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Internal and external potential-field estimation from regional vector data at varying satellite altitude

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SUMMARY

When modelling satellite data to recover a global planetary magnetic or gravitational potential field, the method of choice remains their analysis in terms of spherical harmonics. When only regional data are available, or when data quality varies strongly with geographic location, the inversion problem becomes severely ill-posed. In those cases, adopting explicitly local methods is to be preferred over adapting global ones (e.g. by regularization). Here, we develop the theory behind a procedure to invert for planetary potential fields from vector observations collected within a spatially bounded region at varying satellite altitude. Our method relies on the construction of spatiospectrally localized bases of functions that mitigate the noise amplification caused by downward continuation (from the satellite altitude to the source) while balancing the conflicting demands for spatial concentration and spectral limitation. The 'altitude-cognizant' gradient vector Slepian functions (AC-GVSF) enjoy a noise tolerance under downward continuation that is much improved relative to the 'classical' gradient vector Slepian functions (CL-GVSF), which do not factor satellite altitude into their construction. Furthermore, venturing beyond the realm of their first application, published in a preceding paper, in the present article we extend the theory to being able to handle both internal and external potential-field estimation. Solving simultaneously for internal and external fields under the limitation of regional data availability reduces internal-field artifacts introduced by downward-continuing unmodelled external fields, as we show with numerical examples. We explain our solution strategies on the basis of analytic expressions for the behavior of the estimation bias and variance of models for which signal and noise are uncorrelated, (essentially) space- and band-limited, and spectrally (almost) white. The AC-GVSF are optimal linear combinations of vector spherical harmonics. Their construction is not altogether very computationally demanding when the concentration domains (the regions of spatial concentration) have circular symmetry, for example, on spherical caps or rings—even when the spherical-harmonic bandwidth is large. Data inversion proceeds by solving for the expansion coefficients of truncated function sequences, by least-squares analysis in a reduced-dimensional space. Hence, our method brings high-resolution regional potential-field modelling from incomplete and noisy vector-valued satellite data within reach of contemporary desktop machines.

Key words: Satellite gravity, Satellite magnetics, Fourier analysis, Inverse theory, Spatial analysis

1 INTRODUCTION

Potential fields such as gravity and magnetic fields provide indispensable information about planetary or lunar structure and evolution (Kaula 1968; Lambeck 1988; Langel & Hinze 1998; Merrill et al. 1998). At the scale of the globe for Earth and Moon, and more generally for other planets and their moons, the vast majority of the data is derived from satellite missions (Connerney 2015; Wieczorek 2015). Recording gravity and magnetic fields *in* space is an engineering problem of instrumentation. Mapping such fields *from* space down to the body of interest, separately from any fields generated externally, is a problem of inversion (Plattner & Simons 2015b; Sabaka et al. 2015). Regional modelling is predicated on the ability to include data collected at varying satellite altitude, alleviating noise amplification under 'downward continuation', and, in particular in the case of magnetic field modelling, taking external fields into account. The full estimation problem as we consider it here consists in determining 'best' models—suitable for evaluation at the surface of the planetary body, and geological interpretation as far as accuracy and resolution permit—of an internally generated field noisily observed at a scattered, areally-limited set of locations taken at varying satellite altitude, in the presence of an external field.

Beginning with Gauss (1839), the parameterization of the solution in terms of global basis functions, spherical (Backus et al. 1996) or ellipsoidal (Bölling & Grafarend 2005) harmonics, remains today a popular practical approach (Sneeuw 1994). At the other end of the modelling spectrum are local methods, specifically, those based on gridded sets of monopoles (e.g. O'Brien & Parker 1994), equivalentssource dipoles (e.g. Langel & Hinze 1998), or point masses (e.g. Baur & Sneeuw 2011). In-between those extremes of spectral and spatial selectivity (for a classification, see Freeden & Michel 1999; Freeden et al. 2016) lies a variety of methods that use functions such as radial basis functions (e.g. Schmidt et al. 2007), mascons (e.g. Watkins et al. 2015), spherical cap harmonics (e.g. Thébault et al. 2006; Langlais et al. 2010), spherical-harmonic splines (e.g. Shure et al. 1982; Amirbekyan et al. 2008), and wavelets (Holschneider et al. 2003; Mayer & Maier 2006; Gerhards 2012). Among the constructively 'spatio-spectrally localized' spherical functions (e.g. Lesur 2006) features the general class of 'Slepian functions' (Simons et al. 2006; Plattner & Simons 2014; Simons & Plattner 2015) upon which we build our present work.

Building new bases (or 'frames', in a wider sense) by the judiciously weighted linear combination of spherical harmonics, which most of the above localization methods have in common, provides a natural way to respect the harmonicity of the potential fields under study. When the spherical-harmonic expansion coefficients of a potential field at a certain altitude are 'known', downward continuation to the zero height of the planetary surface, usually approximated by a sphere, amounts to a simple reevaluation via multiplication of the coefficients with factors that depend on the radii of the measurement sphere and the planet (e.g. Blakely 1995; Backus et al. 1996; Dahlen & Tromp 1998). In the case of imperfect knowledge, however, numerical and statistical stability limit the spatial resolution of the reevaluated fields that can be obtained in this way, depending on the relative altitude and the signal-to-noise ratios of the coefficients. Such difficulties are exacerbated if the source of the uncertainty, fundamentally, lies in the original data being available over an incomplete portion of the measurement sphere (Kaula 1967; Xu 1992; Trampert & Snieder 1996; Simons & Dahlen 2006; Schachtschneider et al. 2012). For such problems, inversion methods that rely on spherical-harmonics based localized basis functions confer efficiency and stability, dimensional reduction, and the overall ease and ability to produce and downward-continue regional potential-field models with less statistical a priori information or numerical regularization.

Satellite data coverage is far from being always 'global'. Coverage may be only regional, as is the case over Mercury (Solomon et al. 2001, 2007), or data quality may vary due to spatial variations of signal-to-noise levels or satellite altitude, rendering a geographical restriction of the area of interest desirable. Such was the situation for Mars (Albee et al. 2001), where Plattner & Simons (2015a) selected low-altitude nighttime magnetic-field data for inversion using the 'altitude-cognizant gradient vector Slepian functions' (AC-GVSF) that are the subject of this paper, resulting in a new lithospheric magnetic-field model of the Martian South Pole. They subtracted an external-field model made independently by Olsen et al. (2010a) from the data prior to inversion. In the present paper, we treat the estimation of internally and externally generated fields as an inverse problem that considers both jointly.

Our method traces its history to the 1-D theory of 'prolate spheroidal wave functions' by Slepian & Pollak (1961), its applications in signal processing (Slepian 1983), and especially its extensions to scalar spherical fields by Simons et al. (2006) and Simons & Dahlen (2006), to spherical vector fields by Plattner & Simons (2014), and to gradient vector spherical functions (curl-free potential fields) by Plattner & Simons (2015b). In the above cited works, satellite altitude, though explicitly considered within the context of the inverse problem, was never a factor in the optimization construction of the Slepian functions, and so we will term them 'canonical' or 'classical'. In particular, the functions of Plattner & Simons (2015b) will hereafter be known as 'classical gradient vector Slepian functions' (CL-GVSF). In contrast, the construction by Plattner & Simons (2015a), reformulated here, does incorporate satellite altitude directly, hence their designation ACsGVSF. The CL-GVSF solve a spatial (surface) optimization problem for bandlimited functions, while the AC-GVSF incorporate optimization under downward continuation from satellite altitude. Using the AC-GVSF for satellite-data inversion is different than using the CL-GVSF basis. In the latter case, vector-field measurements are first inverted for a best-fitting model at altitude, and the results are downward-continued afterwards. As Plattner & Simons (2015b) already noted in their sections 7.1 and 7.2, in that case, the model at the planetary surface is potentially biased by power in the high spherical-harmonic degrees leaking in through the downward continuation. This bias is a consequence of using functions that solely optimize spatial concentration within a given region. The general method presented here aims at overcoming these issues.

We construct a basis of functions from linear combinations of 'gradient vector spherical harmonics' by solving an optimization problem that incorporates the satellite altitude at which the data are acquired. We present two versions of an inversion method that use different forms of the AC-GVSF. In the first method we assume that external fields are not present, or have been removed from the data by prior analysis. In our second method, we model external fields simultaneously while solving for the internal field. Only the first approach was used by Plattner & Simons (2015a), and they did not present a complete mathematical analysis, as we do here. Notation and preliminary considerations can be found Section 2. A statement of the problem that we solve is found in Section 3. The body of the paper is arranged around the three questions 'what?', 'how?', and 'why?'. Sections 4–8 cover the question 'what' and touch on the question 'how?'. Sections 9 and 10 answer the question 'why?'. Finally, the Appendix focuses again on the question 'how?', in more detail. More precisely, in Section 4 we present the purely 'internal-field AC-GVSF' that we will use in Section 6 to solve for a potential-field model from purely internal-field regional vector data. Section 5 describes the construction of internal and external 'full-field AC-GVSF' that we use, in Section 7, to solve simultaneously for the internal and external potential field from regional satellite data with varying altitude. We test both methods on a simulated data set in Section 8 and investigate the effect of neglecting to account for an external field. In Sections 9 and 10 we provide a more in-depth analysis of the relationship of our new Slepian functions to the general vector spherical Slepian functions presented by Plattner & Simons (2014) and showcase their mathematical and statistical properties. We summarize our findings in Section 11 and explain methods to significantly decrease the computational costs of high spherical-harmonic degree calculations in the Appendix.

We assume that the magnetic field that we solve for is static, in that we do not incorporate a direct time dependence. To avoid temporal aliasing, the data should be binned into episodic clusters before inversion and then inverted individually using the same set of AC-GVSF. Compared to other regional methods, the AC-GVSF approach has the overall advantage that all calculations happen within a space spanned by bandlimited spherical harmonics, the natural basis for source-free potential fields outside a sphere (Section **??**). Our method can be interpreted as a computationally tractable approximation to the truncated singular-value decomposition of the full spherical-harmonic global problem focused on a chosen region. The AC-GVSF are easy to use, computationally efficient, and work with discrete data collected at varying satellite altitude. A benchmark comparison test with other methods remains desirable but is beyond the scope of this article. Instead we summarize where we discern the main differences with other regional methods. The popular Revised Spherical Cap Harmonic Analysis (R-SCHA) method by Thébault et al. (2006) fits data using basis functions that solve Laplace's equation inside a cone covering the chosen region with appropriate boundary conditions. Our method allows for the separation of internal- and external fields (with bias, as we show in eqs **??**–**??**), which can not be readily achieved using R-SCHA (Thébault et al. 2006). Potential fields obtained using AC-GVSF are naturally expressed as a wavelength-dependent power spectrum, which is not as straightforward with R-SCHA (Thébault et al. 2006). The spherical wavelet methods by Mayer & Maier (2006) and Gerhards (2011, 2012, 2014) provide another powerful regional approach. To the best of our knowledge these methods assume that the satellite orbit is of constant altitude, which is not required in our method. Discrete-source methods such as monopoles (O'Brien & Parker 1994), equivalent dipoles (Langel & Hinze 1998), or point-mass modelling (Baur & Sneeuw 2011) require assigning effective depths (since a true depth cannot be constrained from potential-field measurements) to a finite set of infinitesimal sources distributed over the surface. By solving for the uniquely constrained potential field itself, we obviate parameterizing or discretizing the source altogether.

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