Geophys. J. R. astr. Soc. (1968) **16**, 329–367.

The Normal Modes of a Rotating, Elliptical Earth

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(Received 1968 June 21)

Summary

It is possible to calculate precisely the theoretical eigen-frequencies of any Earth model which is non-rotating, spherically symmetric, and which has an isotropic static stress field and an isotropic dynamic stress–strain relation. In this paper Rayleigh's principle is used to provide a formalism which allows the approximate computation of the normal mode eigen-frequencies of any Earth model which is slowly rotating and slightly aspherical and anisotropic. This formalism is used to compute, correct to second order, the effects of the Earth's angular rotation, and correct to first order, the effects of the Earth's ellipticity of figure on the normal mode eigenfrequencies. It is found that for an arbitrary poloidal or toroidal multiplet, the central ($m = 0$) member of the multiplet is shifted slightly in frequency and that the other members of the multiplet are split apart asymmetrically by the effects of the Earth's rotation and ellipticity. The results may be used to make a preliminary correction for rotation and ellipticity to the Earth's raw normal mode data.

1. Introduction

The elastic-gravitational normal modes of the Earth have been excited by major earthquakes and observed on various low-frequency seismological instruments. Records of these observations can be used to measure the angular frequencies of oscillation of the Earth's normal modes. In recent years it has also become possible, using high-speed computers, to calculate quickly and precisely the theoretical angular frequencies of oscillation of the elastic-gravitational normal modes for a large class of Earth models; namely, for any model having the following characteristics:

(1) the Earth model is spherically symmetric;

(2) the angular velocity of steady rotation is zero;

(3) the dynamic stress–strain relation at every point is perfectly elastic, and furthermore is isotropic;

(4) the static stress field in the equilibrium configuration is at every point isotropic.

Any such model of the Earth will be called a SNREI (spherical, non-rotating, elastic, isotropic) Earth model. For the purpose of computing the theoretical eigen-frequencies, a SNREI Earth model of radius a can be completely characterized by three functions of r, the radial distance from the centre. These three functions are the density $\rho_0(r)$, the bulk modulus $\kappa(r)$, and the shear modulus $\mu(r)$, the latter two an arbitrary SNREI Earth model due to slow angular rotations and small asphericities and anisotropies. For the lower order fundamental normal modes, it is expected that the Earth's rotation and ellipticity are the dominant perturbing effects. The computed rotational and elliptical splitting parameters depend upon the properties ρ_0, κ, μ of the unperturbed SNREI Earth model. In this paper, the eigenfrequencies ${}_n\omega_l^S$ and ${}_n\omega_l^T$ and the associated $2l+1$ dimensional eigenspaces were computed for three different SNREI Earth models, and then Rayleigh's principle and second-order rotational perturbation theory were used to determine the corrections to the eigenfrequencies, correct to first order in the ellipticity ε_a and to second-order in the rotation. The degeneracy of any multiplet ${}_nS_l$ or ${}_nT_l$ is in general completely removed; to zeroeth order the eigenfunctions of a rotating elliptical Earth without geographical variations in properties can be characterized by a single spherical harmonic Y_l^m. The first-order effect of ellipticity and the second-order effect of rotation not only act to shift the entire multiplet but also cause the splitting of a multiplet to be asymmetrical. It is pointed out that another effect of rotation, ellipticity and lateral inhomogeneities is to give rise to the presence of small amplitude first-order displacement fields. In particular there will be poloidal fields at toroidal eigenfrequencies and toroidal fields at poloidal eigenfrequencies.

Acknowledgments

I gratefully thank my research advisors, Professor G. E. Backus and Professor J. F. Gilbert, for suggesting this problem and acknowledge their continual guidance throughout. Both Professors Backus and Gilbert read the manuscript and suggested many improvements. This work was supported by the National Science Foundation under NSF Grant GP-4096.

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1968 *June.*

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