Rayleigh's principle:

$$H = -\rho^{-1}(x)\, T \frac{d^2}{dx^2}$$

$$Hu = \omega^2 u$$

$$\omega^2 = \frac{\langle u, Hu\rangle}{\langle u, u\rangle} = \frac{T \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx}{\int_0^L \rho u^2\, dx} = \frac{\mathcal{V}}{\mathcal{T}}$$

$$\delta \omega^2 = \frac{1}{\mathcal{T}}\, \delta(\mathcal{V} - \omega^2 \mathcal{T})$$

normal mode action:

$$I = \frac{1}{2}(\omega^2 \mathcal{T} - \mathcal{V}) = \frac{1}{2}\int_0^L \left[\rho \omega^2 u^2 - T\left(\frac{\partial u}{\partial x}\right)^2\right] dx$$

$\delta \omega^2 = 0$ iff $\delta I = 0$ as before

$$\delta I = \int_0^L \delta u \left(\rho \omega^2 u + T \frac{d^2 u}{dx^2}\right) dx - \left[\delta u \left(T \frac{du}{dx}\right)\right]_-^+ = 0$$

again need admissibity constraint $\delta u = 0$ on ends

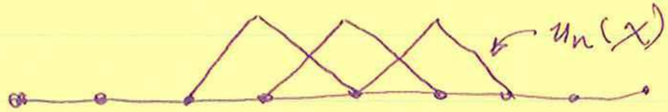
$$\delta I = 0 \text{ iff } -\rho^{-1} T \frac{d^2 u}{dx} = \omega^2 u, \quad u(0) = u(L) = 0$$

Application: Rayleigh-Ritz method — heterogeneous string, $\rho(x)$

Basis functions
$u_1(x), \dots, u_N(x)$

e.g. global $u_n(x) = \sin \frac{n\pi x}{L}$

local – piecewise linear splines



$$u(x) = \sum_{n=1}^{N} q_n u_n(x), \quad q_n \text{ expansion coeffs.}$$

$$I = \frac{1}{2}(\omega^2 \mathcal{T} - \mathcal{V}) = \frac{1}{2}\int_0^L \left(\rho\omega^2 u^2 - T\left(\frac{du}{dx}\right)^2\right) dx$$

$$= \frac{1}{2}\omega^2 \mathbf{q}^T \mathbf{T} \mathbf{q} - \frac{1}{2}\mathbf{q}^T \mathbf{V} \mathbf{q}$$

$$\mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$$

$$V_{ij} = \mathcal{V}(u_i, u_j) = T\int_0^L \frac{du_i}{dx}\frac{du_j}{dx}\, dx$$

$$T_{ij} = \mathcal{T}(u_i, u_j) = \int_0^L \rho(x)\, u_i(x)\, u_j(x)\, dx$$

$\mathbf{T} = \mathbf{T}^T$, $\mathbf{V} = \mathbf{V}^T$ and $\mathbf{T}$ pos. def.

Reduces to previously considered problem

e.g. with a global basis

$$u_n(x) = \sin\frac{n\pi x}{L} \text{ , satisfy bc}$$

$$V_{mn} = \frac{1}{2} n^2 \pi^2 (T/L)\, \delta_{mn} \text{ , diagonal}$$

$$T_{mn} = \int_0^L \rho(x) \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx$$

Easiest in this case (since $T =$ const) to solve

$$V^{-1} T \underline{q} = \omega^{-2} \underline{q}$$

Second application: first-order perturbation

$$\rho \to \rho + \delta\rho \text{ , } \omega \to \omega + \delta\omega \text{ , } u \to u + \delta u$$

Method same as before

$$I(\omega, u, \rho) = I(\omega+\delta\omega, u+\delta u, \rho+\delta\rho) = 0$$

$$I = \frac{1}{2} \int_0^L \left[\rho\omega^2 u^2 - T\left(\frac{du}{dx}\right)^2\right] dx = 0$$

$$\delta_{total} I = \frac{1}{2}\, \delta\omega^2 \int_0^L \rho u^2\, dx$$

$$+ \frac{1}{2} \int_0^L \delta\rho\, \omega^2 u^2\, dx$$

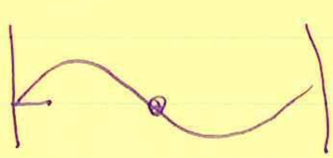
$$+ \underbrace{\int_0^L \delta u \left(\rho\omega^2 u + T\frac{d^2u}{dx^2}\right) dx - \left[\delta u \left(T\frac{du}{dx}\right)\right]_-^+}_{\text{zero by Rayleigh}} = 0$$

$$\delta\omega^2/\omega^2 \cong \frac{2\omega}{\delta\omega} = -\frac{\int_0^L \delta\rho\, u^2\, dx}{\int_0^L \rho u^2\, dx}$$

e.g. a bead of mass m at $x = L/2$

$$\delta\rho = m\,\delta(x - L/2)$$

$$\frac{\delta\omega_n}{\omega_n} = -\left(\frac{m}{\rho L}\right)\sin^2\frac{n\pi}{2} = \begin{cases} -m/\rho L & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$



even: at node

odd: <u>graver</u> oscillation

Example 2: <u>smooth</u> continuous perturbation

0 L

$|\delta\rho| \ll \rho$

$$\frac{\delta\omega_n}{\omega_n} = -\frac{1}{\rho L}\int_0^L \delta\rho(x)\,\sin^2\frac{n\pi x}{L}\,dx$$

$$= -\frac{1}{2\rho L}\int_0^L \delta\rho(x)\left(1 - \cancel{\cos\frac{2n\pi x}{L}}\right)dx$$

zero for $n \gg 1$ high freq. mode

$$\omega_n \approx \frac{n\pi\sqrt{T/\rho}}{L}\left[1 - \frac{1}{2L}\int_0^L (\delta\rho/\rho)\,dx\right]$$

$c = \sqrt{T/\rho}$ is wavespeed

$$\omega_n = \frac{2n\pi}{T}, \quad T = \frac{2L}{c} \qquad \omega_n = \frac{n\pi c}{L}$$

generalized to

$$\omega_n \approx \frac{2n\pi}{T}, \quad T = 2\int_0^L \frac{dx}{c(x)}$$

in this case $\quad c = \sqrt{\frac{T}{\rho + \delta\rho}} \approx c\left[1 - \frac{1}{2}\frac{\delta\rho}{\rho}\right]$

Green function $g(x, x'; t)$ satisfies

$$\rho \partial_t^2 g = T \partial_x^2 g + \delta(x - x')\delta(t)$$

equivalent initial value problem

$$\rho \partial_t^2 g = T \partial_x^2 g$$

$$\rho [\partial_t g]_-^+ = 1, \quad [g]_-^+ = 0$$

gen'l sol'n $\quad g(x, x'; t) = \sum_n (a_n \cos\omega_n t + b_n \sin\omega_n t) u_n(x)$

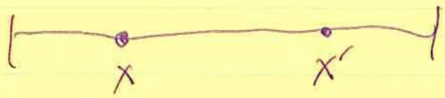
$$\sum_n a_n u_n = 0, \quad \sum_n \omega_n b_n u_n = \rho^{-1}\delta(x - x')$$

orthogonality: $\quad a_n = 0, \quad b_n = \omega_n^{-1} u_n(x')$

$$g(x,x';t) = \sum_n \overbrace{\omega_n^{-1}\sin\omega_n t\; u_n(x')}^{\text{amp. factor}}\;\overbrace{u_n(x)}^{\text{shape of oscillation}}$$

heterogeneous string
oscillating in time

source-receiver reciprocity : $g(x,x';t) = g(x',x;t)$



Fourier domain:

comment on sign: opposite from A&R

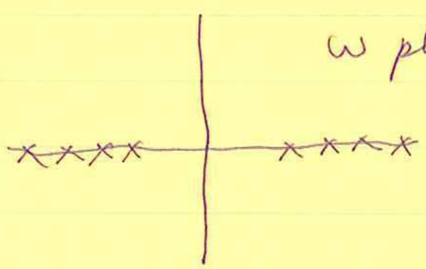
$$g(\omega) = \int_0^\infty g(t)\, e^{-i\omega t}\, dt$$

comment

$$g(x,x';\omega) = \sum_n \frac{u_n(x')\,u_n(x)}{\omega_n^2 - \omega^2}$$

Instructive to verify by working backward

$$g(x,x';t) = \sum_n u_n(x')\,u_n(x)\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}\, d\omega}{(\omega_n-\omega)(\omega_n+\omega)}$$



poles at $\omega = \pm\omega_n$ on real axis

Two ways to eliminate ambiguity

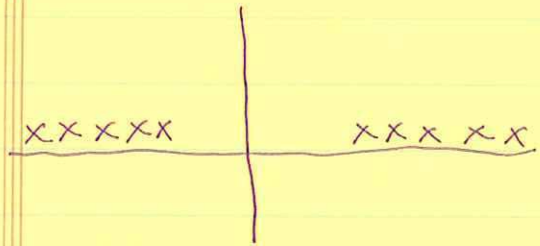
1. real systems _always_ have dissipation

$\omega_n \to \omega_n + i\varepsilon_n$, $\varepsilon_n > 0$

why + ?

since $e^{i\omega_n t} \to e^{i\omega_n t - \varepsilon_n t}$

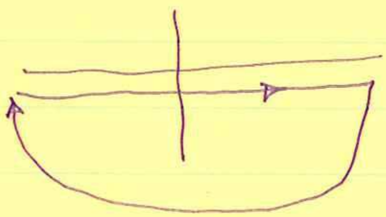
2. causality : response must be zero before $t=0$.



$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}\,d\omega}{(\omega-\omega_n)(\omega+\omega_n)}$$

$t<0$: close in lower halfplane $\mathrm{Im}\,\omega<0$

$\Rightarrow e^{i\omega t}\to 0$ on arc



$$g(x,x';t)=0,\quad t\le 0$$

$t\geq 0$: close in upper halfplane, pick up poles

$$-\frac{1}{2\pi}(2\pi i)\,\Sigma\text{ residues} = \frac{\sin\omega_n t}{\omega_n}$$

Finally, conversion to travelling waves (homog. string)

$$\omega_n=\frac{n\pi c}{L}\quad;\quad c=\sqrt{T/\rho}$$

$$u_n(x)=\sqrt{\frac{2}{\rho L}}\,\sin\frac{n\pi x}{L}$$

$$g(t)=\frac{2}{\rho c}\sum_{n=1}^{\infty}\underbrace{\frac{1}{n\pi}\sin(n\pi x'/L)}_{\text{amp. factor}}\ \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{shape}}\ \underbrace{\sin\left(\frac{n\pi ct}{L}\right)}_{\text{oscillatory in time}}$$

sum over modes

depends on source location and strength ; in this case 1 in FT domain

$$g(\omega) = \frac{2}{\rho L} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x'}{L} \sin \frac{n\pi x}{L}}{\frac{n^2\pi^2 c^2}{L^2} - \omega^2}$$

$$= \frac{L}{\rho \pi^2 c^2} \sum_{n=1}^{\infty} \frac{\cos \frac{n\pi}{L}|x-x'| - \cos \frac{n\pi}{L}(x+x')}{n^2 - \omega^2 L^2/\pi^2 c^2}$$

use the Fourier series identity

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n^2-\alpha^2} = \frac{1}{2\alpha^2} \underbrace{- \frac{\pi}{2} \frac{\cos \alpha(\pi - z)}{\alpha \sin \alpha\pi}}, \quad 0 \le z \le 2\pi$$

this is the Fourier cosine series of this function which satisfies $f(2\pi - z) = f(z)$

$$g(\omega) = \frac{1}{2\rho c} \left\{ \frac{\cos \frac{\omega}{c}[L-(x+x')] - \cos \frac{\omega}{c}[L-|x-x'|]}{\omega \sin \frac{\omega L}{c}} \right\}$$

numerator $\quad \cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

denominator $\quad \frac{1}{\sin \frac{\omega L}{c}} = 2i \left[e^{i\omega L/c} - e^{-i\omega L/c} \right]^{-1}$

$$= 2i\, e^{-i\omega L/c} \left[1 - e^{-2i\omega L/c} \right]^{-1} \qquad \text{geom. series}$$

$$= 2i\, e^{-i\omega L/c} \sum_{m=0}^{\infty} e^{-2im\omega L/c}$$

$d_1 = |x-x'|$ $\quad N_1 = 0$ $\quad$ sketch picture of $d_1 - d_5$

$d_2 = x + x'$ $\quad N_2 = 1$

$d_3 = 2L - (x+x')$ $\quad N_2 = 1$

$d_4 = 2L - |x-x'|$ $\quad N_4 = 2$

$d_{j+4} = 2L + d_j$ $\quad N_{j+4} = N_j + 2$

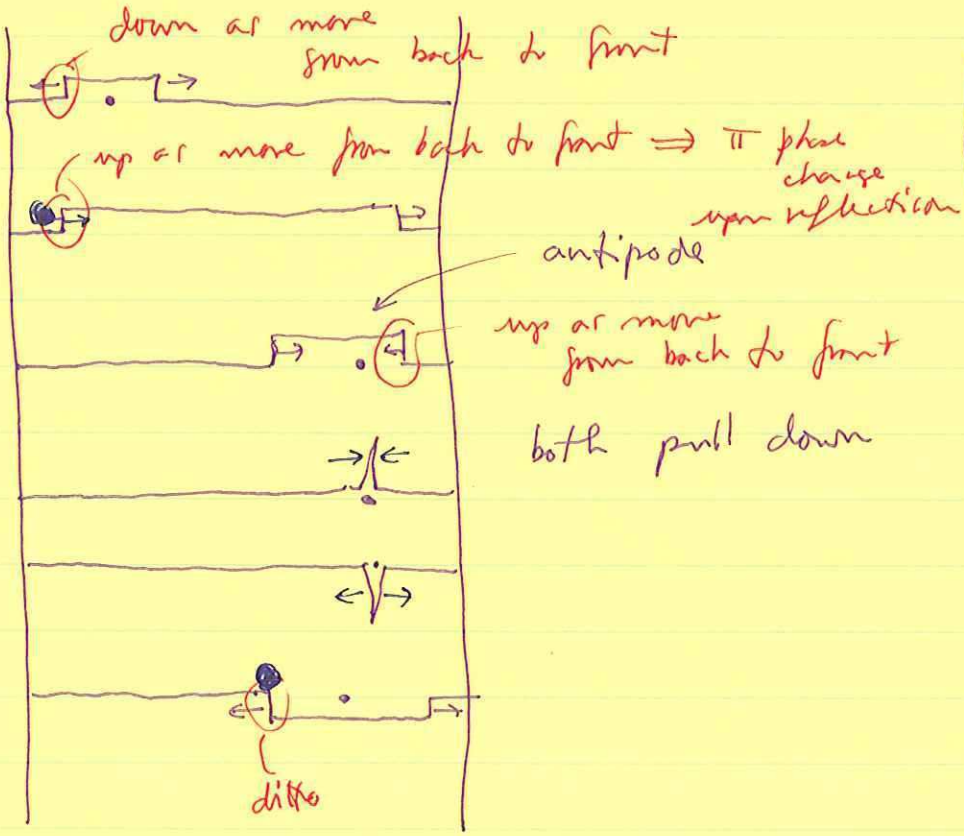
$$g(\omega) = \frac{1}{2\rho c} \sum_{j=1}^{\infty} (i\omega)^{-1} e^{-i\omega d_j/c - iN_j\pi}$$

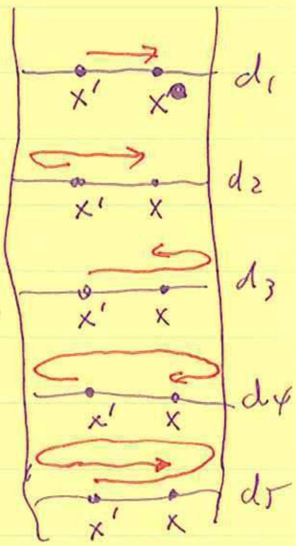
$(-1)^{N_j}$ phase change upon reflection of ends

$$g(t) = \frac{1}{2\rho c} \sum_{j=1}^{\infty} e^{-iN_j\pi} H(t - d_j/c)$$

Heaviside function

sum of propagating step pulses





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