation of energy

$\frac{1}{2}\rho(\partial_t u)^2$: ke density

$\frac{1}{2}T(\partial_x u)^2$: pe density

Hamilton's principle:

$$I = \int_{t_1}^{t_2}\int_0^L L(u, \partial_t u, \partial_x u)\, dx\, dt$$

$$\delta I = 0 \qquad L = \frac{1}{2}\rho(\partial_t u)^2 - \frac{1}{2}T(\partial_x u)^2$$

$$I = \int_{t_1}^{t_2}\int_0^L \left[\frac{1}{2}\rho(\partial_t u)^2 - \frac{1}{2}T(\partial_x u)^2\right] dx\, dt$$

$$\delta I = \delta\int_{t_1}^{t_2} dt \int_0^L L(u, \partial_t u, \partial_x u)\, dx$$

$$= \int_{t_1}^{t_2} dt \int_0^L \left(\frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial(\partial_t u)}\partial_t \delta u + \frac{\partial L}{\partial(\partial_x u)}\partial_x \delta u\right) dx$$

$$= \int_0^L \underbrace{\left[\frac{\partial L}{\partial(\partial_t u)} \delta u \right]_{t_1}^{t_2}}_{\to\, 0 \text{ since } u(x,t_1) \text{ and } u(x,t_2) \text{ fixed}} dx$$

$$+ \int_{t_1}^{t_2} \underbrace{\left[\frac{\partial L}{\partial(\partial_x u)} \delta u \right]_0^L}_{\to\, 0 \text{ see below}} dt$$

$$+ \int_{t_1}^{t_2} dt \int_0^L \delta u \left[\frac{\partial L}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial(\partial_t u)}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial(\partial_x u)}\right) \right] dx$$

natural bc is $\frac{\partial L}{\partial(\partial_x u)} = T(\partial_x u) = 0$ free ends

instead we must impose an admissibility constraint upon δu : namely $\delta u(0,t) = \delta u(L,t) = 0$

Then I stationary for arbitrary admissible δu iff

$$\underbrace{\cancel{\frac{\partial L}{\partial u}}}_{\text{zero in our case}} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial(\partial_t u)}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial(\partial_x u)}\right) = 0$$

$$\rho\, \partial_t^2 u - T \partial_x^2 u = 0$$

Normal mode solutions: $u(x,t) = \underbrace{u(x)}_{\text{eigenfcn: mode shape}} e^{i\overbrace{\omega}^{\text{eigen-freq}} t}$

comment on FT
convention $\neq$ Aki & Richards

$$-T\frac{d^2u}{dx^2} = \rho\omega^2 u \quad \text{with } u(0)=u(L)=0$$

eigenvalue problem, $\rho > 0$ analogous to π pos-def.

ke inner product

$$\langle u, u' \rangle = \int_0^L \rho(x)\, u(x)\, u'(x)\, dx$$

Rewrite as

$$-\rho^{-1} T \frac{d^2u}{dx^2} = \omega^2 u, \quad H = -\rho^{-1} T \frac{d^2}{dx^2}$$

Operator $H = -\rho^{-1} T \frac{d^2}{dx^2}$ Hermitian w.r.t. $\langle , \rangle$

Proof easy: 2 integrations by parts

$$\langle u, \overbrace{-\rho^{-1} T \frac{d^2}{dx^2}}^{H} u' \rangle = -T \int_0^L u \frac{d^2u'}{dx^2}\, dx$$

$$= \underbrace{-T\left[u\frac{du'}{dx}\right]_0^L}_{\text{zero by bc.}} + T\int_0^L \frac{du}{dx}\frac{du'}{dx}\, dx$$

$$= \underbrace{T\left[\frac{du}{dx}u'\right]_0^L}_{\text{ditto}} - T\int_0^L \frac{d^2u}{dx^2} u'\, dx$$

$$= \langle \overbrace{-\rho^{-1} T \frac{d^2}{dx^2}}^{H} u, u' \rangle \quad \text{qed}$$

intimately involves bc — typical of field theories

Now easy to show that ω^2 real and

$$(\omega^2-\omega'^2)\langle u,u'\rangle = (\omega^2-\omega'^2)\int_0^L \rho u u' dx = 0$$

orthogonality end here class #2

intimate association: Hermitian, real eigenvalues, energy conservation

Example: $\rho(x)=\rho$, homogeneous

$$-(T/\rho)\frac{d^2u}{dx^2} = \omega^2 u, \qquad u(0)=u(L)=0$$

$$\left.\begin{aligned} \omega_n &= \frac{n\pi}{L}\sqrt{T/\rho} \\ u_n(x) &= \sqrt{\frac{2}{\rho L}}\,\sin\left(\frac{n\pi x}{L}\right) \end{aligned}\right\} n=1,2,\ldots$$

fundamental first overtone at twice frequency

orthogonality $\displaystyle \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \delta_{nm}$

$$\langle u_n, u_m\rangle = \delta_{nm}$$

Class #3

Hunb's question — when is it useful to seek normal mode solutions?

$$\rho(x)\, \partial_t^2 u = T \partial_x^2 u$$

b.c. $u(0,t) = u(L,t) = 0$

Instead seek a separable solution

$$u(x,t) = f(x)\, g(t)$$

$$\rho f \frac{d^2 g}{dt^2} = T g \frac{d^2 f}{dx^2}$$

$$\underbrace{\frac{1}{Tg} \frac{d^2 g}{dt^2}}_{\text{fcn of } t \text{ only}} = \underbrace{\frac{1}{\rho f} \frac{d^2 f}{dx^2}}_{\text{fcn of } x \text{ only}}$$

$\Rightarrow$ must both be = to a constant, which we call $-\omega^2/T$

$$\underbrace{\frac{1}{g} \frac{d^2 g}{dt^2}}_{\Rightarrow\ g \sim e^{i\omega t}} = -\omega^2 = \frac{T}{\rho} \frac{1}{f} \frac{d^2 f}{dx^2}$$

$$T \frac{d^2 f}{dx^2} = -\rho \omega^2 f$$

But what if $T = T(t)$: tuning your violin while you play it.

Then $$\frac{1}{T(t)}\frac{1}{g}\frac{d^2g}{dt^2} = \text{const} = \frac{1}{\rho(x)}\frac{1}{f}\frac{d^2f}{dx^2}$$

In general it is permissible ~~to~~ to seek normal mode solutions whenever none of the parameters describing the system or model depend on ~~the~~ time

In the discrete case

$$\mathbb{T}\ddot{\underline{q}} + \mathbb{V}\underline{q} = 0$$

ke: $T = \frac{1}{2}\dot{\underline{q}}^T \mathbb{T} \dot{\underline{q}}$

pe: $V = V_0 + \frac{1}{2}\underline{q}^T \mathbb{V} \underline{q}$

$\mathbb{T}$ and $\mathbb{V}$ are time-independent

This the hallmark of a conservative system.

Then do homogeneous string

4^{th} lecture: Thurs Feb 15

tensors as multilinear functionals
taken from mimeographed notes
plus Appendix A of D&T.

5th class: 20 Feb

linear functional $f(\underline{v}) \to$ scalar

$\uparrow$ slot

isomorphic to vectors

$$f(\underline{v}) = \underline{f}\cdot\underline{v} \quad \text{for all } \underline{v}$$

multilinear functional $T(\underline{u},\underline{v})$

tensor product: $TS(\underline{u},\underline{v},\underline{x},\underline{y})$
$= T(\underline{u},\underline{v})\,S(\underline{x},\underline{y})$ order $p+\underline{q}$

write without $\otimes$ $T\otimes S$

trace $\mathrm{tr}\,T = T(\hat{x}_i,\hat{x}_i) = T(\hat{x}_i',\hat{x}_i')$

transpose $T^T(\underline{u},\underline{v}) = T(\underline{v},\underline{u})$

components $T_{ij} = T(\hat{x}_i,\hat{x}_j)$
just like $f_i = \underline{f}\cdot\hat{x}_i = f(\hat{x}_i)$

$\hat{x}_{i_1}\cdots\hat{x}_{i_q}$ a basis for space of tensors of order q - dimension 3^q

e.g. $T = T_{ij}\,\hat{x}_i\hat{x}_j$

$(TS)_{ijkl} = T_{ij}S_{kl}$

$\mathrm{tr}\,T = T_{ii}$

identity tensor $\underline{\underline{I}}(\underline{u}, \underline{v}) = \underline{u}\cdot\underline{v})$

$I_{ij} = \delta_{ij}$, isotropic tensor

alternating tensor $\Lambda(\underline{u}, \underline{v}, \underline{w}) = \underline{u}\cdot(\underline{v}\times\underline{w})$

$\Lambda_{ijk} = \pm \varepsilon_{ijk}$ rt handed / left handed

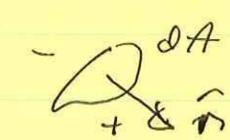
Most general isotropic tensor $a\underline{\underline{I}}$, $a\delta_{ij}$

Change of basis : $T_{ij}' = (\hat{x}_i' \cdot \hat{x}_j)(\hat{x}_j' \cdot \hat{x}_k) T_{kl}$

Then: tensors of order $q=2$ as linear operators

End with physical examples of tensors. Common in linear constitutive relations

Ohm's law: $\underline{J} = \underline{\underline{\sigma}}\cdot\underline{E}$
thermal conductivity: $\underline{H} = -\underline{\underline{K}}\cdot\nabla\theta$
dielectric tensor: $\underline{D} = \underline{\underline{\varepsilon}}\cdot\underline{E}$
inertia tensor: $\underline{L} = \underline{\underline{I}}\cdot\underline{\omega}$
stress tensor: ~~[illegible]~~



$\underline{f} = (\hat{n}\, dA)\cdot\underline{\underline{T}}$

or since $\underline{\underline{T}} = \underline{\underline{T}}^T$

$\underline{f} = \underline{\underline{T}}\cdot(\hat{n}\, dA)$