MAXIMUM-LIKELIHOOD ANALYSIS OF PLANETARY ROUGHNESS

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Introduction

What numbers "capture" the spatial statistics of topography? If it were a stationary, white, and Gaussian stochastic process: mean and variance would be sufficient. But "whiteness" is a strong assumption. Most analysis methods use varying "baselines" over which to compute means and variances. Those approaches subscribe to topography as a correlated process, and detailed knowledge requires the estimation of the parameters of its covariance function. We present a spectral-domain "Whittle" maximum-likelihood procedure for the estimation of the parameters of a Matérn form, whose parameters (variance, smoothness, range) define the shape of the covariance function. We treat finite sampling effects in simulation and estimation. We determine the estimation variance of all parameters. The "best" estimate may not be "good enough"; we test whether the "model" itself warrants rejection. We illustrate the methodology on geologically mapped patches of Venus.

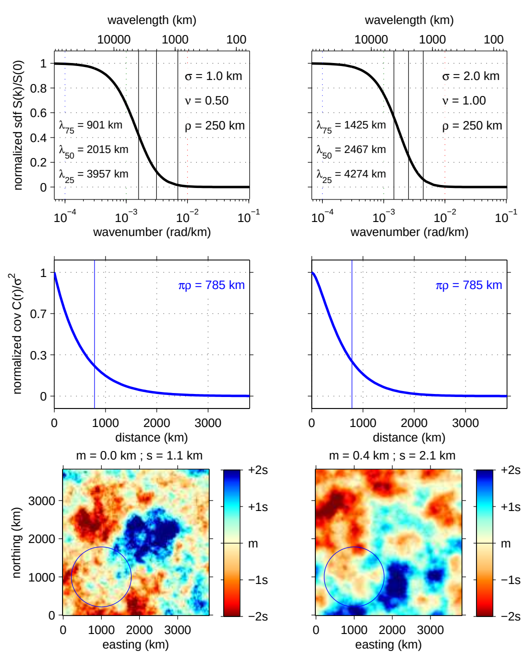

Figure 1: Isotropic random fields generated from Matérn covariance structures with specific variances σ^2, differentiabilities (smoothnesses) ν, and correlation lengths (ranges) ρ, as indicated. (*Top:*) Normalized spectral densities. (*Middle:*) Normalized spatial covariances. (*Bottom:*) Field realizations.

Matérn Model

We draw on Sections 2.1 and Appendix A.6 of [1]. Referring furthermore to [2], [3], and [4], we define the relation between the spatial and spectral topographical quantities of interest, $\mathcal{H}(\mathbf{x})$ and $d\mathcal{H}(\mathbf{k})$.

The two-dimensional (demeaned) planetary topography $\mathcal{H}(\mathbf{x})$ is considered to be a stationary, zero-mean, Gaussian random field. We assume there exists a spectral increment process $d\mathcal{H}(\mathbf{k})$ according to which

$$\mathcal{H}(\mathbf{x}) = \iint e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathcal{H}(\mathbf{k}). \qquad (1)$$

The integration is over the space of all wave vectors $\mathbf{k}$.

The power-spectral density $\mathcal{S}(\mathbf{k})$ relates to the spectral covariance of the process $d\mathcal{H}(\mathbf{k})$ via

$$\langle d\mathcal{H}(\mathbf{k})\, d\mathcal{H}^*(\mathbf{k}')\rangle = \mathcal{S}(\mathbf{k})\, d\mathbf{k}\, d\mathbf{k}'\, \delta(\mathbf{k},\mathbf{k}'). \qquad (2)$$

In the special case of isotropy, $\mathcal{S}(\mathbf{k}) = \mathcal{S}(k)$ only depends on the scalar wavenumber $k = \|\mathbf{k}\|$, and by integrating over the polar angles, the spatial covariance $\mathcal{C}(\|\mathbf{x}-\mathbf{x}'\|)$ is dependent on distance, not direction, and

$$\langle \mathcal{H}(\mathbf{x})\mathcal{H}^*(\mathbf{x})\rangle = 2\pi \int \mathcal{S}(k)\, k\, dk = \mathcal{C}(0) = \sigma^2. \qquad (3)$$

We model planetary topography as a member of the isotropic Matérn spectral class. Under this very general and widely applied model, the isotropic two-dimensional spectral density $\mathcal{S}(k)$ assumes the parameterized form

$$\mathcal{S}(k) = \frac{\sigma^2 \nu^{\nu+1} 4^\nu}{\pi(\pi\rho)^{2\nu}} \left(\frac{4\nu}{\pi^2\rho^2} + k^2\right)^{-\nu-1}. \qquad (4)$$

See Fig. 1. Our unknowns will be the three Matérn parameters, generically denoted $\theta > 0$, collected in the set

$$\boldsymbol{\theta} = [\sigma^2\ \nu\ \rho]^T. \qquad (5)$$

The 'variance', σ^2, indeed satisfies eq. (3). At short spatial wavelengths, when k grows large, the spectrum $\mathcal{S}(k)$ decays at a rate that depends on ν, which controls the mean-squared differentiability of the process. We refer to ν as the 'smoothness', and ρ is the 'range'. The behavior at the longest spatial wavelengths, for small k, is controlled by the combined efffect of σ^2 and ρ. Normalized for the variance, the 'fluctuation scale' is

$$\frac{\mathcal{S}(0)}{\sigma^2} = \frac{\pi\rho^2}{4}. \qquad (6)$$

Estimation Technique

The discrete Fourier transform of the measurements of $\mathcal{H}(\mathbf{x})$ obtained on a rectangular $M \times N$ grid is

$$H(\mathbf{k}) \equiv \frac{1}{2\pi}\left(\frac{\Delta x \Delta y}{MN}\right)^{\frac{1}{2}} \sum_{\mathbf{x}} \mathcal{H}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}, \qquad (7)$$

and on this Nyquist grid we identify eq. (1) with

$$\mathcal{H}(\mathbf{x}) \equiv \frac{2\pi}{(MN\Delta x \Delta y)^{\frac{1}{2}}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} H(\mathbf{k}). \qquad (8)$$

The likelihood $\bar{\mathcal{L}}(\boldsymbol{\theta})$ of observing the data $\mathcal{H}(\mathbf{x})$, under the spectral model (4), is given in terms of the Fourier coefficients $H(\mathbf{k})$ in eq. (7) and a version of eq. (2), blurred into a new quantity $\bar{\mathcal{S}}$ that incorporates grid effects. Summed over all the wavenumbers K, we have

$$\bar{\mathcal{L}}(\boldsymbol{\theta}) = -\frac{1}{K} \sum_{\mathbf{k}} \left[\ln \bar{\mathcal{S}}(k) + \bar{\mathcal{S}}^{-1}(k)\, |H(\mathbf{k})|^2\right]. \qquad (9)$$

Eq. (9) is the quantity that we maximize under positivity constraints for the parameter vector $\boldsymbol{\theta}$, defining the maximum-likelihood estimate $\hat{\boldsymbol{\theta}}$ to be the maximizer of the vector $\bar{\boldsymbol{\gamma}}$ of first derivatives of the blurred likelihood,

$$\bar{\boldsymbol{\gamma}}(\hat{\boldsymbol{\theta}}) = \mathbf{0}. \qquad (10)$$

Performance

Fig. 2 shows the behavior of the maximum-likelihood estimators of the Matérn parameters (σ^2, ν, ρ) over 40 lattice simulations conducted on square data fields composed of up to $M = N = 128$ pixels of equal size $\Delta x = \Delta y = 10$ km. With growing field size, the estimates reveal themselves to be asymptotically unbiased. Moreover, they are asymptotically normally distributed.

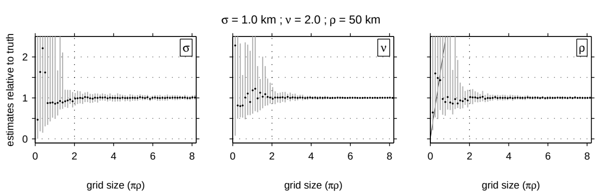

Figure 2: Behavior of the maximum-likelihood estimators for the true values listed in the title. Grey bars: 5–95th percentiles of the estimates. Black filled circles: mean estimates. The slanted line in the righmost panel is the data patch size.

Furthermore, we test that the distributions of the residuals, the quadratic form in eq. (9), are

$$X(\mathbf{k}) = \bar{\mathcal{S}}^{-1}(k)\, |H(\mathbf{k})|^2 \sim \chi^2_2/2. \qquad (11)$$

Application to Venus

We determined the Matérn parameters applicable to the topography of Venus, and found that the model holds across the planetary surface. Figs 3 and 4 show generic examples. The small smoothnesses ν indicate that the shape of the spatial covariance function is far from Gaussian, as might be assumed in other techniques, which would, if they assumed such a squared-exponential form for the covariance, seek to determine best-fitting parameters for a model that ought to be rejected from the start.

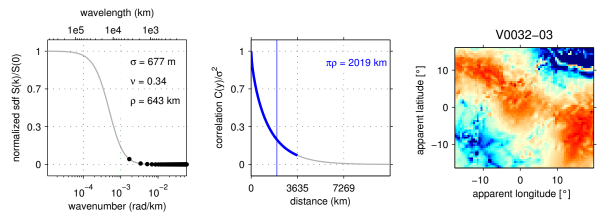

Figure 3: Matérn parameters, spectral and spatial covariance functions determined by maximum-likelihood analysis over the Venusian region *Laimdota Planitia*.

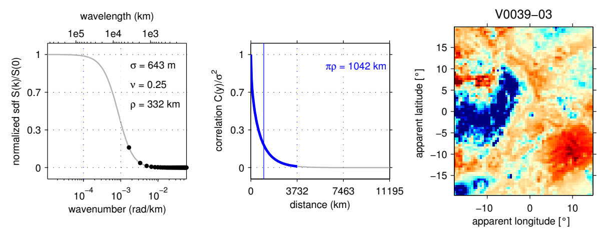

Figure 4: Matérn parameters, spectral and spatial covariance functions determined by maximum-likelihood analysis over the Venusian region *Lowana Planitia*.

Summary and Discussion

We have developed an efficient computational method for the characterization of the spatial statistics of planetary data fields. The model is a Matérn form which flexibly parameterizes a suite of covariance shapes with differing variances, smoothnesses, and decorrelation lengths. Ongoing work involves the classification of the entire Venusian terrain to attach geological significance to the varying statistical regimes.

References

[1] F. J. Simons & S. C. Olhede (2013) *Geophys. J. Int.*, doi: 10.1093/gji/ggt056. [2] D. B. Percival & A. T. Walden (1993) *Cambridge U. Press*, NY. [3] M. L. Stein (1999) *Springer Ser. Statistics*, NY. [4] E. Vanmarcke (2010) *World Scientific*, Singapore