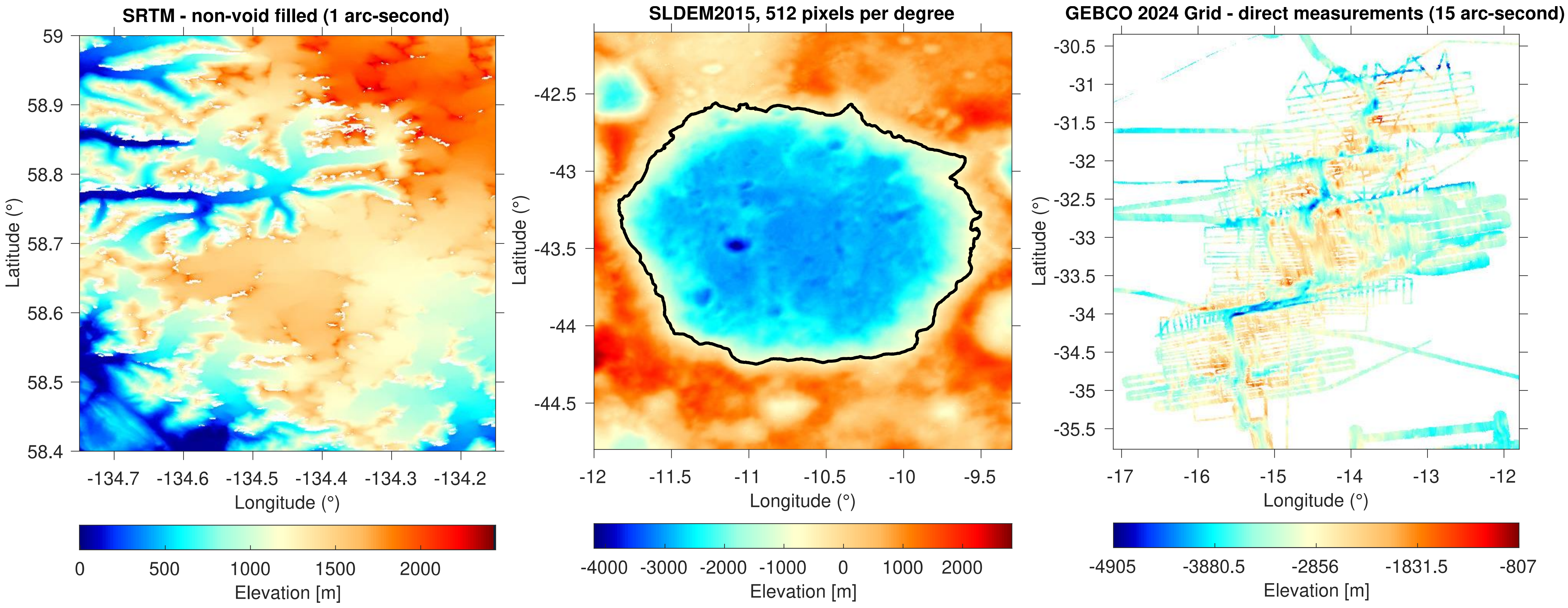
Finite geospatial observations of surfaces and media are often sampled under the constraints of sparsity, irregular boundaries, or structure.



We take spatial data sets as Gaussian random processes and study their stationary covariance structure in the spatial and spectral domain.

	Spatial domain $\longleftrightarrow$	Spectral domain
Observation	$\mathcal{H}(\mathbf{x})$	$d\mathcal{H}(\mathbf{k})$
Stationary covariance	$\langle \mathcal{H}(\mathbf{x})\mathcal{H}^*(\mathbf{x}') \rangle$	$\langle d\mathcal{H}(\mathbf{k}) d\mathcal{H}^*(\mathbf{k}') \rangle$
Stationary parametric autocovariance	$\mathcal{C}_{\boldsymbol{\theta}}(\mathbf{x}-\mathbf{x}')$	$\mathcal{S}_{\boldsymbol{\theta}}(\mathbf{k})$

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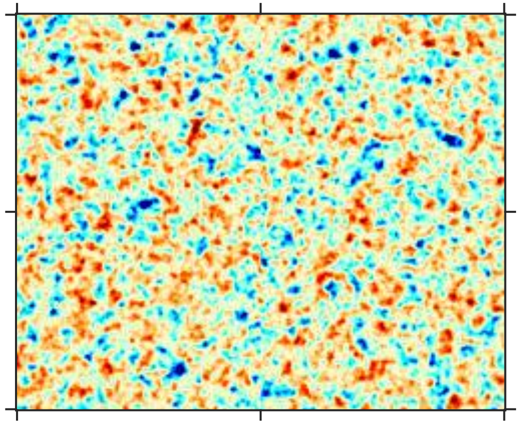
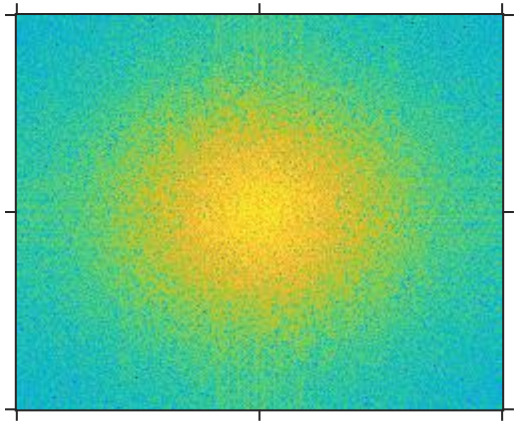
Spatial field



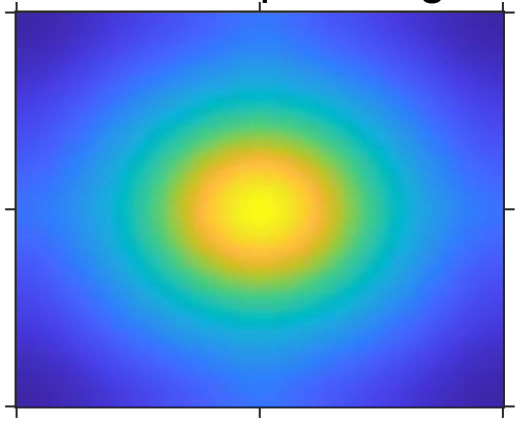
$\mathcal{H}(\mathbf{x})$

Empirical periodogram



$|H(\mathbf{k})|^2$

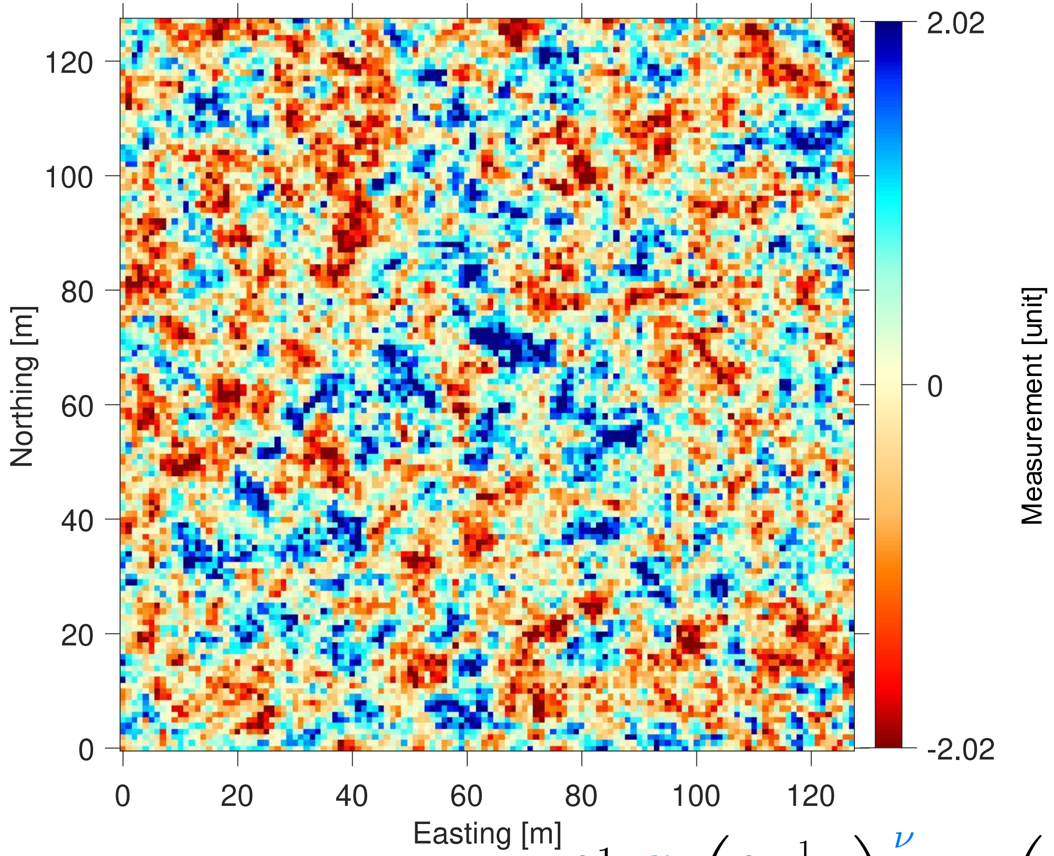
Theoretical periodogram



$\mathcal{S}_{\boldsymbol{\theta}}(\mathbf{k})$

We consider a flexible parametric covariance structure for sampled finite fields.

variance, $\sigma^2 = 1\ [\text{unit}]^2$; **smoothness,** $\nu = 0.50$; **range,** $\rho = 1$ m



Isotropic Matérn **spatial** covariance
x: lag distances

$$C_{\boldsymbol{\theta}}(r) = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{2\nu^{\frac{1}{2}}}{\pi\rho} r \right)^{\nu} K_{\nu} \left(\frac{2\nu^{\frac{1}{2}}}{\pi\rho} r \right) \quad (1)$$

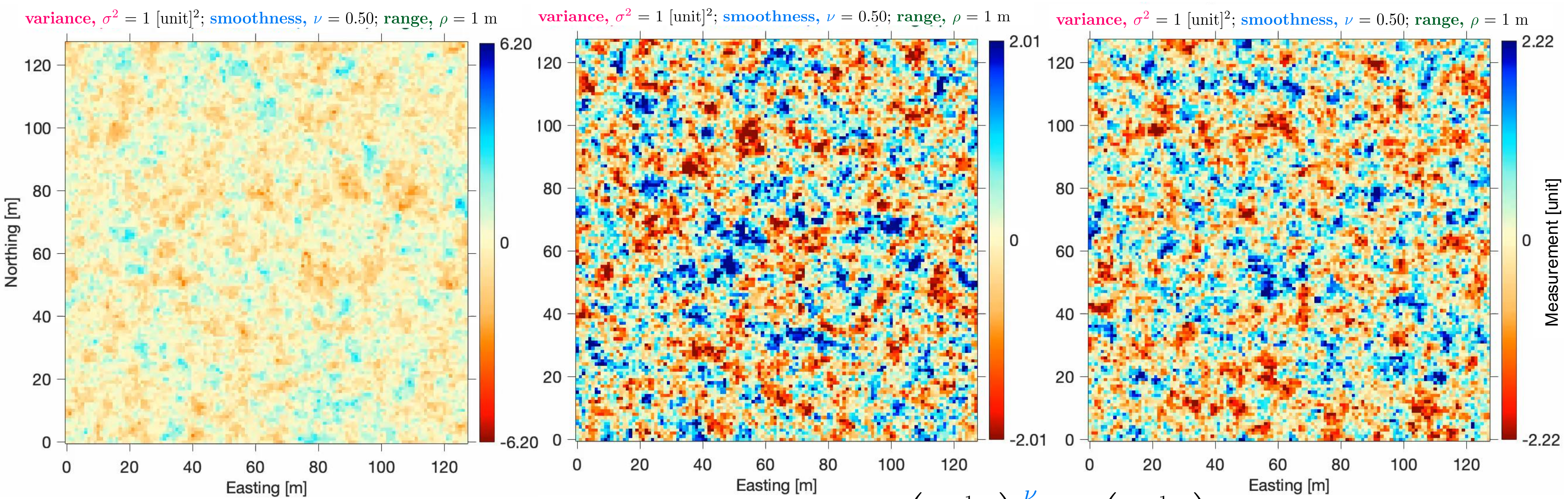
where $r = ||\mathbf{x} - \mathbf{x}'||_{\mathbf{2}}$ and $\boldsymbol{\theta} = [\sigma^2\ \nu\ \rho]$

Isotropic Matérn **spectral** density
k: wave vector, *d*: dimension

$$S_{d,\boldsymbol{\theta}}(k) = \frac{\sigma^2}{\pi^{d/2}} \frac{\Gamma(\nu + d/2)}{\Gamma(\nu)} \left(\frac{4\nu}{\pi^2\rho^2} \right)^{\nu} \left(\frac{4\nu}{\pi^2\rho^2} + k^2 \right)^{-\nu-d/2} \quad (2)$$

We consider a flexible parametric covariance structure for sampled finite fields.

Change in variance | Change in smoothness | Change in range



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We estimate the unbiased parametric covariance from spatial data that minimizes the debiased Whittle likelihood and we quantify its uncertainty analytically.

Debiased Whittle Likelihood function

$$\bar{\mathcal{L}}(\boldsymbol{\theta}) = -\frac{1}{K}\sum_{\mathbf{k}}\left[\ln \bar{\mathcal{S}}_{\boldsymbol{\theta}}(\mathbf{k}) + \frac{|H(\mathbf{k})|^2}{\bar{\mathcal{S}}_{\boldsymbol{\theta}}(\mathbf{k})}\right] \quad (3)$$

Empirical periodogram depends on the field's data and sampling

$$|H(\mathbf{k})|^2 = \frac{(2\pi)^{-2}}{\sum_{\mathbf{x}} g^2(\mathbf{x})}\left|\sum_{\mathbf{x}} g(\mathbf{x})\mathcal{H}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\right|^2 \quad (4)$$

Blurred spectral density depends on the field's sampling and spatial autocovariance

$$\bar{\mathcal{S}}_{\boldsymbol{\theta}}(\mathbf{k}) = \frac{(2\pi)^{-d}}{\sum_{\mathbf{x}} g^2(\mathbf{x})}\sum_{\mathbf{y}}\left(\sum_{\mathbf{x}} g(\mathbf{x})g(\mathbf{x}+\mathbf{y})\right)\mathcal{C}_{\boldsymbol{\theta}}(\mathbf{y})e^{-i\mathbf{k}\cdot\mathbf{y}} \quad (5)$$

The estimator optimizes the likelihood (eq. 3)

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \bar{\mathcal{L}}(\boldsymbol{\theta}) \quad (6)$$

The estimator is debiased in expectation

$$\mathbb{E}\{\hat{\boldsymbol{\theta}}\} = \boldsymbol{\theta}_0 \quad (7)$$

The covariance of the estimator can be calculated analytically

$$\begin{aligned}\mathrm{cov}\{\hat{\boldsymbol{\theta}}\} = K^{-2}&\left(\sum_{\mathbf{k}} \frac{\partial \bar{\mathcal{S}}_{\boldsymbol{\theta}}(\mathbf{k})}{\partial \boldsymbol{\theta}}\frac{1}{\bar{\mathcal{S}}_{\boldsymbol{\theta}}(\mathbf{k})}\frac{\partial \bar{\mathcal{S}}_{\boldsymbol{\theta}'}(\mathbf{k})}{\partial \boldsymbol{\theta}'}\frac{1}{\bar{\mathcal{S}}_{\boldsymbol{\theta}'}(\mathbf{k})}\right)^{-1}\\ &\times \sum_{\mathbf{k}}\sum_{\mathbf{k}'} \frac{\partial \bar{\mathcal{S}}_{\boldsymbol{\theta}}(\mathbf{k})}{\partial \boldsymbol{\theta}}\frac{\mathrm{cov}\{|H(\mathbf{k})|^2, |H(\mathbf{k}')|^2\}}{\bar{\mathcal{S}}^2_{\boldsymbol{\theta}}(\mathbf{k})\bar{\mathcal{S}}^2_{\boldsymbol{\theta}'}(\mathbf{k}')}\frac{\partial \bar{\mathcal{S}}_{\boldsymbol{\theta}'}(\mathbf{k}')}{\partial \boldsymbol{\theta}'}\\ &\times\left(\sum_{\mathbf{k}'} \frac{\partial \bar{\mathcal{S}}_{\boldsymbol{\theta}}(\mathbf{k}')}{\partial \boldsymbol{\theta}}\frac{1}{\bar{\mathcal{S}}_{\boldsymbol{\theta}}(\mathbf{k}')}\frac{\partial \bar{\mathcal{S}}_{\boldsymbol{\theta}'}(\mathbf{k}')}{\partial \boldsymbol{\theta}'}\frac{1}{\bar{\mathcal{S}}_{\boldsymbol{\theta}'}(\mathbf{k}')}\right)^{-1}\end{aligned} \quad (8)$$

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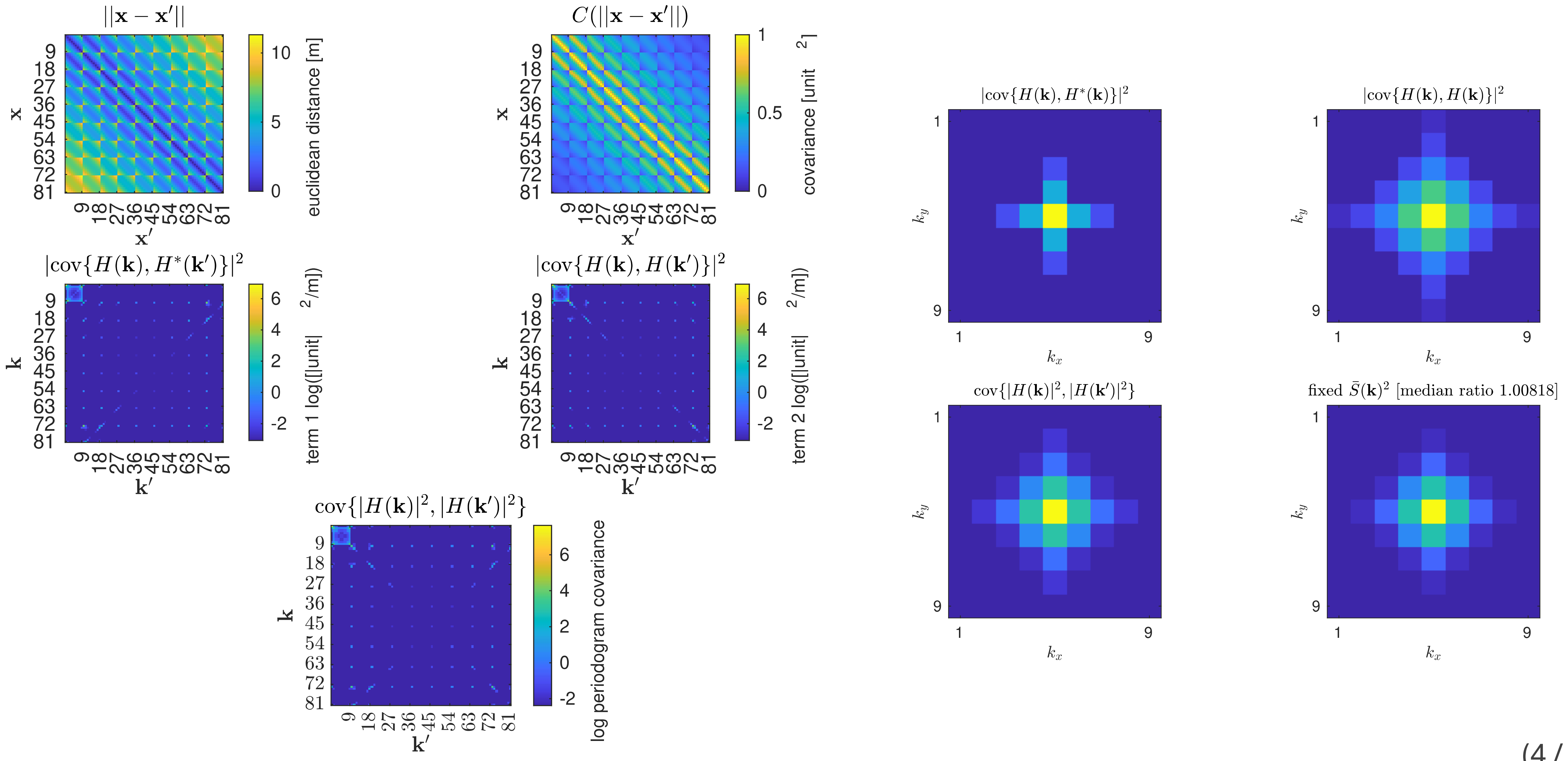
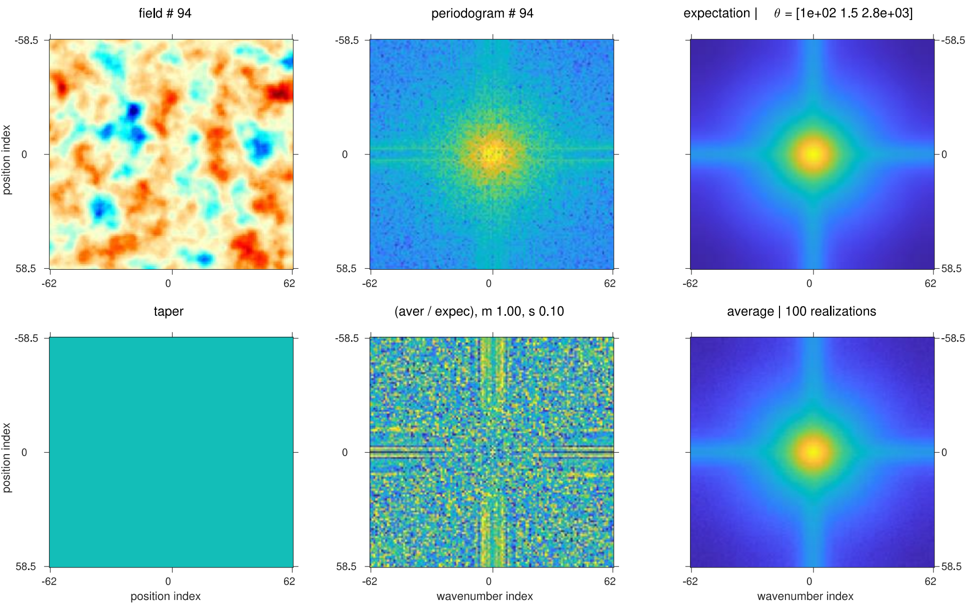
$$\mathrm{cov}\{|H(\mathbf{k})|^2, |H(\mathbf{k}')|^2\} = |\mathrm{cov}\{H(\mathbf{k}), H^*(\mathbf{k}')\}|^2 + |\mathrm{cov}\{H(\mathbf{k}), H(\mathbf{k}')\}|^2$$

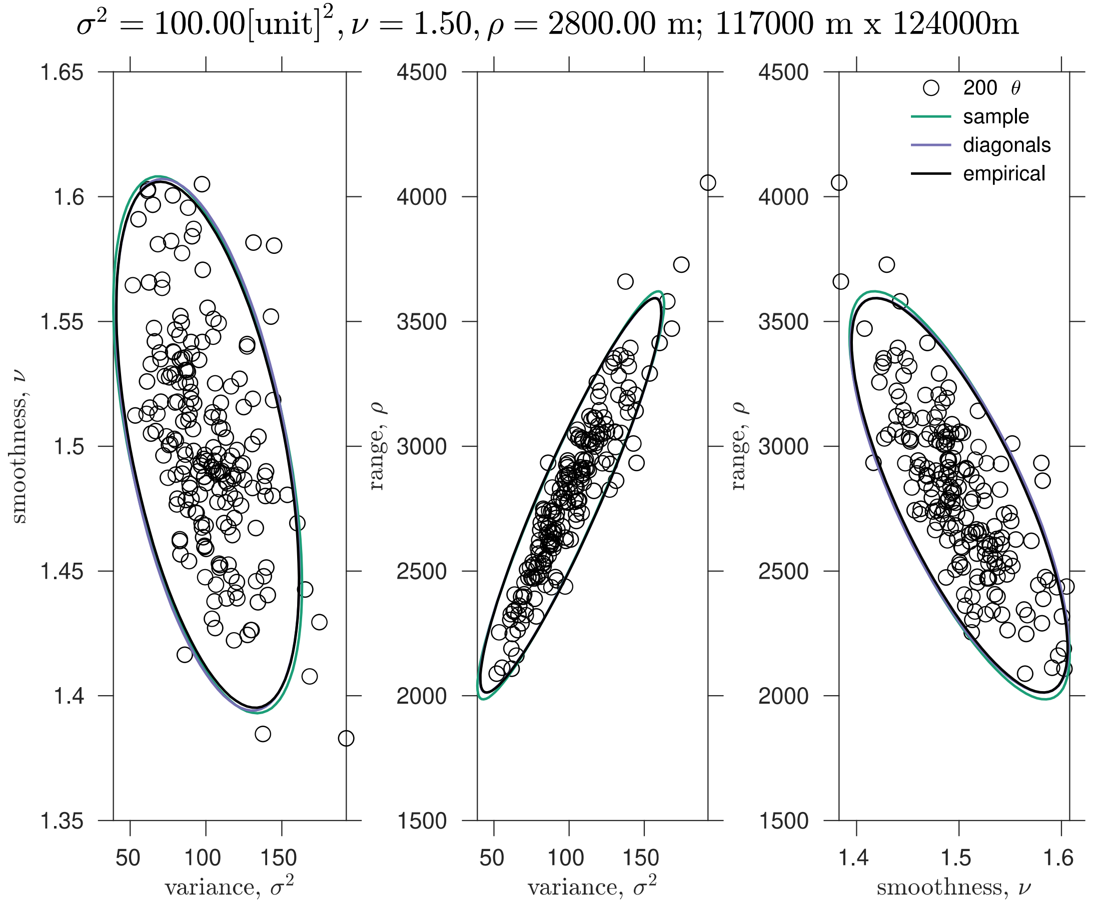
$$\mathrm{cov}\{|H(\mathbf{k})|^2, |H(\mathbf{k}')|^2\} = |\left(\mathbf{U}(\mathbf{U}\mathcal{C}_{\mathbf{x}}\mathbf{U}^T)^*\mathbf{U}\right)^*|^2 + |\mathbf{U}(\mathbf{U}\mathcal{C}_{\mathbf{x}}\mathbf{U}^T)\mathbf{U}|^2$$

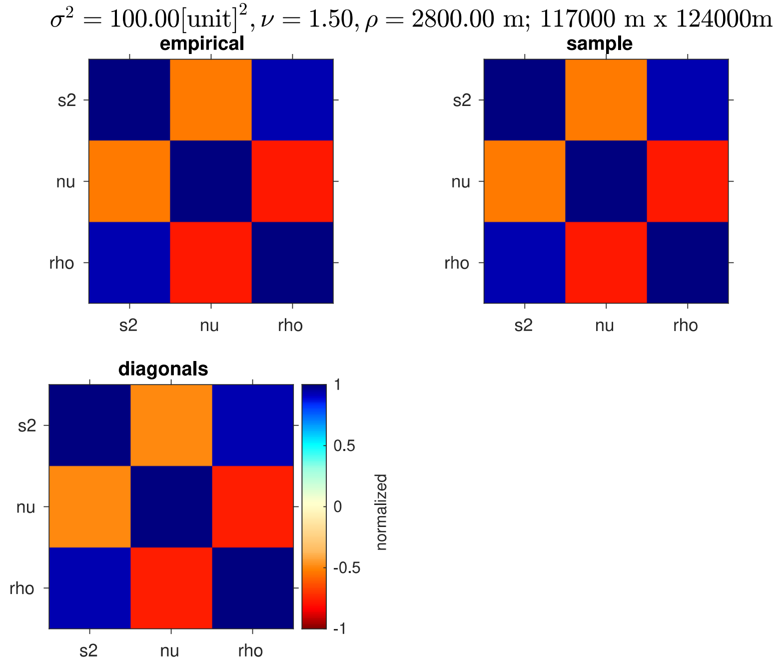


Estimator covariance behavior for fields with **regular sampling** (100% observed)



The estimator minimizes the debiased Whittle likelihood function, which compares the empirical periodogram with the blurred spectral density, both tapered by the data sampling.

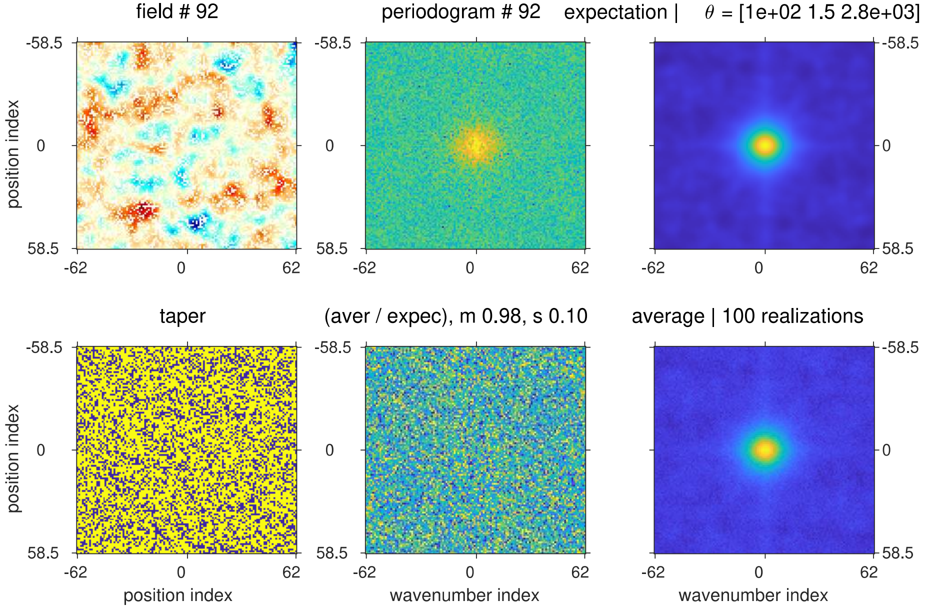




Correlation structure for each of the parameter covariance calculation methods.

Ensemble of estimates with error ellipses calculated from the singular values of the empirical and analytical parameter covariance.

Estimator covariance behavior for fields with **sparse sampling** (66.7% observed)



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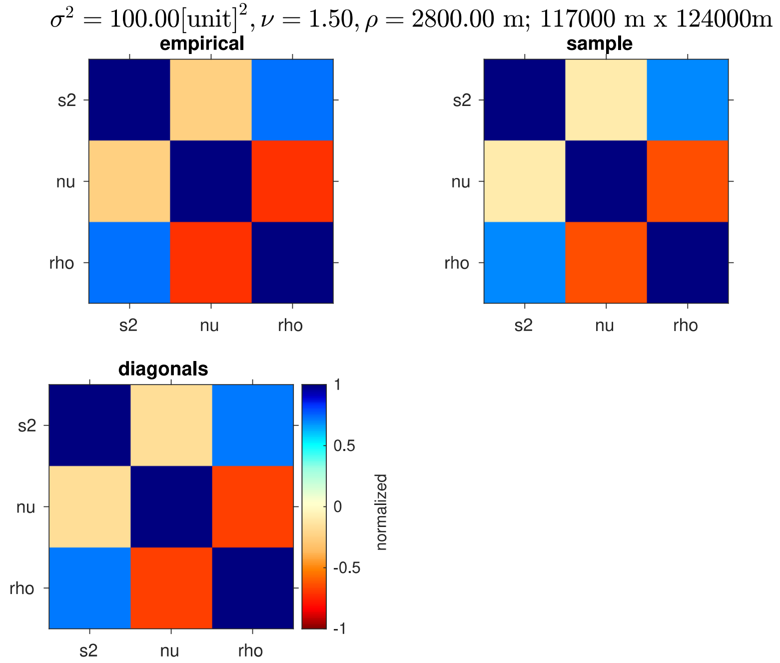

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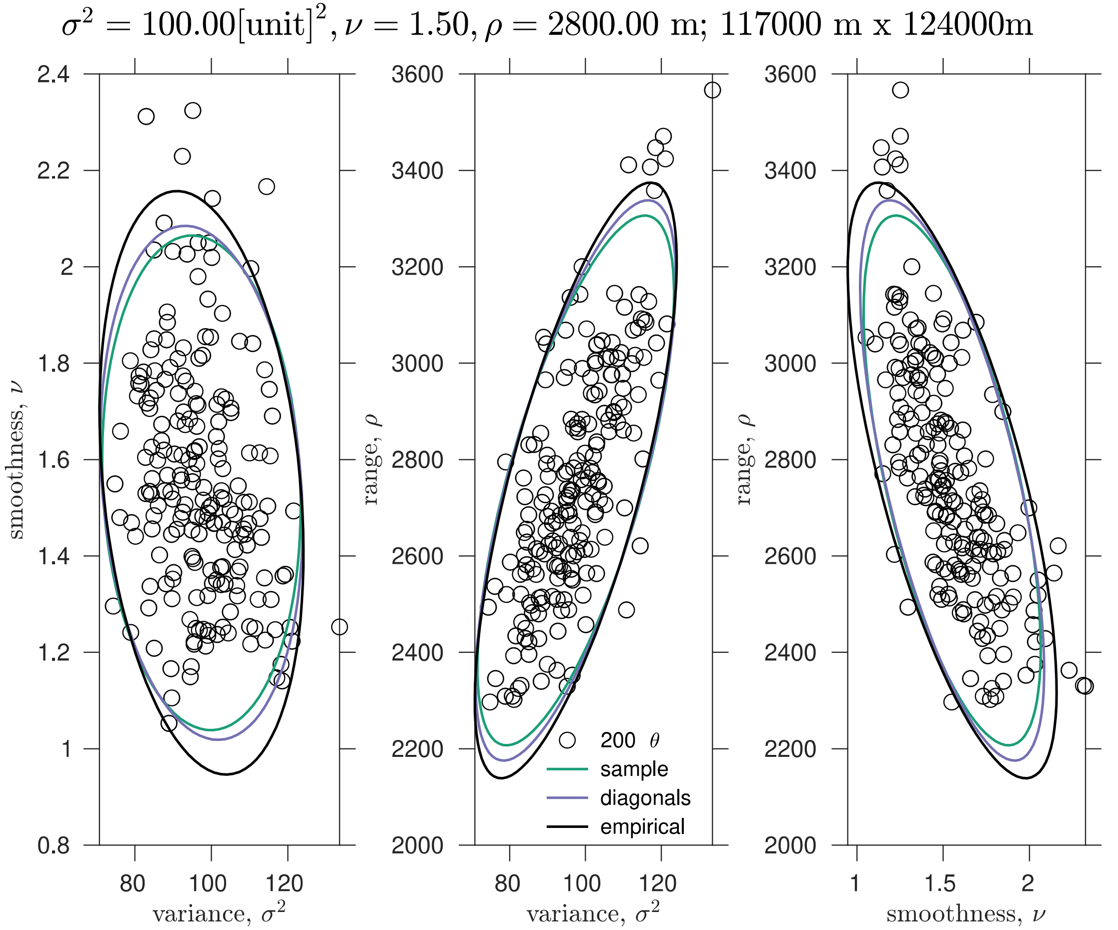

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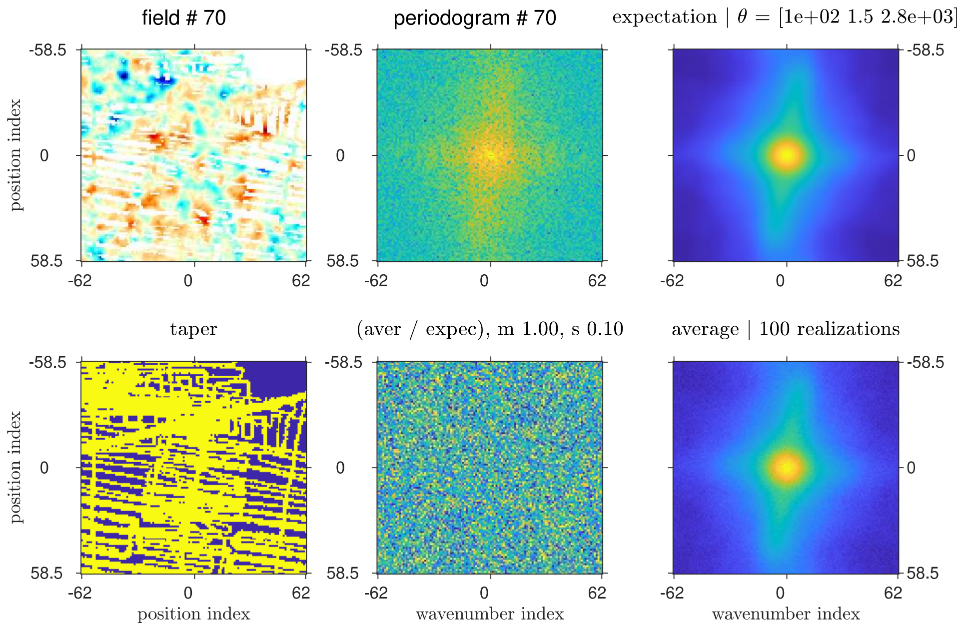
Estimator covariance behavior for fields with **structured sampling** (66.7% observed)



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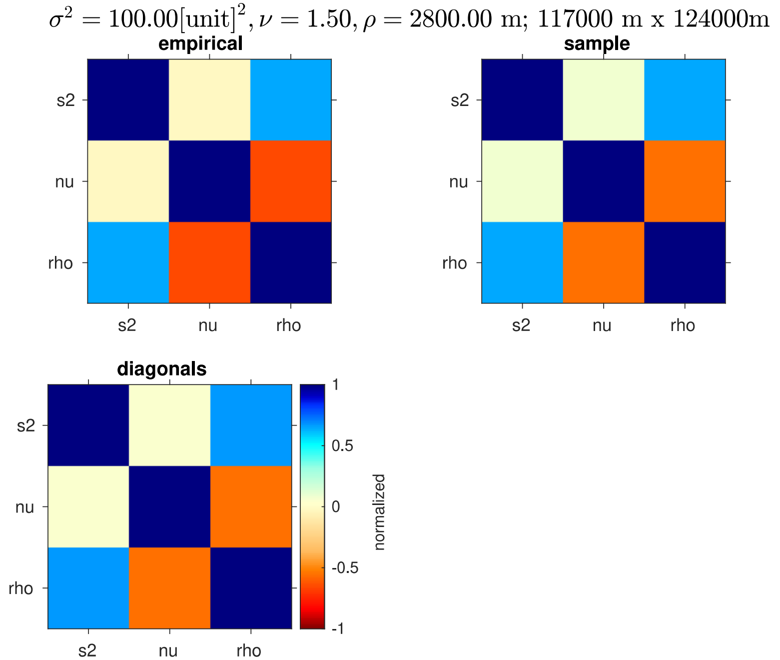

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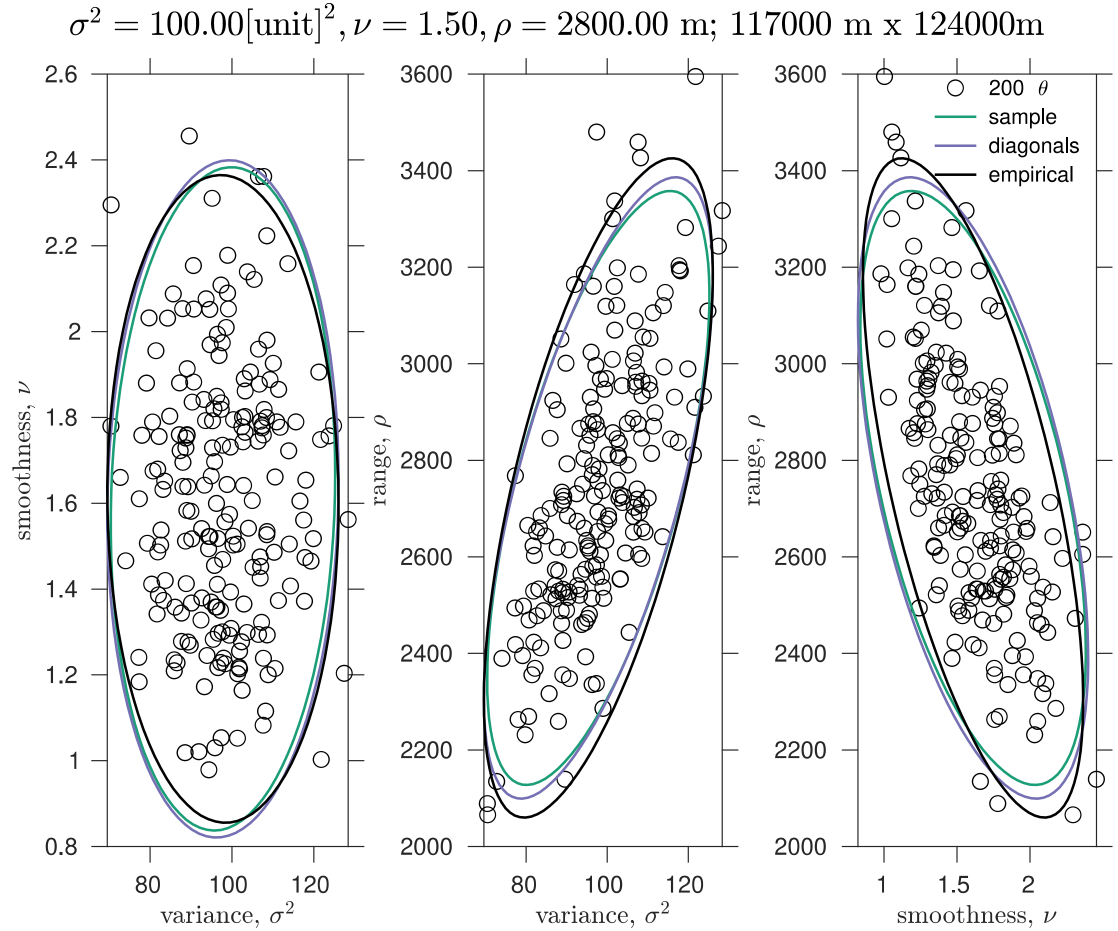
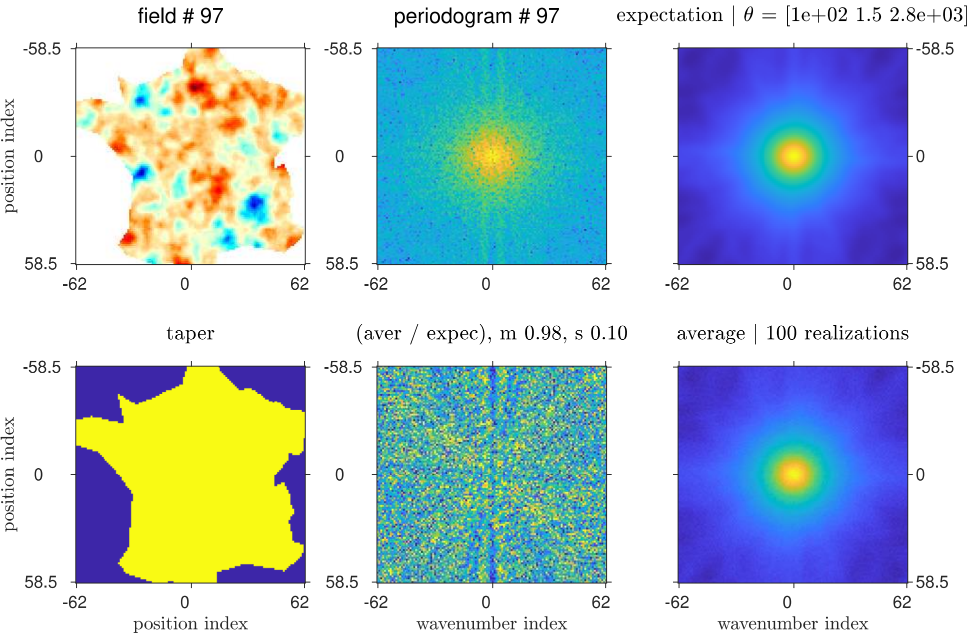


Ensemble of estimates with error ellipses calculated from the singular values of the empirical and analytical parameter covariance.

Estimator covariance behavior for fields with an **irregular boundary** (66.7% observed)



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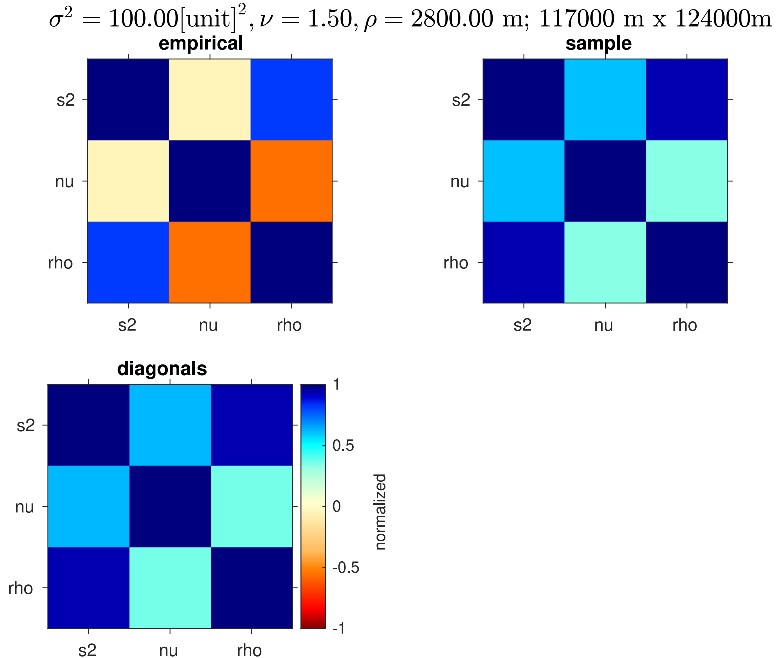

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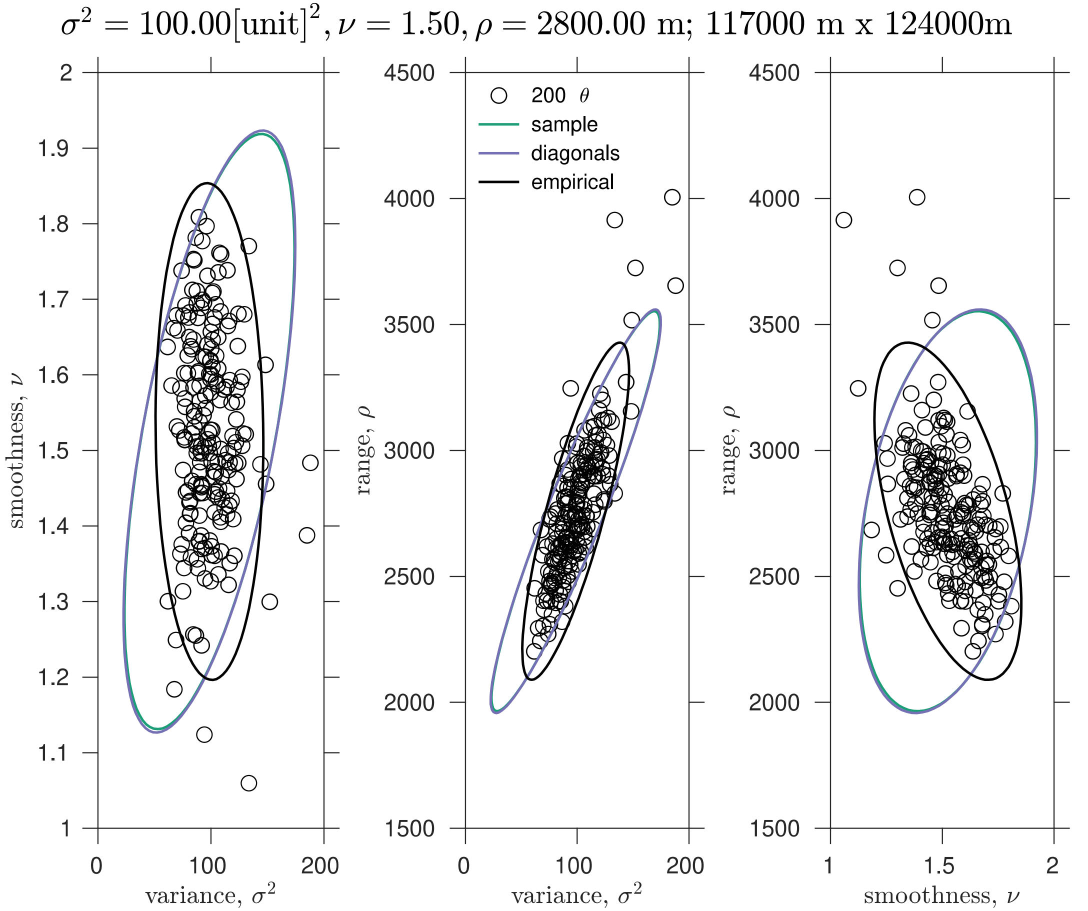

Ensemble of estimates with error ellipses calculated from the singular values of the empirical and analytical parameter covariance.