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2014

Fundamentals of Continuum Mechanics
Fall Term 1973

I. Introductory Remarks

Matter is made up of atoms. In many situations, especially in many geophysical situations, this well-known fact is of little importance.
A continuum is a mathematical idealization of a very large number of atoms or molecules in which nearby atoms behave alike.
For many (but not all) purposes, mantle, core, atmosphere, oceans, rocks in lab, steel girders, and lumps of jello can be treated as continua

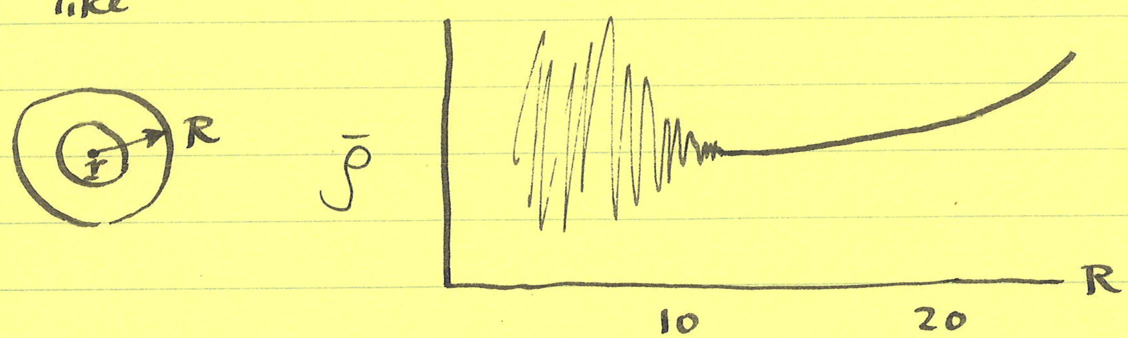
A continuum is a whole region of space occupied by continuous matter.

The study of continua is the study of fields. We think of matter not as a collection (a lattice or a jumble) of atoms, but rather as a continuous field. We will investigate the response of this matter field to external influences which are also fields.

Consider the nature of this approximation to real materials. In a <u>real</u> material made up of atoms we can compute mean <u>density</u> of any region V as $\bar{\rho} = \frac{1}{V}\sum m_i$, the sum over all atoms in V.

Say $V = \frac{4}{3}\pi R^3$

Choose a fixed point $\underline{r} \in V$, center V on $\underline{r}$ and let $R \to 0$. The mean density will look like



At first $\rho(\underline{r})$ tends to a limit, then fluctuates wildly. Units are mean interatomic distance.

We define the <u>mass density</u> $\rho(\underline{r}, t)$ of the continuum to be the limit. Such a defn not precise but useful if we are dealing with distances $\gg$ the interatomic distance.

When we speak of a <u>particle</u> in studying continua, we actually mean a macroparticle, a blob containing relatively many atoms. But we consider there to be such a macroparticle at every pt. $\underline{r}$ in that part of Euclidean 3-space occupied by the continuum. This is important to realize since we will later speak of internal degrees of freedom of the particles of a continuum

The field point of view is very useful, indeed predominant, in geophysics. We idealize matter as a field and consider its response (also described by fields) to such external influences as gravitational or electromagnetic fields. So long as the external influence and the response are such that nearby atoms really do behave roughly alike, this macroscopic field theory is a highly useful one. A macroscopic theory can however tell us nothing about the molecular const. of the continuum or about the way atoms interact: unsatisfying to the solid state physicist, but perfect for the seismologist, meteorologist, oceanographer, engineer.

Another name for our subject might be <u>macroscopic</u> <u>field</u> <u>theory</u>. We are interested in the interaction of gravitational and electromagnetic fields with macroscopic matter. Concepts of interest are <u>mechanical</u>, <u>thermodynamical</u>, and <u>electromagnetic</u>

<u>Mechanical concepts</u>: motion or flow fields, stress or force fields

<u>Thermodynamic concepts:</u> we are forced to consider, in general, e.g. it is well known that motion produces changes in temperature; temperature fields, entropy fields, energy fields, heat flow. The systematic inclusion of thermodynamics into the foundations of our theory is probably our most difficult task.

<u>Electromagnetic concepts</u>: the interaction of macroscopic matter with electromagnetic fields; $\exists$ many interesting thermoelectric, thermomagnetic, etc. effects. We will <u>not</u> study E+M; we will have our hands full just studying <u>thermomechanical</u> applications. The most important <u>geophysical</u>

problem we shall thus neglect is <u>magnetohydrodynamics</u>

Two remarks on the place of macroscopic field theory within physics

1. provides a useful idealization for problems involving macroscopic time and space scales. Problems involving <u>ionosphere</u> and above often require discussion of dist. and times > mean free path and times. Continuum description thus not adequate. One requires details about behavior of atoms which depend on fact that nearby atoms do <u>not</u> behave alike.

2. more importantly, as we shall see, a necessary element of continuum theory is the use of so-called <u>constitutive relations</u> which describe the physical properties of the type of continuum under discussion.

   <u>Example</u>: a good example is <u>Ohm's law</u> In rigid conductors, for small fields, it is a very good approx. that

$$\underline{J} = \sigma \underline{E}$$

σ is called the <u>electrical</u> <u>conductivity</u>. It is a physical property of the particular continuum under discussion. In a given discussion it is assumed known as a function of (say) temperature θ.

Ohm's law is not a law, merely a useful idealization in many circumstances. It will, e.g. break down at very large $\underline{E}$ fields

<u>Constitutive relations</u> are an essential ingredient of continuum theory. We will find when we write down all the <u>laws</u> of physics for continua, that we do <u>not</u> achieve a closed system of equations. We must append <u>constitutive</u> <u>relations</u> which <u>define</u> the physical nature of the particular type of continuum under discussion.

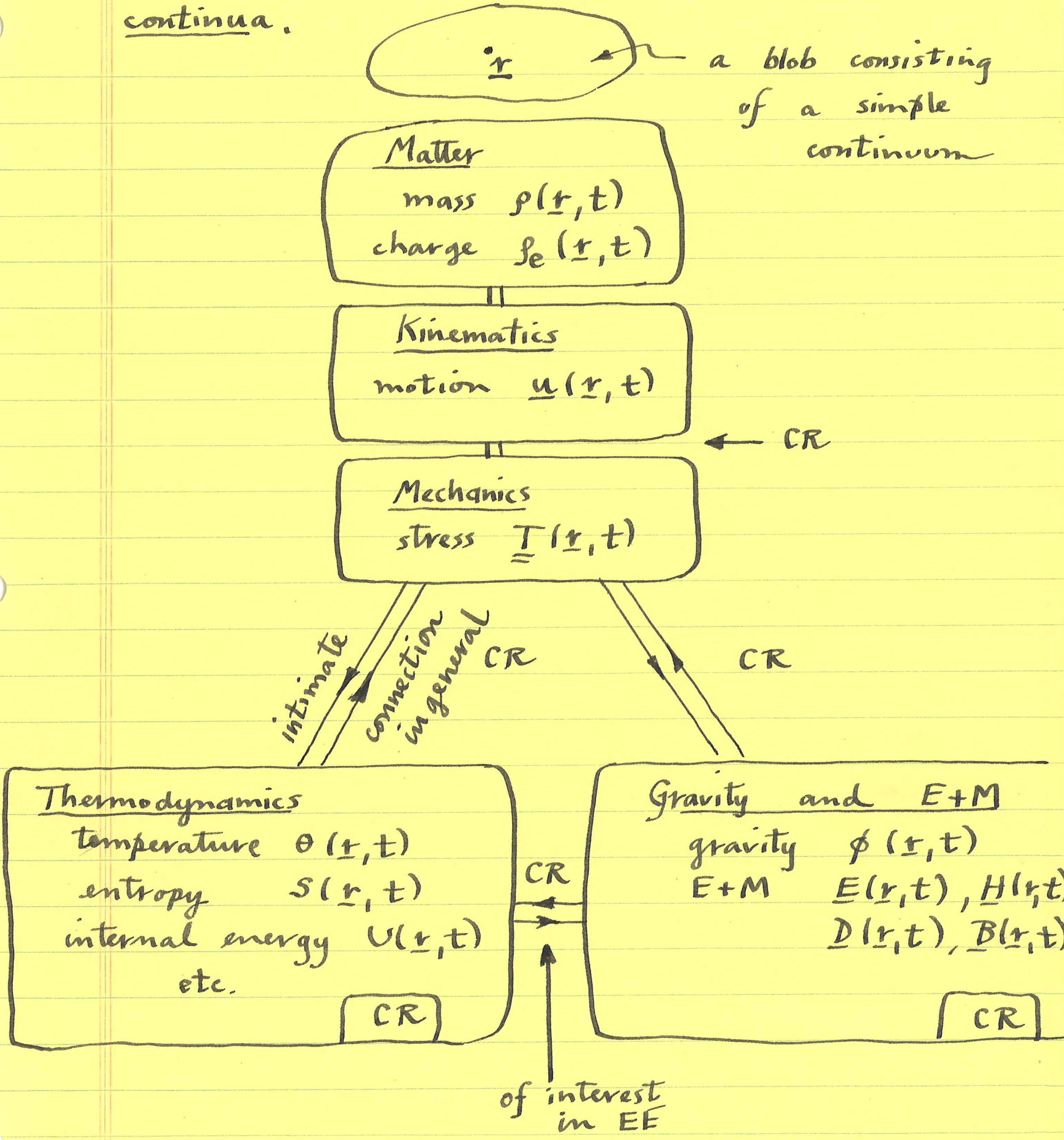
Here is the essential link between macroscopic field theory and atomic physics. Constitutive relations always involve <u>parameters</u> which are physical properties. In macroscopic physics these are assumed <u>known</u>

and characteristic of the material in question. (Stress-strain relations and thermodynamic equations of state provide other examples). A major goal of microscopic physics (solid state physics, kinetic theory of gases, e.g.) is to deduce these physical properties from first principles.

Examples: electrical and thermal conductivity of solids, dielectric and magnetic properties of solids, specific heats of solids and gases, molecular viscosity of gases, etc.

We will restrict discussion almost entirely to simple continua (single component, single phase). We will not have time to consider electromagnetic effects for lack of time. We will also not consider relativistic effects, of little application in geophysics (not so in astrophysics).

An outline of field theory for simple continua.



CR: require constitutive relations

Lecture two:

II. Review of mathematical tools:

List of important concepts (refer to fall term 1971 notes)

Vector space: $\underline{A} \cdot \underline{B}$ and $\underline{A} \times \underline{B}$

Arrays

Kronecker δ

Alternating symbol ___ aside on $\underline{A} \times \underline{B}$ not a pseudovector

Summation convention

Basis of a vector space

e.g. $\{0; \hat{x}_1, \hat{x}_2, \hat{x}_3\}$ for R_3

Inner Product Space

length of a vector

Schwarz and Δ inequalities

Orthogonality

Orthonormal basis

useful for computation p. 15

Change of orthonormal basis

e.g. in R_3 say $\exists$ a vector $\underline{v}$

$$v = v_j \hat{x}_j = v_i' \hat{x}_i'$$

then $v_i' = (\hat{x}_i' \cdot \hat{x}_j) v_j$

if we define an orthogonal matrix Q by $Q_{ij} = (x_i' \cdot \hat{x}_j)$

then $v_i' = Q_{ij} v_j$, or

$$\begin{pmatrix} v' \end{pmatrix} = \begin{pmatrix} Q \end{pmatrix} \begin{pmatrix} v \end{pmatrix}$$

Operations on Vector Spaces (Mappings)

$f: U \to V$

<u>Linear</u> Operators or Mappings

we will be especially concerned with

$f: R^3 \to R$ vectors

$f: R^3 \to R^3$ Tensors of order 2

Linear functionals on R^3

isomorphic to vectors $f(\underline{v}) = \underline{f} \cdot \underline{v}$

Multilinear functionals (tensors)

make a point of the <u>invariance</u> of the defn.

Four important operations

scalar mult. } vector space
sum }
tensor product
trace

This is all there is to <u>tensor algebra</u>

Cartesian components of tensors

A basis for $\otimes^q R^3$

Component arithmetic

Change of orthonormal basis p.73

Second order tensors as Linear Operators

important theorem on p. 75

example: constitutive relations (linear)

also, a <u>physical law</u> $\underline{L} = \underline{\underline{I}} \cdot \underline{\omega}$

e.g. $\underline{J} = \underline{\underline{\sigma}} \cdot \underline{E}$, or

$\underline{J} = \sigma(\underline{E})$ <u>linear operator</u>

The transpose

Operator Products

The Gibbs Notation (we will make constant use of this).

The components or the <u>matrix</u> of a linear
operator

<u>Inverse</u> of a linear operator

Eigenvalues and eigenvectors of a linear
operator

Three particular kinds of operators

1. antisymmetric

   the wedge operator: we will make use of this

   1-1 correspondence to vectors

2. orthogonal   det $= \pm 1$

   proper: rigid rotations

   improper: " " + inversion

Euler's theorem

3. symmetric

   3 real eivs

   Cart. axis system of eigenvectors

   positive definiteness

Polar Decomposition Theorem p. 117.6

<u>Norm</u> of a linear operator (has right properties p. 119)
(Can now do calculus)

<u>Small</u> Operators

Tensor differential calculus

Vector integral calculus

Vanishing integral theorem

Moving volume of integration.

~~[illegible]~~

Last time briefly reviewed a few basic formulae from vector and tensor integral calculus.

We used without defining expression $\nabla \underline{T}$ and $\nabla \cdot \underline{T}$ (grad and div of a tensor of order q.)
Last term we talked a great deal about tensor algebra but not at all about tensor calculus.
It is probably fairly obvious, but here we shall define very briefly $\nabla \underline{T}$ and $\nabla \cdot \underline{T}$, and show how one would go about computing them

Tools of tensor differential calculus:

Defn (gradient of a tensor field)
Let $T(\underline{r})$ be a tensor field of order q defined on R^3 (or some open set $V \in R^3$)
$T : R^3 \to \otimes^q R^3$
The field $T(\underline{r})$ is said to be differentiable at $\underline{r} \in V$ if $\exists$ a tensor ~~field of or~~ of order $q+1$ depending on $\underline{r}$, and denoted $\nabla T(\underline{r})$, such that

$$\boxed{T(\underline{r} + \underline{dr}) = T(\underline{r}) + \underline{dr} \cdot \nabla T(\underline{r}) + O(|dr|^2)}$$

note that in general $\underline{dr} \cdot \nabla T(\underline{r}) \neq \nabla T(\underline{r}) \cdot \underline{dr}$

<u>note the defn is</u> <u>coordinate-free</u>

$\left(O(|\underline{dr}|^2) \to \epsilon(\underline{r}, \underline{dr}) \ni \epsilon / |\underline{dr}| \to 0 \text{ as } |\underline{dr}| \to 0 \right)$

$\boxed{\text{note that if } T(\underline{r}) \text{ is } \cancel{\text{of}} \text{ of order } q \text{, then } \nabla T(\underline{r}) \text{ is of order } q+1}$

$\boxed{\begin{array}{l}\underline{\text{Defn}} \text{ (divergence of a tensor field). The notation } \nabla \cdot T(\underline{r}) \text{ will be used to denote } tr_{12}(\nabla T(\underline{r})) \\ \text{note that if } T(r) \text{ is of order } q, \ \nabla \cdot T(\underline{r}) \text{ is of order } q-1\end{array}}$

To <u>compute</u> $\nabla T(\underline{r})$ and $\nabla \cdot T(\underline{r})$

Choose a Cart. axis system $\hat{x}_1, \hat{x}_2, \hat{x}_3$

Then

$$T(r) = T_{j_1 \dots j_q}(\underline{r}) \, \hat{x}_{j_1} \dots \hat{x}_{j_q}$$

expanded in terms of 3^q polyads

then
$$\nabla T(\underline{r}) = \nabla T_{i j_1 \dots j_q} \, \hat{x}_i \hat{x}_{j_1} \dots \hat{x}_{j_q}$$

$$\boxed{\text{where} \quad \nabla T_{i j_1 \dots j_q} = \partial_i T_{j_1 \dots j_q}}$$

and
$$\nabla \cdot T(\underline{r}) = (\nabla \cdot T)_{j_1 \dots j_{q-1}} \, \hat{x}_{j_1} \dots \hat{x}_{j_{q-1}} \quad \text{where}$$

$$\boxed{(\nabla \cdot T)_{j_1 \dots j_{q-1}} = \partial_i T_{i j_1 \dots j_{q-1}}}$$

As an example, let $\underline{\underline{T}}(\underline{r})$ be of order two

$$\underline{\underline{T}}(\underline{r}) = T_{ij}(\underline{r})\,\hat{x}_i\,\hat{x}_j$$
$$\underline{\underline{\underline{\nabla T}}}(\underline{r}) = (\nabla T)_{kij}(\underline{r})\,\hat{x}_k\,\hat{x}_i\,\hat{x}_j$$
$$\nabla\cdot\underline{\underline{T}}(\underline{r}) = (\nabla\cdot T)_j(\underline{r})\,\hat{x}_j$$

where

$$(\nabla T)_{kij}(\underline{r}) = \partial_k T_{ij}(\underline{r})$$
$$(\nabla\cdot T)(\underline{r}) = \partial_i T_{ij}(\underline{r})$$

6. Tools of vector integral calculus

1. Gauss' theorem

Let V be a region with surface ∂V and outward unit normal $\hat{n}$. Let $\underline{v}(\underline{r})$ be a cont. differ. vector field in $V \cup \partial V$. Then

$$\int_V \nabla \cdot \underline{v} \, dV = \int_{\partial V} \hat{n} \cdot \underline{v} \, dA$$

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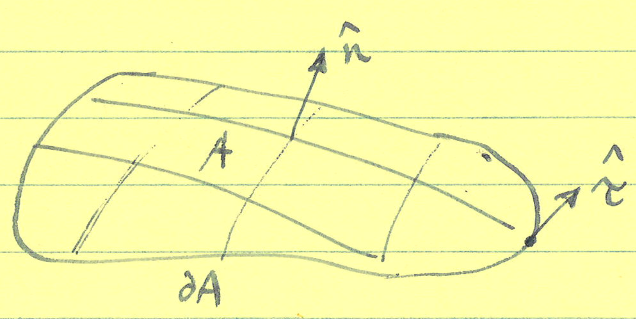
note: $\underline{v}$ may be replaced by $\underline{\underline{M}}$, a tensor field, note $\hat{n} \cdot \underline{\underline{M}} \neq \underline{\underline{M}} \cdot \hat{n}$
$\hat{n}$ replaced ∇

2. Stokes' theorem

Let A be a two sided surface with edge curve ∂A, let $\hat{n}$ be the unit normal one side of A and $\hat{\tau}$ be the unit tangent to ∂A $\Rightarrow$ $\hat{n} \times \hat{\tau}$ points into A. Let $\underline{v}(\underline{r})$ be a cont. differ. vector field on A. Then

right hand rule



$$\int_A (\nabla \times \underline{v}) \cdot \hat{n} \, dA = \int_{\partial A} v \cdot \hat{\tau} \, dl$$

<u>Gauss' theorem</u> is in fact valid for tensor fields K of any order

~~For a vector~~

Let V be a region, surface ∂V, normal $\hat{n}$

$K(\underline{r})$ a cont. differ $q+1$ – order tensor field in V. Then

$$\int_V \nabla \cdot K \, dV = \int_{\partial V} \hat{n} \cdot K \, dA$$

proof: choose a Cart. axis system $\hat{x}_1, \hat{x}_2, \hat{x}_3$

~~expan~~ generates an orthonormal basis for $\otimes^{q+1} R^3$

$$K = K_{i j_1 \cdots j_q} \hat{x}_i \hat{x}_{j_1} \cdots \hat{x}_{j_q}$$

define 3^q vectors $\underline{K}_{j_1 \cdots j_q} = K_{i j_1 \cdots j_q} \hat{x}_i$ (actually 3^q vectors)

Gauss' thm for vectors $\Rightarrow \int_V \nabla \cdot \underline{K}_{j_1 \cdots j_q} \, dV = \int_{\partial V} \hat{n} \cdot \underline{K}_{j_1 \cdots j_q} \, dA$

mult. by $\hat{x}_{j_1} \cdots \hat{x}_{j_q}$ and sum

$$\boxed{\int_V \nabla \cdot K \, dV = \int_{\partial V} \hat{n} \cdot K \, dA}$$

note that for K of second order or $>$, $\hat{n} \cdot \underline{\underline{K}} \neq \underline{\underline{K}} \cdot \hat{n}$

$\boxed{\text{important: replace } \nabla \cdot \text{ by } \hat{n}}$

3. Change of variables of integration

Suppose u^1, u^2, u^3 and v^1, v^2, v^3 are ~~the~~ two systems of curvilinear coordinates in a three-dim region V. Let $f(\underline{r})$ be a fcn defined for all $\underline{r} \in V$. Let V_u be the set of triples (u^1, u^2, u^3) belonging to pts in V and V_v the set of triples (v^1, v^2, v^3) belonging to pts in V (A system of curvilinear coords on V is a 1–1 mapping from some subset of R^3 onto V, with certain smoothness cond.

Define the <u>Jacobian determinant</u> $\dfrac{\partial(v^1, v^2, v^3)}{\partial(u^1, u^2, u^3)}$ to mean

$$\det \begin{vmatrix} \frac{\partial v^1}{\partial u^1} & \frac{\partial v^2}{\partial u_1} & \cdot \\ \frac{\partial v^1}{\partial u^2} & \cdot & \cdot \\ \cdot & \cdot & \frac{\partial v^3}{\partial u^3} \end{vmatrix}$$

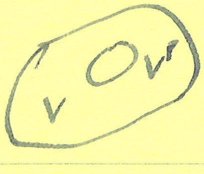
Then

$$\int_{V_v} f\, dv^1 dv^2 dv^3 = \int_{V_u} f \left| \frac{\partial(v_1, v_2, v_3)}{\partial(u_1, u_2, u_3)} \right| du_1 du_2 du_3$$

4. Vanishing integral theorem

If $f(\underline{r})$ is a cont. scalar fcn in a volume V and if for <u>every</u> volume V' contained inside V

$$\int_{V'} f(\underline{r})\, dV = 0$$



then $f(\underline{r}) = 0$ identically in V.

Consequently if $h(\underline{r})$ and $g(\underline{r})$ are two cont. scalar fcns in $V \ni$

$$\int_{V'} g(\underline{r})\,dV = \int_{V'} h(\underline{r})\,dV$$

for all $V' \in V$, then $g(\underline{r}) = h(\underline{r}) \quad \forall\ \underline{r} \in V$

proof: suppose that for some $\underline{r}_0 \in V$, $f(\underline{r}_0) > 0$. then by continuity of f $\exists$ $\epsilon > 0$ so small that if

$\|\underline{r} - \underline{r}_0\| < \epsilon$ then $|f(\underline{r}) - f(\underline{r}_0)| < \frac{1}{2} f(\underline{r}_0)$ but then

$f(\underline{r}_0) - f(\underline{r}) < \frac{1}{2} f(\underline{r}_0)$ so

$$f(\underline{r}) > \frac{1}{2} f(\underline{r}_0)$$

$\therefore$ $f(\underline{r}) > \frac{1}{2} f(\underline{r}_0)$ everywhere in the ball $S_\epsilon(\underline{r}_0)$

but then

$$\int_{S_\epsilon(\underline{r}_0)} f\,dV > \left[\tfrac{1}{2} f(\underline{r}_0)\right]\left[\text{volume of } S_\epsilon(\underline{r}_0)\right]$$

$$= \frac{2\pi}{3} \epsilon^3 f(\underline{r}_0) > 0, \quad \text{a contradiction}$$

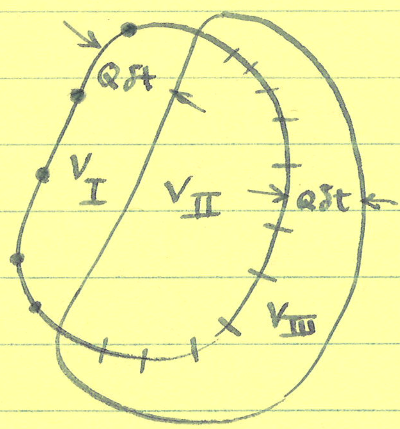
5. time derivative of an integral over a moving volume.

Suppose that a fcn $f(\underline{r}, t)$ is given. Suppose that a region $V(t)$ is given which depends on time in such a way that at any instant at any point $\underline{r}$ on the surface $\partial V(t)$, $\partial V(t)$ is moving outward $\perp$ to itself with speed $Q(\underline{r}, t)$

$$\boxed{Q(\underline{r},t) \equiv \text{normal outward velocity}}$$

then

$$\boxed{\frac{d}{dt}\int_{V(t)} f(\underline{r},t)\,dV = \int_{V(t)} \frac{\partial f(\underline{r},t)}{\partial t}\,dV + \int_{\partial V(t)} f(\underline{r},t)\,Q(\underline{r},t)\,dA}$$



proof: we want to compute

$$\lim_{\delta t \to 0} \frac{1}{\delta t}\left[\int_{V(t+\delta t)} f(\underline{r},t+\delta t)\,dV - \int_{V(t)} f(\underline{r},t)\,dt\right]$$

now $\displaystyle\int_{V(t+\delta t)} f\,dV = \int_{V_{II}} f\,dV + \int_{V_{III}} f\,dV$ and

$$\int_{V(t)} f\,dV = \int_{V_{II}} f\,dV + \int_{V_{I}} f\,dV$$

thus

$$\int_{V(t+\delta t)} f(\underline{r},t+\delta t)\,dV - \int_{V(t)} f(\underline{r},t)\,dV =$$

$$\int_{V_{II}} [f(\underline{r},t+\delta t) - f(\underline{r},t)]\,dV$$

$$+ \int_{V_{III}} f(\underline{r},t+\delta t)\,dV - \int_{V_{I}} f(\underline{r},t)\,dV$$

furthermore $\int_{V_{III}} f\,dV \cong \int_{A_{III}} f\,Q\,\delta t\,dA$ where

A_{III} is marked

and $\int_{V_I} f\,dV \cong -\int_{A_I} f\,Q\,\delta t\,dA$ where A_I is

thus

$$\frac{1}{\delta t}\left[\int_{V(t+\delta t)} f(\underline{r}, t+\delta t) - \int_{V(t)} f(\underline{r}, t)\,dV\right] =$$

$$\int_{V_{II}} \left[\frac{f(\underline{r}, t+\delta t) - f(\underline{r}, t)}{\delta t}\right] dV$$

$$+ \int_{A} f\,Q\,dA + \text{terms or order } \delta t$$

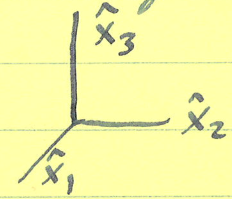
now take limit as $\delta t \to 0$ and result follows.

III. Kinematics of a continuum

We discuss first: how to describe the motion of a continuum.
We begin with

2. Kinematics of a pt. mass

We restrict ourselves for remainder of term to <u>non-relativistic</u> limit



pick origin $t = 0$ of time

origin O of space, and Cart. axis system $\hat{x}_1, \hat{x}_2, \hat{x}_3$

the motion of a pt. mass is completely described by 3 coordinates as a fcn of time

$x_1(t)$, $x_2(t)$, $x_3(t)$ or a vector posn fcn of time

$$\underline{r}(t) = x_1(t)\,\hat{x}_1 + x_2(t)\,\hat{x}_2 + x_3(t)\,\hat{x}_3 = x_i(t)\,\hat{x}_i$$

<u>By defn</u> a <u>point mass</u> is a physical object <u>completely</u> described by

1. its mass m
2. its position $\underline{r}(t)$

Otherwise an object <u>not</u> a pt. mass, e.g. a rigid body requires also inertia tensor + 3 Euler angles as fcn(t)

no pt. masses in nature – good approx for some problems, e.g. motion of a planet about $\odot$

The history of N pt. masses is completely known if N vector-valued fcns of time are given

$\underline{r}_n(t)$ $\quad n = 1, \cdots N$

$\underline{r}_n(t) \equiv$ posn of n^{th} particle at time t.

3. Kinematics of a continuum. <u>Lagrangian description</u>

A continuum is a region of space filled with continuous matter. Its history of motion is completely described by giving the position of each one of its (infinitesimal – on our scale, not on an atomic scale) as fcn of time.
We require an ∞ no. of particle <u>labels</u>.
Since nearby particles behave roughly alike we <u>label</u> particles not arbitrarily but by the positions they occupied in space at time $t=0$.

$\boxed{\text{The particle labeled } \underline{x} \text{ was at posn } \underline{x} \text{ at } t=0}$

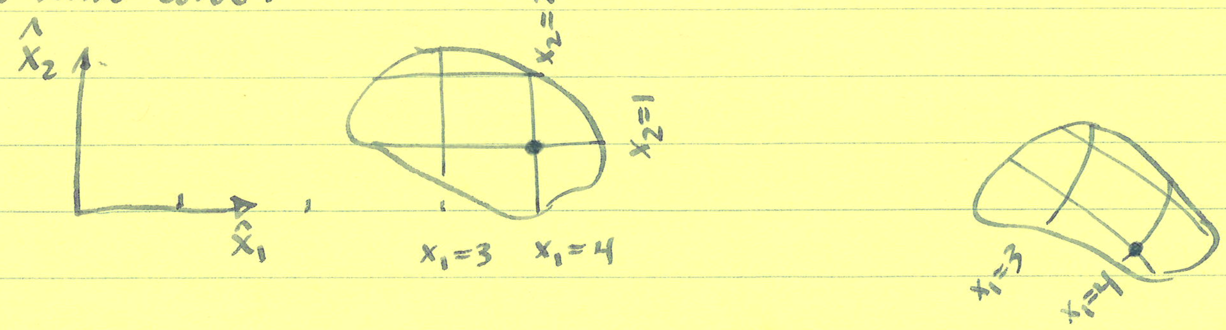
The history of motion is completely described by

$\underline{r}_{\underline{x}}(t)$ for each particle $\underline{x} \in V$

i.e. the position vector at time t / $\underline{r}_{\underline{x}}(t)$ of the particle at $\underline{x}$ at $t=0$
We will write this $\underline{r}(\underline{x}, t)$

$\underline{r}(\underline{x}, t) \equiv$ posn occupied at ~~this~~ time t by the inf. particle which was at $\underline{x}$ at time $t=0$.
The vector $\underline{x}$ labels the particle throughout its history.

We can draw a picture of the motion of a 2-dim cont.



at $t=0$ we paint coord lines into the material

dot • ~~ ~~ is particle $x_1=4$, $x_2=1$

At t its posn is $r_1 \cong 9$, $r_2 \simeq -1$

But its label remains the same throughout the motion

delete for a moment

At any time t, the velocity vector $\underline{u}$ of the particle labelled $\underline{x}$ is the time deriv. of its posn vector. In carrying out the differ. we must hold $\underline{x}$ fixed as we are ~~ ~~ watching the particle labelled $\underline{x}$. Use D/Dt for reasons to be explained later to denote this

$$\boxed{\begin{aligned} \underline{u}(\underline{x},t) &= \frac{D}{Dt}\underline{r}(\underline{x},t) \\ u_i(\underline{x},t) &= \frac{D}{Dt} r_i(\underline{x},t) \end{aligned}}$$

or

$$\boxed{D/Dt \equiv \left(\frac{\partial}{\partial t}\right)_x}$$

The acceleration vector $\underline{a}(\underline{x},t)$ of the particle $\underline{x}$ is

$$\boxed{\underline{a}(\underline{x},t) = \frac{D}{Dt}\underline{u}(\underline{x},t) = \frac{D^2}{Dt^2}\underline{r}(\underline{x},t)}$$

in component form $\quad a_i(\underline{x},t) = \frac{D}{Dt} u_i(\underline{x},t)$
$$= \frac{D^2}{Dt^2} r_i(\underline{x},t)$$

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This is not quite yet our defn of a continuum.
Not all vector valued fcns $\underline{r}(\underline{x},t)$ describe the motion of a continuum.

$$\boxed{\text{By definition} \quad \underline{r}(\underline{x},0) = \underline{x}}$$

Also we assume, as part of the defn of a continuum that except at a few isolated places (points, lines, surfaces)

$$\boxed{\underline{r}(\underline{x},t) \text{ is cont. differ. as fcn of } \underline{x} \text{ \& } t}$$

this amounts to assuming the material does not tear and does not develop kinks

We assume the three fcns $r_i(x_1, x_2, x_3, t)$ have partial derivatives $\partial r_i/\partial x_j$ and $\partial r_i/\partial t = Dr_i/Dt$

We further assume that when r_1, r_2, r_3, t are given, $\exists$ exactly one solution to the three simult. eqns

$$r_1(x_1, x_2, x_3, t) = r_1$$
$$r_2(x_1, x_2, x_3, t) = r_2$$
$$r_3(x_1, x_2, x_3, t) = r_3$$

i.e. in vector notation
given $\underline{r}$ and t, $\exists$ ~~only~~ exactly one soln to $r(x, t) = \underline{r}$

physically this means two particles $\underline{x}_1$, $\underline{x}_2$ different at $t=0$ can never occupy the same position at a later time: two particles can't be at the same place at the same time

Now given $\underline{r}$ and t let $\underline{x}$ be the unique soln to $\underline{r}(\underline{x}, t) = \underline{r}$, then $\underline{x} = \underline{x}(\underline{r}, t)$ ~~x~~ since $\underline{x}$ depends only on $\underline{r}$ and t

We assume further that $\underline{x}(\underline{r}, t) = \underline{x}(r_1, r_2, r_3, t)$ is cont. differ.

no kinking or tearing if cont. is run backward in time.

Remark: if we know where a particle was at any time t_0 we can find where it will be at t_1

if it was at $\mathbf{r}_0$ at t_0 we solve

$$\mathbf{r}(\mathbf{x}, t_0) = \mathbf{r}_0 \quad \text{for } \mathbf{x} \text{ obtaining}$$

$\mathbf{x}(\mathbf{r}_0, t_0)$. Then the position of the particle ($\mathbf{x}$) at t_1 is

$$\mathbf{r}_1 = \mathbf{r}\left(\mathbf{x}(\mathbf{r}_0, t_0), t_1\right)$$

we call our ~~three~~ conditions the identity condition, separation condition, and the smoothness condition.

Now define velocity $\mathbf{u}(\mathbf{x}, t)$ and acceleration $\mathbf{a}(\mathbf{x}, t)$ of the $\mathbf{x}$ particle

4. Kinematics of a continuum. The Eulerian description

Lagrangian is ~~[illegible]~~ seismometer descr.

If the position $\mathbf{r}(\mathbf{x}, t)$ is known for all t and all particles $\mathbf{x}$ we have the so-called Lagrangian description of the motion.

The weather bureau description (incomplete) provides the velocity $\mathbf{u}$ of the wind at all pts $\mathbf{r}$ in space for all t. This is the Eulerian description of the motion.

great interest in history of continuum mechanics

Eulerian description $\underline{u}_E(\underline{r},t) \equiv$ the velocity $\underline{u}$ at time t of that particle which is at posn $\underline{r}$ at time t.

Lagrangian: attn centered on posn of ind. labeled particles as they move about

Eulerian: we watch the velocities of the various material particles as they move by us.

→ Truesdell: Lag. descr. by Euler 1762, Eulerian by d'Alembert 1752

Theorem: the Lagrangian and Eulerian descriptions are equivalent in that each may be obtained from the other.

we focus attn on a part. p. R

say we desire $\underline{u}(\underline{r},t)$ at $\underline{r}=\underline{R}$

The fcn $\underline{u}_L(\underline{x},t) = \frac{\partial}{\partial t}\underline{r}(\underline{x},t)$ previously introduced is not the Eulerian description since $\underline{x}$ is the particle label (the initial posn) rather than the present position.

But to obtain Eulerian description we simply calc. $\underline{r}$ from $\underline{x}$ by solving $\underline{r}(\underline{x},t)=\underline{R}$ for $\underline{x}=\underline{X}$

$$\boxed{\begin{aligned} &\underline{u} = \underline{u}_E(\underline{R},t) = \underline{u}_L(\underline{X}(\underline{R},t),t) \quad \text{or} \\ &\underline{u} = \underline{u}_E(\underline{R}(\underline{X},t),t) = \underline{u}_L(\underline{X},t) \end{aligned}}$$

$\underline{u}_E$ and $\underline{u}_L$ are different fcns, $\underline{u}_E$ does not have the same functional dependence on $\underline{R},t$

Focus attn on the particle at $\underline{X}$ at time $t=0$

as $\underline{u}_L$ does on $\underline{x}, t$.

Hence given the Lagrangean description $\underline{r}(\underline{x}, t)$
we can find Eulerian $\underline{u}(\underline{r}, t)$ at any $\underline{r}$ ($\underline{R}$ was arbitrary)
How about vice-versa

Given $\underline{u}(\underline{r}, t)$ to determine $\underline{r}(\underline{x}, t)$
Focus attn on a particular fluid particle at $\underline{X}$ at $t=0$. At t its posn is $\underline{R}(t)$ and its velocity $d\underline{R}/dt$.
$\underline{R}(t)$ is a fcn of t only as we fix attention on only one fluid particle

But from the defn of u_E, its velocity is also $\underline{u}_E(\underline{R}, t)$

$$\boxed{d\underline{R}/dt = u_E(\underline{R}, t)}$$

or in component form

$$\frac{dR^1}{dt} = u_E^{\,1}(R^1(t), R^2(t), R^3(t), t)$$

$$\frac{dR^2}{dt} = u_E^{\,2}(R^1(t), R^2(t), R^3(t), t)$$

$$\frac{dR^3}{dt} = u_E^{\,3}(R^1(t), R^2(t), R^3(t), t)$$

three ordinary first order differential equations for three unknown $R^1(t)$, $R^2(t)$, $R^3(t)$

From theory of diff. eqn. given some very mild smoothness conditions on $u_E^{\,i}(r,t)$
(Coddington and Levinson, 15–19), we know

$$\boxed{\begin{array}{l}\forall\ \underline{X} = (X_1, X_2, X_3) \quad \exists \text{ exactly one soln} \\ \underline{R}(t) \text{ such that } \underline{R}(0) = X \text{ or} \\ R^i(0) = x^i \quad i = 1,2,3\end{array}}$$

This solution obviously depends on the initial posn $\underline{X}$, we write it as $\underline{R}(\underline{X}, t)$
But $\underline{R}(\underline{X}, t)$ is the position at time t of the particle at $\underline{X}$ at $t = 0$, i.e. the Lagrangian description of the motion

Hence $\underline{r}(\underline{x}, t)$ is uniquely determined by $u_E(r,t)$

5. The substantial derivative.

If a physical quant. M (e.g. air temperature) is given as a fcn of particle $\underline{x}$ and time t we say we have a Lagrangian description of M.

If M is given as a fcn of spatial posn $\underline{r}$ and time t we have an Eulerian description of M

M may be a scalar, vector, or a tensor field. The relation between the Eulerian and Lagrangian descriptions is

$$\boxed{M_L(\underline{x},t) = M_E(\underline{r}(\underline{x},t),t)}$$

$$\boxed{M_E(\underline{r},t) = M_L(\underline{x}(\underline{r},t),t)}$$

our previous eqn $u_L(\underline{x},t) = u_E(\underline{r}(\underline{x},t),t)$ a special case

note M_L and M_E are <u>clearly different fcns or rules</u>.

Often we will merely write $M(\underline{r},t)$ to mean $M_E(\underline{r},t)$ and $M(\underline{x},t)$ to mean $M_L(\underline{x},t)$

More notation. By convention we write

$\partial M/\partial t$ to mean $\frac{\partial}{\partial t} M(\underline{r},t)$ which is $\frac{\partial}{\partial t} M_E(\underline{r},t)$

DM/Dt to mean $\frac{\partial}{\partial t} M(\underline{x},t)$ which is $\frac{\partial}{\partial t} M_L(\underline{x},t)$

We also write $\partial_t M$ and $D_t M$

Both $D_t M$ and $\partial_t M$ (another form) are partial derivatives

$\partial_t M$ is the rate at which M changes with time at a fixed posn in space
$D_t M$ is the rate at which M changes with time at a fixed particle

$D_t M$ is called the <u>substantial derivative</u> or Lagrangian derivative

The relation between $D_t M$ and $\partial_t M$ may be readily obtained

hold particle $\underline{x}$ fixed
make a small change δt in t
this produces a small change $\underline{\delta r} = [D_t \underline{r}(\underline{x}, t)]\delta t$ in the posn of particle labeled $\underline{x}$

$+ O(\delta t)^2$

the change $\delta M_L = M_L(\underline{x}, t+\delta t) - M_L(\underline{x}, t)$ is

$$\delta M_L(\underline{x}, t) = \underline{\delta r} \cdot \nabla M_E + \delta t\, \partial_t M_E + O(\delta t)^2$$

(this comes from $M_L(\underline{x}, t) = M_E(\underline{r}(\underline{x}, t), t)$

$$= \delta t\,[D_t \underline{r} \cdot \nabla M_E + \partial_t M_E] + O(\delta t)^2$$

See over instead

$$M_L(\underline{x}, t+\delta t) = M_E(\underline{r}(\underline{x}, t+\delta t), t+\delta t) = M_E(\underline{r} + \underline{\delta r}, t+\delta t)$$

divide by δt and take limit as $\delta t \to 0$

$$D_t M_L(\underline{x},t) = \partial_t M_E(\underline{r},t) + D_t \underline{r}(\underline{x},t) \cdot \nabla M_E(\underline{r},t)$$

$$D_t \underline{r}(\underline{x},t) = \underline{u}_L(\underline{x},t) \quad \text{is the particle velocity}$$
$$= \underline{u}_E(\underline{r}(\underline{x},t),t)$$
$$= \underline{u}_E(\underline{r},t)$$

$$\boxed{D_t M_L(\underline{x},t) = \partial_t M_E(\underline{r},t) + \underline{u}_E(\underline{r},t) \cdot \nabla M_E(\underline{r},t)}$$

often written

$$\boxed{\begin{array}{l} D_t M = \partial_t M + \underline{u} \cdot \nabla M \quad \text{or} \\ D_t = \partial_t + \underline{u} \cdot \nabla \end{array}}$$

The calculation is nothing more than the chain rule for partial differentiation.

$$M_L(\underline{x},t) = M_E(\underline{r}(\underline{x},t),t)$$

swap

$$D_t M_L(\underline{x},t) = \nabla M_E \cdot D_t \underline{r}(\underline{x},t) + \partial_t M_E$$

$$D_t M_L(\underline{x},t) = \partial_t M_E(\underline{r},t) + \underline{u}_E(\underline{r},t) \cdot \nabla M_E(\underline{r},t)$$

example: $M = \underline{u}$, the velocity — Stophere 2Feb.

$D\underline{u}/Dt = \partial \underline{u}/\partial t + \underline{u} \cdot \nabla \underline{u}$ — acceleration $a = D\underline{u}/Dt$

Lecture # 2 review

Discussed the kinematics (description of motion) of a continuum

Two common types of description

Lagrangian: fix attn on ind. particles, specify their posn as fcn of time $\underline{r}(\underline{x},t)$

$\underline{x} \equiv$ posn at time $t = 0$

Eulerian: fix attn on fixed pts. in space, specify velocity of particle at that point at time t

$\underline{u}_E(\underline{r},t)$ or $\underline{u}(\underline{r},t)$

We also speak of an Eulerian and a Lagrangian description of any physical quantity, any tensor quantity M

$$M_L(\underline{x},t) \quad \text{and} \quad M_E(\underline{r},t)$$

$M_E(\underline{r},t)$ is how we actually measure things in lab or field. Seismometers measure $\underline{r}(\underline{x},t)$, in

Relation between E. and L. descriptions is principle

$$\boxed{M_L(\underline{x},t) = M_E(\underline{r}(\underline{x},t),t)}$$

Defined substantial derivative $D/Dt \equiv$ rate of change as seen by an ind. particle. Use chain rule for partial diff. on above

$$D_t M_L(\underline{x},t) = \partial_t M_E(\underline{r},t) + \underline{u}_E(\underline{r},t)\cdot\nabla M_E(\underline{r},t)$$

physical interpretation of this eqn is clear

Rate of change D/Dt of some quantity as seen by an observer on an ind. particle due to two causes 1. a change of the property at fixed pts in space 2. the motion of the particle thru a gradient of the property.

II. Dynamics of a continuum

~~We have lear~~

1. We have learned how to describe the motion of a cont. Now we wish to learn how to predict its motion given its initial state and the external influences to which it is subjected.

We seek laws of motion which are, as far as possible, independent of the molecular constitution of our material. Thus the wider is the applicability of our predictions, but the smaller is the amount of info. obtained about the mol. const. from an experimental verification of our ~~results~~ predictions.

We discuss the the application to continua of the ~~three~~ four general laws of Newtonian mechanics:

conservation of mass
conservation of linear ~~momentum~~
conservation of angular momentum
conservation of energy

To apply these laws we must fix attn on a particular parcel of matter

consider a volume $V(0)$ at $t=0$
At time t this volume will have moved to occupy a new region $V(t)$. As t varies $V(t)$ will change but it will always consist of the same material particles

Any region $V(t)$ which varies with time so as to always be occupied by the same material particles is said to <u>move with the material</u>.

Given a region $V(t)$ moving with the material in a cont. with $\mathcal{L}$. description $\underline{r}(\underline{x},t)$. Recall we have assumed that $\underline{r}(\underline{x},t)$ is cont. differ. as is $\underline{x}(\underline{r},t)$ <u>the smoothness cond.</u> It is a rigorous mathematical consequence of cont. differ. and a simple theorem from pt. set topology that:

The surface $\partial V(t)$ ~~always~~ of the region $V(t)$ always consists of the same material particles

The proof is not very revealing. The assertion is <u>not</u> true if $\underline{r}(\underline{x},t)$ is not continuous., e.g.

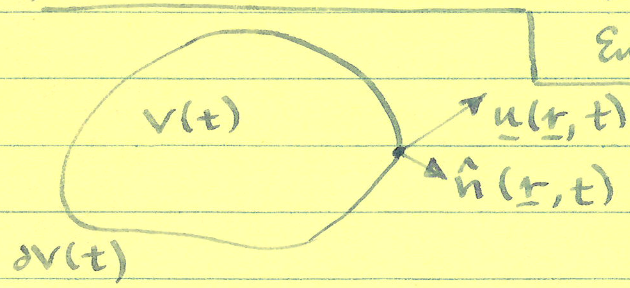
Say the cont. at $t=0$ consists of two hemispheres.

$t = 0$ $\quad$ $t = t_1$ $\quad$ $t = t_2$ $\quad$ $t = t_3$

obviously some particles on $\partial V(0)$ are not on $\partial V(t_3)$

Consider a region $V(t)$ moving with the material, surface $\partial V(t)$, unit outward normal $\hat{n}(\underline{r},t)$. Now $\partial V(t)$ moves with the velocities of the particles which make it up. At a pt $\underline{r}$ on $\partial V(t)$, the <u>normal</u> velocity of the surface is

$$\boxed{Q(\underline{r},t) = \hat{n}(\underline{r},t)\cdot \underbrace{\underline{u}(\underline{r},t)}_{\text{Eulerian velocity}}}$$



Consider a physical quantity whose <u>density per unit volume</u> is $\phi(\underline{r},t)$

e.g. density $\rho(\underline{r},t)$, salt conc. in sea H_2O $S(\underline{r},t)$, etc., <u>in general any extensive thermodynamic variable</u>.

ϕ-stuff $\equiv$ mass

~~as an example let $\underline{\underline{T}}(\underline{r})$ be of order two~~

$$\cancel{\underline{\underline{T}}(\underline{r}) = T_{ij}(\underline{r})\,\hat{x}_i\hat{x}_j}$$

$$\cancel{\nabla\underline{\underline{T}}(\underline{r}) = (\nabla T)_{kij}(\underline{r})\,\hat{x}_k\hat{x}_i\hat{x}_j}$$

$$\cancel{\nabla\cdot\underline{\underline{T}}(\underline{r}) = (\nabla\cdot T)_j(\underline{r})\,\hat{x}_j}$$

~~where~~

$$\cancel{(\nabla T)_{kij}(\underline{r}) = \partial_k T_{ij}(\underline{r})}$$

$$\cancel{(\nabla\cdot T)_j(\underline{r}) = \partial_i T_{ij}(\underline{r})}$$

Now the final thing which we did last time was to:

Consider a volume $V(t)$ moving with the same material (i.e., always containing the same material particles)

There was some confusion here last time due mostly to my failure to point out I am only considering a <u>one-component</u> system.

Consider some <u>extensive</u> quantity (e.g. mass, energy, momentum, entropy) whose density per unit volume is $\phi(\underline{r}, t)$ <u>Eulerian description</u> (ϕ-stuff)

The time rate of change of the total amt of ϕ-stuff in the volume V is then

we used theorem on differ. integrals over moving volumes, and then used Gauss' theorem to derive this

$$\boxed{\frac{d}{dt}\int_{V(t)} \phi(\underline{r}, t)\, dV = \int_{V(t)} \left[\frac{\partial \phi}{\partial t} + \nabla\cdot(\phi\underline{u}) \right] dV}$$

$u_E(\underline{r}, t)$ Eulerian velocity total amount of ϕ-stuff in $V(t)$ is

$$\int_{V(t)} \phi(\underline{r}, t)\, dV$$

Now apply thm. on differentiating integrals over moving volumes

Time rate of change of the amount of ϕ-stuff in the volume $V(t)$ moving with the material is

$$\frac{d}{dt}\int_{V(t)} \phi(\underline{r}, t)\, dV = \int_{V(t)} \frac{\partial \phi}{\partial t}\, dV + \int_{\partial V(t)} \phi(\underline{u}\cdot\hat{n})\, dA$$

↑ convert to vol. $\int dV$ using Gauss' theorem

Reynolds transport theorem

$$\boxed{\frac{d}{dt}\int_{V(t)} \phi(\underline{r}, t) = \int_{V(t)} \left[\frac{\partial \phi}{\partial t} + \nabla\cdot(\phi\underline{u})\right] dV \quad \star}$$

stop here 4 Feb.

2. Conservation of mass: Eulerian form

Eulerian form can be written almost immediately. Given a cont. with Eulerian description $\underline{u}(\underline{r}, t)$ and whose density has Eulerian description $\rho(\underline{r}, t)$

$V(t)$ moving with the material, let $M(t)$ be the total mass in $V(t)$

A law of Newtonian (not Einstein) physics is that mass can't be created or destroyed
Hence $M(t)$ must in fact be ind. of t
or $dM/dt = 0$
Now

$$M(t) = \int_{V(t)} \rho(\underline{r}, t)\, dV \quad \text{and by } \star$$

$$\frac{dM}{dt} \underset{\substack{\uparrow \\ \text{math. identity}}}{=} \int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) \right] dV \underset{\substack{\uparrow \\ \text{physical law}}}{=} 0$$

$dM/dt = 0$ a physical law
$dM/dt = \int_{V(t)} [\partial_t \rho + \nabla \cdot (\rho \underline{u})] dV$ a math. identity

equate

$$\int_{V(t)} [\partial \rho / \partial t + \nabla \cdot (\rho \underline{u})]\, dV = 0$$

Fix the time t, then $\star\star$ is true for every region $V(t)$ in the material
Assuming (as we shall) the integrand is cont., the vanishing integral theorem asserts that the integrand must vanish at time t

But t is arbitrary, hence for all t

$$\boxed{\partial\rho/\partial t + \nabla\cdot(\rho\underline{u}) = 0}$$

Eulerian form of law of cons. of mass
Called the <u>continuity equation</u>
The argument has been given in detail as it is typical of arguments we shall use to deduce other laws.

Another form
$$\nabla\cdot(\rho\underline{u}) = \rho\nabla\cdot\underline{u} + \nabla\rho\cdot\underline{u}$$

$$\boxed{D\rho/Dt + \rho\nabla\cdot\underline{u} = 0}$$

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Special case of an <u>incompressible</u> fluid or material.
The local density at the various particles cannot change
<u>Question: how can the density of compr. ~~particle~~ material change, i.e. how can the density of a single particle ever vary?</u>
<u>Answer: particles we discuss are infinitesimal macroparticles</u>, contain many atoms, many internal d.f., just compress the atoms closer together
An <u>incompressible</u> material is one for which

if τ is defined as the specific volume of the material (the volume occupied by one gram) then $\rho\tau = 1$

$$\boxed{\tau = \rho^{-1}}$$

$\ln\rho + \ln\tau = 0$

$$D\rho/Dt + \rho\,\nabla\cdot\underline{u} = 0 \quad \text{or}$$
$$D\ln\rho/Dt + \nabla\cdot\underline{u} = 0$$

$D\ln\tau/Dt - \nabla\cdot\underline{u} = 0$ or

$$\boxed{\frac{1}{\tau}\frac{D\tau}{Dt} = \nabla\cdot\underline{u}}$$

physical interpretation $\frac{1}{\tau}\frac{D\tau}{Dt}$ is the relative time rate of change of local volume at a given infinitesimal macroparticle.
Thus $\nabla\cdot\underline{u}$ is a measure of local rate of change of volume

An incompressible material is one for which

$$\frac{1}{\tau}\left(D\tau/Dt\right) = 0 \quad \text{or} \quad \nabla\cdot\underline{u} = 0$$

this cannot be done

$$D\rho/Dt = 0$$

for an incomp. material

$$\boxed{\nabla \cdot \underline{u} = 0 \quad \text{cont eqn for incompr material}}$$

note: the derivation is independent of whether or not $\rho =$ const. in space
Many books do not point this out, e.g. Officer

3. Lagrangian form of law of cons. of mass

We will talk more about the Eulerian form in a second, first:

$\underline{r}(\underline{x}, t)$ Lagrangian description of a cont. with density $\rho_L(\underline{x}, t) = \rho_E(\underline{r}, t)$
$V(t)$ moving with material, $M(t) \equiv$ mass in $V(t)$

$$M(t) = \int_{V(t)} \rho_E(\underline{r}, t)\, dr_1\, dr_2\, dr_3 \qquad \text{as before}$$

now change variable of integration to x_1, x_2, x_3

$$M(t) = \int_{V(0)} \rho_L(\underline{x},t) \left| \frac{\partial(\underline{r})}{\partial(\underline{x})} \right| dx_1 dx_2 dx_3$$

but $$M(0) = \int_{V(0)} \rho_L(\underline{x},0)\, dx_1 dx_2 dx_3$$

law of cons. of mass $M(t) = M(0) \quad \forall t$

$\therefore$ at any t the integrals are $=$ for any $V(0)$, thus integrands must ~~vanish~~ be equal everywhere in the material

$$\boxed{\rho_L(\underline{x},t) \left| \frac{\partial(\underline{r})}{\partial(\underline{x})} \right| = \rho_L(\underline{x},0)}$$

Provides interpretation of Jacobian of a transformation.

Lagrangian form of law of cons. of mass

4. Conservation laws in general

Suppose we have two scalar fields $\phi(\underline{r},t)$ and $k(\underline{r},t)$ and a vector field $\underline{K}(\underline{r},t)$ which satisfy the equation

$$\boxed{\partial\phi/\partial t + \nabla\cdot\underline{K} = k}$$

$$\int_V \frac{\partial \phi}{\partial t}\, dV + \int_V \nabla \cdot \underline{K}\, dV = \int_V k\, dV$$

we choose a volume V fixed in space and integrate over it
Since V is fixed in space

$$\int_V \frac{\partial \phi}{\partial t}\, dV = \frac{d}{dt}\int_V \phi\, dV$$

and Gauss says that $\int_V \nabla \cdot \underline{K}\, dV = \int_{\partial V} \hat{n} \cdot \underline{K}\, dA$

$\hat{n} \equiv$ unit outward normal

$$\boxed{d/dt\left[\int_V \phi\, dV\right] = \int_V k\, dV - \int_{\partial V} \hat{n} \cdot \underline{K}\, dA}$$

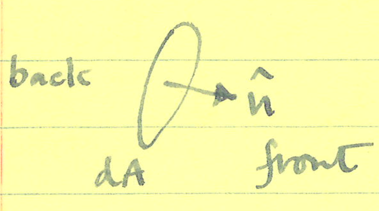
we think of V as containing a substance ϕ-stuff whose density per unit volume is ϕ
The amount of ϕ-stuff in V is $\int_V \phi\, dV$

$k \equiv$ rate of production of ϕ-stuff per unit volume per second

$\underline{K} \equiv$ current density, or flux, of ϕ-stuff in space per unit area per second

To see this consider

A small area dA fixed in space

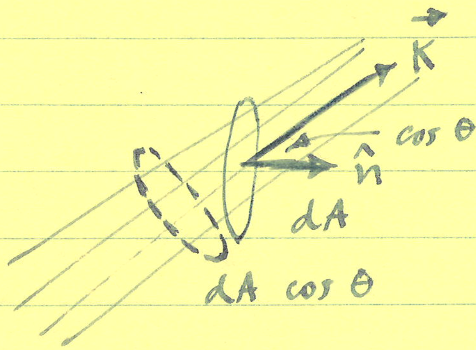


the amount of ϕ-stuff which flows thru dA from the back (-)

to the front (+) per second is

$$|\underline{K}| \underbrace{dA \cos\theta}$$

projection of dA on a plane normal to $\underline{K}$



or $\hat{n} \cdot \underline{K}\, dA$

If $\phi = \rho$, ϕ-stuff is mass
The cont. eqn. is

$$\partial\rho/\partial t + \nabla\cdot(\rho\underline{u}) = 0$$

a problem asks you to consider such a term

$k = 0$ no mass is produced or created (or destroyed)
$\underline{K} = \rho\underline{u}$ but there is a flow of mass
$\rho\underline{u}$ mass flux per square cm per sec.

whenever we have an equation of the form

$$\partial\phi/\partial t + \nabla\cdot\underline{K} = k$$

$\phi, \underline{K}, k$ can be so interpreted. This is the general form of a conservation law. All our conservation laws will come out in this form (momentum, ang. momentum, energy)

• sometimes ϕ - stuff can be a tensor field of order q

ϕ, k	tensor fields of order q
K	tensor field of order $q+1$
such that	$\partial\phi/\partial t + \nabla\cdot\underline{K} = k$

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introduce a Cartesian axis system $\hat{x}_1, \hat{x}_2, \hat{x}_3$.
Then ϕ, k, K may be written in terms of their basis of q^{th} order polyads

$$\phi = \phi_{j_1 \cdots j_q} \hat{x}_{j_1} \cdots \hat{x}_{j_q}$$

$$k = k_{j_1 \cdots j_q} \hat{x}_{j_1} \cdots \hat{x}_{j_q}$$

$$K = K_{i j_1 \cdots j_q} \hat{x}_i \hat{x}_{j_1} \cdots \hat{x}_{j_q}$$

then $\quad \partial_t \phi_{j_1 \cdots j_q} + \partial K_{j_1 \cdots j_q} / \partial r_i = k_{j_1 \cdots j_q}$

$\phi_{j_1 \cdots j_q}$ stuff $\equiv$ the amt of the component $\phi_{j_1 \cdots j_q}$ per unit volume.

$k_{j_1 \cdots j_q} \equiv$ its rate of production per unit vol. per sec.

$K_{i j_1 \cdots j_q} \equiv$ its current density

e.g. $\quad \phi$ - stuff could be momentum if $q = 1$

$\phi = \rho \underline{u}$

we define a vector $\vec{K}_{j_1 \cdots j_q} = K_{i j_1 \cdots j_q} \hat{x}_i$, then

$$\partial_t \phi_{j_1 \cdots j_q} = \nabla\cdot \underline{K}_{j_1 \cdots j_q} + k_{j_1 \cdots j_q}$$

now for any component we have

$$\frac{d}{dt}\int_V \phi_{j_1 \cdots j_q}\, dV = \int_V k_{j_1 \cdots j_q}\, dV - \int_{\partial V} dA\, \hat{n}\cdot \underline{K}_{j_1 \cdots j_q}$$

now the polyad $\hat{x}_{j_1} \cdots \hat{x}_{j_q}$ is ind. of $\hat{r}$ and t, mult. above eqn by this and sum

$$\boxed{d/dt \left[\int_V \phi\, dV\right] = \int_V k\, dV - \int_{\partial V} \hat{n}\cdot K\, dA \qquad (\star)}$$

so the interpretation of the cons. eqn. is valid for tensors also

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<u>Another way to prove $(\star)$ probably better for lecture</u>.

Gauss theorem is valid for tensor fields of any order

$$\int_V \nabla\cdot K\, dV = \int_{\partial V} \hat{n}\cdot K\, dA$$

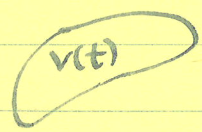
in general $q \geq 2$ $\hat{n}\cdot K \neq K\cdot\hat{n}$ important to replace $\nabla\cdot$ by $\hat{n}$

The derivation of $(\star)$ for <u>scalar</u> ϕ-stuff used only Gauss' theorem. Hence valid for tensor ϕ stuff as well

5. Conservation of momentum: application of Newton's Laws

Derivation very similar to that of conservation of momentum.

region $V(t)$ moving with the material



Let $\mathcal{F}$ be the total force exerted on the material in V by everything outside it ~~[illegible]~~

Newton's law of motion is that the time rate of change of ~~any~~ the momentum of any collection of particles is equal to the total external force acting on all the particles.

$V(t)$ always contains the same particles, hence we may apply Newton's law to it

We apply Newton's law to a parcel of matter.

$$d/dt \int_{V(t)} \rho \underline{u} \, dV = \mathcal{F} \qquad \text{or} \qquad \frac{d}{dt} \int_{V(t)} \rho u_j \, dV = \mathcal{F}_j$$

apply our math. identity for ϕ-stuff with $\phi = \rho u_i$

$$\int_{V(t)} \left[\frac{\partial}{\partial t}(\rho u_j) + \nabla \cdot (\rho u_j \underline{u}) \right] dV = \mathcal{F}_j$$

Lecture # 4 Review

Last time talked about the general form of a conservation law for a field quantity (Eulerian form)

$\phi(\underline{r},t) \equiv$ density (per unit volume) of ϕ-stuff

$k(\underline{r},t) \equiv$ rate of production per unit volume of ϕ-stuff

$\underline{K}(\underline{r},t) \equiv$ current density or flux density per unit area per second, in space.

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \underline{K} = k$$

$$\underbrace{\frac{d}{dt}\int_V \phi \, dV}_{\text{rate of change of net amt. of } \phi\text{-stuff in } V} = \underbrace{\int_V k \, dV}_{\text{net amt. produced by spontaneous creation (or destruction) in } V} - \underbrace{\int_{\partial V} \hat{n} \cdot \underline{K} \, dA}_{\text{net flow in or out of } V}$$

V fixed in space

We derived one conservation law, conservation of mass

$$\partial \rho / \partial t + \nabla \cdot (\underbrace{\rho \underline{u}}_{\text{mass flux density}}) = 0$$

rewrite in form

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \nabla \cdot \underline{u}$$

if $\exists$ a pt $\underline{r} \in$ space such that $\nabla \cdot \underline{u}(\underline{r},t)$ at time t there is a non-zero <u>divergence</u> of the Eulerian velocity, then the particle $\underline{x}$ at position $\underline{r}$ at time t is experiencing a relative change in density or volume.

We were in the midst of deriving another conservation law
<u>By applying Newton's second law of motion to a parcel of matter</u>

$V(t)$ moving with volume

$$\frac{d}{dt}\left[\int_{V(t)} \rho u_i \, dV\right] = \int_{V(t)} \left[\frac{\partial}{\partial t}(\rho u_i) + \nabla\cdot(\rho u_i \underline{u})\right] dV = \mathcal{F}_i$$

better to use index j.

i^{th} comp. of net momentum of parcel of matter within $V(t)$

$\mathcal{F}_i$ = i^{th} component of net <u>external</u> force on $V(t)$ (internal forces cancel by Newton's third law – action = reaction)

We must express $\mathcal{F}_i$ as an integral over $V(t)$

to use the vanishing integral thm. we must be able to express $\mathcal{F}_i$ as a volume integral over $V(t)$. We will spend quite a while learning how to do this and our discussion will eventually lead us to postulate the existence of a stress tensor.

stop here 7 Feb.

6. Body forces and surface forces

In general $\underline{\mathcal{F}}$ can be broken up into two kinds of forces

A force acting on a continuum is a body force if $\exists$ a vector field $\underline{f}(\underline{r})$ defined throughout such that the force on the matter in any region V is

$$\underline{\mathcal{F}} = \int_V \underline{f} \, dV$$

The field $\underline{f}$ is the body force density or body force per unit volume.

Examples of body forces are

(1) the gravitational force due to an external grav. potential then

$$\underline{\mathcal{F}} = -\int_V \rho \nabla \phi \, dV$$

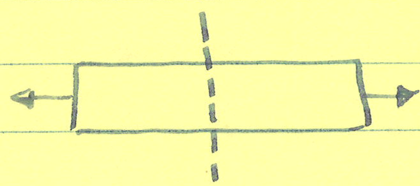
Note: must use Newton's third law to incorporate this

(2) similarly, the grav. force of the various parts of the continuum acting on one another.

(3) the Lorentz force $\underline{J} \times \underline{B}$ acting on an electrically conducting continuum carrying current density $\underline{J}$ in a magnetic field $\underline{B}$

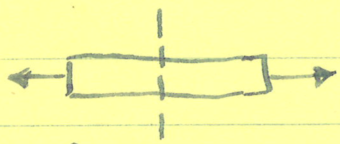
Body forces are a good model for the forces exerted by distant matter, but in addition we expect that the atoms just inside the boundary $\partial V(t)$ of a volume $V(t)$ will experience a very strong short-range force from the atoms just outside $\partial V(t)$. We shall call such a force a <u>surface force</u>.

Consider a steel bar with an imaginary cut



pull on it — the surface forces exerted by the atoms one side of the cut on the atoms on the other side are the forces that prevent us from tearing the bar apart. Let's consider these short-range interatomic forces further. <u>Our goal is a model of them as independent as possible of the molecular const. of the continuum. What are their salient characteristics?</u>

it is an experimentally observed fact that, e.g. in pulling on a rod until it breaks one finds that the force required for breaking or the breaking strength is $\propto$ to the cross sectional area and independent of the areal shape



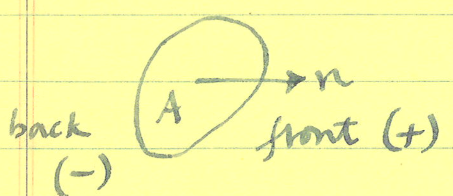
breaking strength divided by area A = a constant
This implies that the surface forces acting across the imaginary surface are also $\propto$ to the area across which they act.

We now give a heuristic argument which leads to the same conclusion. The proportionality to cross-sectional area A will be seen to be a result of the short range nature of the forces in question.

We then postulate as a basic postulate of our defn of a continuum that $\exists$ surface forces $\propto$ to the area across which they act. The postulate is assumed to be valid for all continua we discuss, fluid, solid, etc.

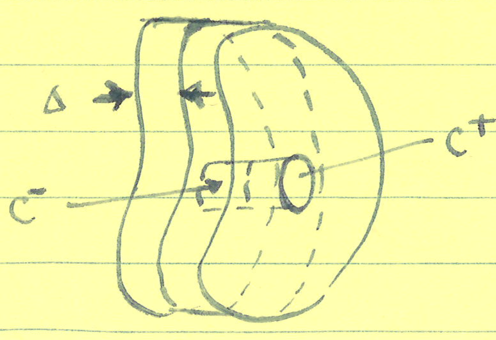
Now: the heuristic argument; we will see once again the dual nature of the continuum approximation.

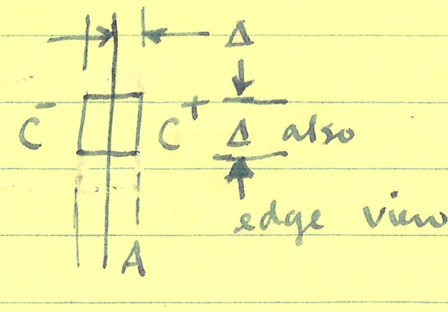
When atoms come within an atomic diameter δ, the forces they exert on each other become very large. Say δ = length scale for short-range interatomic forces.
Choose a length $\Delta \approx$ say $5\delta - 10\delta$
Examine a small ~~patch~~ plane patch area A unit normal $\hat{n}$. Define front, back



take the area large enough so that if $A \sim \pi d^2$, then $d \gg \Delta$ $\qquad d = d(A) \gg \Delta$
also take $d \ll$ macroscopic scale length.

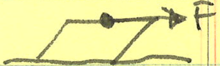
All the atoms in front of A exert a force on all the atoms in back of A, but this force only becomes large when atoms are within $\sim \Delta$
Most of this force is thus due to atoms in front of Δ and within Δ on those behind A and within Δ
Further, any cylinder C^+ of atoms in front of A if its largest dimension is Δ will act mostly on a similar cylinder behind A.
Therefore if $d(A) \gg \Delta$, the total force exerted across A will be roughly proportional to A





the force exerted across A will thus be ~ proportional to A.

or better



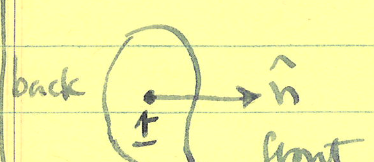
There is no reason to expect the force to be in the direction of $\hat{n}$. Why? Say we pull in this way so as to shear our rod

the force will be of the form $\underline{F}(\underline{r}, t, \hat{n})A$ if $d(A)$ is $\gg \Delta$ and if $d(A) \ll$ macroscopic distances in which macroscopic variables change (density, temp, velocity, etc.)
We see again the nature of the continuum approx. and are reminded that particles in a continuum are macroparticles

the force exerted across A is $\propto$ A, the prop. const. is $\underline{F}(\underline{r}, t, \hat{n})$ depends on time, position, and orientation of A.

Now how about in our idealized continuum. The above discussion suggests the following

If dA is an infinitesimal plane element of area at $\underline{r}$ with unit normal $\hat{n}$, then the force $d\underline{\mathcal{F}}_S$ (s for surface) exerted by the matter just in front of dA on the matter just behind dA is $\propto$ to dA



dA

$$d\underline{\mathcal{F}}_S = \underline{F}(\underline{r}, t, \hat{n})\, dA$$

the constant of proportionality (force per unit area) is called the traction on dA, it depends on $\underline{r}$, t, and orientation $\hat{n}$ of dA.

The total force on the matter just inside $\partial V(t)$ by the matter ~~outs~~ just outside $\partial V(t)$ is

$$\underline{\mathcal{F}}_{\partial V} = \int_{\partial V} \underline{F}(\underline{r}, t, \hat{n}(\underline{r}))\, dA$$

Now returning, we assume that the only forces acting on $V(t)$ are idealized body forces $\underline{f}(\underline{r}, t)$ and idealized surface forces $F(\underline{r}, t, \hat{n}(\underline{r}))$. This assumption essentially another continuum postulate.

$$\underline{\mathcal{F}} = \int_{V(t)} \underline{f}(\underline{r}, t)\, dV + \int_{\partial V(t)} \underline{F}(\underline{r}, t, \hat{n}(\underline{r}))\, dA$$

we now have

$$\int_{V(t)} \left[\frac{\partial}{\partial t}(\rho u^i) + \nabla \cdot (\rho u^i \underline{u}) - f^i \right] dV = \int_{\partial V(t)} F^i(\underline{r}, t, \hat{n}(\underline{r}))\, dA \qquad (*)$$

must still express this as a volume integral – our next task

7. The stress tensor

This equation contains an apparent paradox. Consider the eqn for a series of regions V_l all of the same shape but having maximum diameters l which shrink to zero about a point $\underline{r}_0$

As $l \to 0$, the l.h.s. (volume integral) behaves like l^3, while we would ordinarily expect the r.h.s. (surface integral) to behave like l^2. Yet the two are always equal.

There must be some property of $\underline{F}(\underline{r}, t, \hat{n})$ which gives rise to some kind of cancellation so that the r.h.s. (surface integral) goes to zero like l^3 as $l \to 0$.

The argument is somewhat complicated but it will be given in detail since it will be used again.

That property is expressed by the following theorem

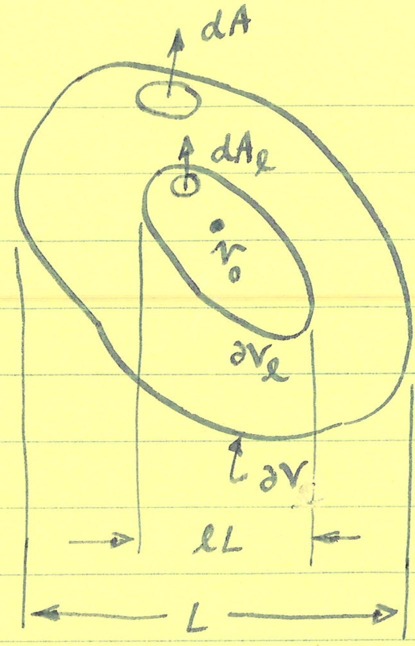
Theorem: Let V be any region in a continuum and $\underline{r}_0$ a fixed point in V. Let $\hat{n}(\underline{r})$ be the unit outward normal to ∂V at $\underline{r}$ on ∂V.

Then

$$\boxed{\int_{\partial V} \underline{F}(\underset{\substack{\uparrow \\ \text{note}}}{\underline{r}_0}, t, \hat{n}(\underline{r}))\, dA = 0}$$

There are two cond on $\underline{F}$
1. cont.
2. must satisfy eqn *

proof:



We have not been two careful about continuity, differentiability, etc. but in fact we must assume that $\underline{F}(\underline{r}, t, \hat{n}(\underline{r}))$ is ~~differentiable~~ continuous, and the proof exploits this continuity.

For any $\ell < 1$ let V_ℓ be obtained from V by shrinking V by the factor ℓ in all its dimensions about $\underline{r}_0$ (one preserves the relative location of $\underline{r}_0$)

∂V_ℓ is the bdry of V_ℓ.

Then on ∂V_ℓ

$$F_i(\underline{r}, t, \hat{n}(\underline{r})) = F_i(\underline{r}_0, t, \hat{n}(\underline{r})) + \epsilon_i^{(\ell)}$$

where

$$\epsilon_i^{(\ell)} \to 0 \quad \text{as} \quad \ell \to 0$$

(we must assume $F_i(\underline{r}, t, \hat{n}(\underline{r}))$ cont. fcn of $\underline{r}$)

We must also assume it sat. $(\star)$

let $q_i = \frac{\partial}{\partial t}(\rho u_i) + \nabla \cdot (\rho u_i \underline{u}) - f_i$

then our eqn so far looks like, when integral is over V_ℓ.

$$\int_{\partial V_\ell} F_i(\underline{r}_0, t, \hat{n}(\underline{r}))\, dA_\ell = \int_{V_\ell} q_i\, dV - \int_{\partial V_\ell} \epsilon_i^{(\ell)}\, dA_\ell \qquad (*)$$

now first we assert that for $\ell \leq 1$

(1) $$\int_{\partial V_\ell} F_i(\underline{r}_0, t, \hat{n}(\underline{r}))\, dA_\ell = \ell^2 \int_{\partial V} F_i(\underline{r}_0, t, \hat{n}(\underline{r}))\, dA \qquad (**)$$

look at the figure. Think of small patches dA on ∂V with shrunken image dA_ℓ on ∂V_ℓ. The normals to dA and dA_ℓ are the same, moreover $dA_\ell = \ell^2 dA$
Also
$\underline{F}(\underline{r}_0, t, \hat{n}(\underline{r}))$ has the same value on dA_ℓ and dA hence

$$\int_{\partial V_\ell} \underline{F}\, dA_\ell = \lim \sum_{\partial V_\ell} \underline{F}\, dA_\ell = \lim \ell^2 \sum_{\partial V} \underline{F}\, dA$$
$$= \ell^2 \int_{\partial V} \underline{F}\, dA$$

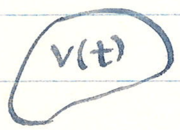
now we have step (2) of the proof

(2) subst $**$ in $*$ and divide by ℓ^2

$$\int_{\partial V} F_i(\underline{r}_0, t, \hat{n}(\underline{r}))\, dA = \frac{1}{\ell^2} \int_{V_\ell} q_i\, dV - \frac{1}{\ell^2} \int_{\partial V_\ell} \epsilon_i^{(\ell)}\, dA$$

Lecture # 5 Review

We are in the midst of deriving the conservation law which expresses Newton's second law of motion



volume $V(t)$ moving with the material

$$\frac{d}{dt}\int_{V(t)} \rho \underline{u}\, dV = \underline{\mathcal{F}}$$

We postulate that two kinds of forces are acting

1. body forces e.g. gravitation
2. surface forces – due to short range nature these are proportional to area

$$\underline{\mathcal{F}} = \underbrace{\int_{V(t)} \underline{f}(t)\, dV}_{\text{body force density (per unit volume)}} + \int_{\partial V(t)} \underbrace{\underline{F}(\underline{r}, t, \hat{n}(\underline{r}))}_{\text{traction}}\, dA$$

body force density (per unit volume)

traction (surface force per unit area) exerted by material outside $\partial V(t)$ on material immediately inside $\partial V(t)$

We wish to transform the surface integral to a volume integral.

both terms on r.h.s. $\to 0$

volume integral $\propto l^3$ and $l^3/l^2 \to 0$ as $l \to 0$

surface integral $\propto l^2 \epsilon_i^{(l)}$ and since $\epsilon_i^{(l)} \to 0$
$(l^2/l^2)\, \epsilon_i^{(l)} \to 0$ as $l \to 0$

Thus the l.h.s. $\to 0$ as $l \to 0$
But the l.h.s. is <u>independent of l</u>. <u>Hence it is identically zero</u>.

$$\boxed{\int_{\partial V} \underline{F}(\underline{r}_0, t, \hat{n}(\underline{r}))\, dA = 0}$$

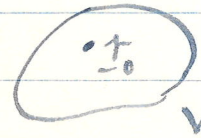
stop here
9 Feb.

Now we apply the above eqn to a very special volume and thereby deduce the existence in general of a <u>stress tensor</u>.
This argument was first made in 1823 by Cauchy.
The volume considered is the so-called <u>Cauchy tetrahedron</u>.

The <u>physical interpretation</u> is simple. If the stress field at all $\underline{r} \in \partial V$ is the same as at $\underline{r}_0$ then the <u>net</u> total surface force vanishes. Hence if the stress field is independent of position, the total surface force on any closed volume is zero

We noted an apparent paradox in our form of Newton's law
The volume integral terms $\to 0$ like l^3 and the surface integral, one would think, would $\to 0$ like l^2.

We then exploited this very paradox to the prove the following theorem



V arbitrary volume, surface ∂V

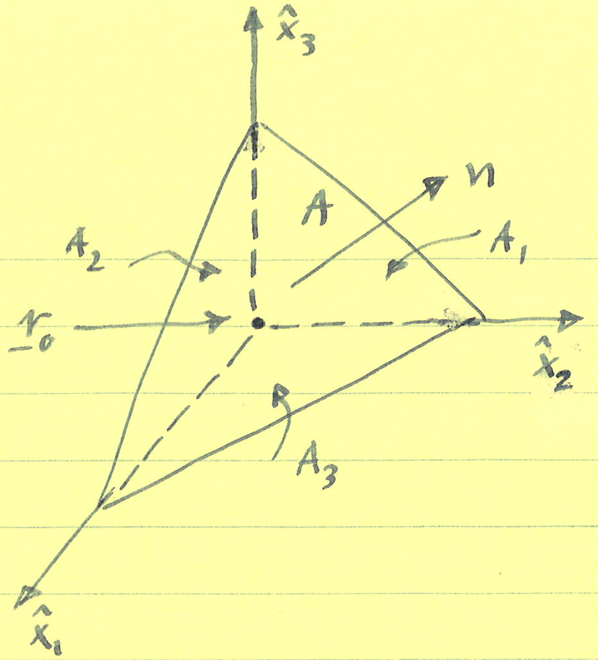
$$\boxed{\int_{\partial V} \underline{F}(\underline{r}_0, t, \hat{n}(\underline{r}))\, dA = 0}$$

$\underline{F}(\underline{r}_0, t, \hat{n}(\underline{r})) \equiv$ traction at $\underline{r}_0$ across dA with unit normal $\hat{n}(\underline{r})$

$\underline{r}_0$ is arbitrary. Theorem true for every $\underline{r}_0 \in V$.
Physical interpretation of theorem:

1. if stress field is homogeneous, so that $\underline{F}(\underline{r}, t, \hat{n}(\underline{r})) = \underline{F}(\underline{r}_0, t, \hat{n}(\underline{r}))$ (a uniformly stressed body — no stress gradients), then total force on every volume $= 0$
2. consider infinitesimally small volumes V_l then stress field $\to$ homogeneous as $l \to 0$. The theorem asserts that infin. small volumes feel no net forces and thus suffer no accelerations. Otherwise their accel. would be infinite (!)

Now we apply theorem to a special volume — Cauchy tetrahedron (Cauchy 1823). We will deduce stress tensor.



<u>Cauchy tetrahedron</u> with corner at the pt. $\underline{r}_0$ and three edges parallel to three coordinate axis vectors $\hat{x}_1, \hat{x}_2, \hat{x}_3$ and an oblique face with normal $\hat{n}$

$\hat{n} = n_i \hat{x}_i$

unit outward normals to other faces are $-\hat{x}_1, -\hat{x}_2, -\hat{x}_3$.

Let their areas be A_1, A_2, A_3, area of oblique face is A.

The $\perp$ projection of $\hat{n}$ on $\hat{x}_i$ is n_i while the $\perp$ projection of A on the plane with normal $-\hat{x}_i$ is A_i

hence $\boxed{A_i = A\, n_i}$

Now apply our theorem to the Cauchy tetrahedron

$$A\underline{F}(\underline{r}_0, t, \hat{n}) + \sum_{i=1}^{3} A_i\, \underline{F}(\underline{r}_0, t, -\hat{x}_i) = 0$$

divide by A to obtain

$$\underline{F}(\underline{r}_0, t, \hat{n}) = -\sum_{i=1}^{3} n_i \underline{F}(\underline{r}_0, t, -\hat{x}_i)$$

let $\hat{n} = \hat{x}_j$, $n_j = \delta_{ij}$ so

$$\boxed{\underline{F}(\underline{r}_0, t, \hat{x}_j) = -\underline{F}(\underline{r}_0, t, -\hat{x}_j)}$$

valid for any $\hat{x}_j$

valid for any pt. $\underline{r}_0$

This is <u>Newton's third law of action equals reaction</u>. The force exerted by the material in front on the material behind is equal and opposite to the force exerted by the material behind on the material in front

Using the above we get the prettier form

$$\boxed{\underline{F}(\underline{r}_0, t, \hat{n}) = \sum_{i=1}^{3} n_i \underline{F}(\underline{r}_0, t, \hat{x}_i)}$$

In a continuum at any pt $\underline{r}_0$ at any time t if the traction on three mutually $\perp$ faces ~~is known, then~~ infinitesimal surfaces is known, then the ~~stress on any other infin~~ traction on any other infinitesimal surface may be calculated simply from the orientation of that surface.

now a new notation, define nine scalar fields

$$\boxed{T_{ij}(\underline{r},t) = F_j(\underline{r},t,\hat{x}_i)}$$

$T_{ij}(\underline{r},t)$ is the j-component of the traction (surface force per unit area) on the face with unit normal $\hat{x}_i$ at pt. $\underline{r}$ and time t

If at any $\underline{r}, t$ the nine $T_{ij}(\underline{r},t)$ then the traction $\underline{F}(\underline{r},t\,\hat{n})$ on an infinitesimal surface with any unit normal $\hat{n}$ can be calculated immediately

dA

$\hat{n}$

skip to next page

$$\boxed{F_j(\underline{r},t,\hat{n}) = T_{ij}(\underline{r},t)\, n_i}$$

swap order

We shall now demonstrate that ~~This means ...~~ $T_{ij}(\underline{r},t)$ are the nine Cartesian components of a Tensor field $\underline{\underline{T}}(\underline{r},t)$ (a second order tensor)

This will be called (aha!) the stress tensor

(skip to next page)

$$\underline{\underline{T}}(\underline{r},t) = T_{ij}(\underline{r},t)\,\hat{x}_i\hat{x}_j$$

$$\underline{F}(\underline{r},t,\hat{n}) = \underline{\underline{T}}^T(\underline{r},t)\cdot\hat{n} = \hat{n}\cdot\underline{\underline{T}}(\underline{r},t)$$

our proof of existence of stress tensor
is very general
valid for all situations, not necessarily
equil. situations (motion of continuum, etc.)

8. The momentum equation

We are now in a position to derive the conservation of momentum equation.
We can now express the total external surface forces acting on a volume $V(t)$ in terms of a volume integral over $V(t)$.

$$\underline{\mathcal{F}}_S = \int_{\partial V(t)} \underline{F}(\underline{r}, t, \hat{n}(\underline{r}))\, dA$$

$\uparrow$
total external force on
$V(t)$ due to surface forces

now
$$F_j(\underline{r}, t, \hat{n}(\underline{r})) = T_{ij}(\underline{r}, t)\, n_i(\underline{r})$$
$$= [T_{ij}(\underline{r}, t)\, \hat{x}_i] \cdot \hat{n}(\underline{r})$$

so
$$\underbrace{\mathcal{F}_S^j}_{j^{th}\text{ comp.}} = \int_{\partial V(t)} \underbrace{[T_{ij}(\underline{r}, t)\, \hat{x}_i]}_{\text{vector}} \cdot \hat{n}\, dA \qquad \text{now by Gauss}$$

$$= \int_{V(t)} \nabla \cdot [T_{ij}(\underline{r}, t)\, \hat{x}_i]\, dV$$

or written another way

$$\mathcal{F}_s^j = \int_{V(t)} \frac{\partial T_{ij}}{\partial r_i} dV$$

Newton's law may therefore be written

$$\int_{V(t)} \left[\frac{\partial}{\partial t}(\rho u_j) + \nabla\cdot(\rho u_j \underline{u}) \right] dV = \int_{V(t)} \left[f_j + \partial_i T_{ij} \right] dV$$

now $V(t)$ is arbitrary. By vanishing integral eqn equate integrands.

$$\frac{\partial}{\partial t}(\rho u_j) + \nabla\cdot(\rho u_j \underline{u}) = f_j + \overset{\nabla\cdot(T_{ij}\hat{x}_i)}{\partial_i T_{ij}}$$

or $$\frac{\partial}{\partial t}(\rho \underline{u}) + \nabla\cdot(\rho \underline{u}\,\underline{u}) = \underline{f} + \nabla\cdot\underline{\underline{T}}$$

$$\boxed{\frac{\partial}{\partial t}(\rho \underline{u}) + \nabla\cdot(\rho \underline{u}\,\underline{u} - \underline{\underline{T}}) = \underline{f}}$$

has the form of a general conservation law

ϕ - stuff this time is a vector, the momentum

$\phi = \rho \underline{u}$ momentum density per unit volume

$k = \underline{f}$ production rate of momentum per sec

$\underline{K} = \rho \underline{u}\underline{u} - \underline{\underline{T}} \equiv$ current density of momentum in space

The current density of momentum in space is a second order tensor $\rho \underline{u}\underline{u} - \underline{\underline{T}}$

It has two parts:

1. transport of momentum by material flow $\rho \underline{u}\underline{u}$
2. transport of momentum by stress $\underline{\underline{T}}$
   note that even if $\underline{\underline{T}}(\underline{r}, t)$ is not time-varying, so long as there are stresses, there will be a net (steady) flow of momentum

You might be slightly more comfortable with the component form of the momentum eqn.

$$\boxed{\frac{\partial}{\partial t}(\rho u_j) + \nabla \cdot (\rho u_j \underline{u} - T_{ij} \hat{x}_i) = f_j}$$

$$\boxed{\frac{\partial}{\partial t}(\rho u_j) + \partial_i (\rho u_i u_j - T_{ij}) = f_j}$$

ϕ-stuff now a scalar

momentum flux density $\rho u_j \underline{u} - T_{ij} \hat{x}_i$

a vector as before

Stop here 11 Feb.

Lecture #6 Review

Last time we discussed the stress tensor
A tensor field – a function of space and time

In the <u>Eulerian description</u> $\underline{\underline{T}}_E(\underline{r},t)$
Called the Eulerian stress or the Cauchy stress
One can also define a <u>Lagrangian stress</u>
but it turns out that the useful one to
define is <u>not</u> just

$$\underline{\underline{T}}_L(\underline{x},t) = \underline{\underline{T}}_E(\underline{r}_L(\underline{x},t),t)$$

much more useful to discuss the force per
unit <u>undeformed</u> area. We probably won't
get into this. Very useful in the theory of
finite elasticity or the theory of elasticity
in presence of initial stress.

Stress tensor $\underline{\underline{T}}_E(\underline{r},t)$ is that linear operator
which ~~assigned~~ to an infinitesimal patch
$\hat{n}\,dA$ the net force $\underline{F}(\underline{r},t,\hat{n}(\underline{r}))\,dA$ acting on
or across that patch

$$\boxed{\underline{F}(\underline{r},t,\hat{n})\,dA = \hat{n}\cdot\underline{\underline{T}}(\underline{r},t)\,dA}$$

we then finally derived the Eulerian form of the momentum conservation law

$$\boxed{\partial_t(\rho \underline{u}) + \nabla\cdot(\rho\underline{u}\,\underline{u} - \underline{\underline{T}}) = \underline{f}}$$

ϕ stuff $\equiv$ momentum

$\phi \equiv$ momentum density $= \rho\underline{u}$

$k \equiv \underline{f}$ body force density
production rate of momentum

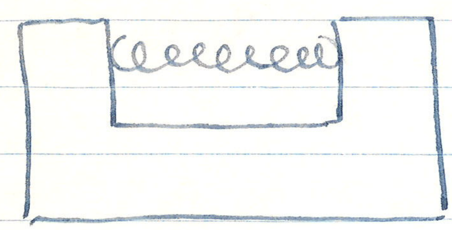
$\underline{K} \equiv \rho\underline{u}\,\underline{u} - \underline{\underline{T}}$ current density or flux density of momentum

flux due to motion of continuum (advected momentum)

flux due to stress — may be present even in the absence of motion

in a statically stressed element, one can think of there as being a continuous (steady) flow of momentum

e.g.



$\partial_t = 0$

$\underline{u} = 0$

$\underline{f} = 0$ (neglect grav.)

note eqn of motion (momentum eqn) here reduces to $\nabla\cdot\underline{\underline{T}} = \nabla\cdot\underline{K} = 0$. There are no sinks or sources of momentum (physical interpretation)

The mom. eqn. can be simplified – it contains buried within it a copy of the continuity eqn.
The momentum equation is often written in a slightly more compact form

$$\frac{\partial}{\partial t}(\rho u_j) + \partial_i(\rho u_i u_j - T_{ij}) = f_j$$

$$\left[\frac{\partial \rho}{\partial t} + \partial_i(\rho u_i)\right] u_j + \rho \frac{\partial u_j}{\partial t} + \rho u_i \partial_i u_j$$

$$= \partial_i T_{ij} + f_j$$

zero from continuity eqn

also $\partial u_j / \partial t + u_i \partial_i u_j = D u_j / D t$

$$\rho \left[\partial u_j / \partial t + u_i \partial_i u_j \right] = \partial_i T_{ij} + f_j$$

$$\boxed{\begin{array}{l} \rho \dfrac{D u_j}{D t} = \dfrac{\partial T_{ij}}{\partial r_i} + f_j \quad \text{or} \\[2ex] \rho \dfrac{D \underline{u}}{D t} = \nabla \cdot \underline{\underline{T}} + \underline{f} \end{array}}$$

The acceleration of the particle instantaneously at a pt. $\underline{r}$ can be produced only by a body force $\underline{f}(\underline{r}, t)$ or a stress divergence $\nabla \cdot \underline{\underline{T}}(\underline{r}, t)$

Note: this is an Eulerian conservation law
$\underline{\underline{T}}(\underline{r}, t) = \underline{\underline{T}}_E(\underline{r}, t)$ is an Eulerian description of the stress field.

We can define a set of nine scalar fields $T_{ij}(\underline{r}, t)$ by means of the relation

$$T_{ij}(\underline{r}, t) = F_j(\underline{r}, t, \hat{x}_i)$$

with respect to any Cartesian axis system

Given two axis systems

$\hat{x}_1, \hat{x}_2, \hat{x}_3$ let

$$T_{ij}(\underline{r}, t) = F_j(\underline{r}, t, \hat{x}_i) = \hat{x}_j \cdot \underline{F}(\underline{r}, t, \hat{x}_i)$$

$$T_{ij}'(\underline{r}, t) = F_j'(\underline{r}, t, \hat{x}_i') = \hat{x}_j' \cdot F(\underline{r}, t, \hat{x}_i)$$

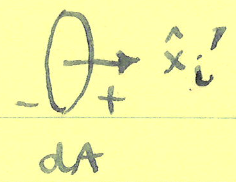
If $T_{ij}(\underline{r}, t)$ and $T_{ij}'(\underline{r}, t)$ (both same $\underline{r}, t$) are to be the components of a Tensor field quantity $\underline{\underline{T}}(\underline{r}, t)$ then $T_{ij}(\underline{r}, t)$ and $T_{ij}'(\underline{r}, t)$ must be related in a certain way

(the components of a tensor transform in a specific way under a change in coord. system.

$$T_{ij}'(\underline{r}, t) = (\hat{x}_i' \cdot \hat{x}_k)(\hat{x}_j' \cdot \hat{x}_\ell)\, T_{k\ell}(\underline{r}, t)$$

Let us demonstrate that $T_{ij}(\underline{r}, t)$ and $T_{ij}'(\underline{r}, t)$ are so related.

Let dA be a small element of area with normal $\hat{x}_i'$ located at $\underline{r}_0$



then $\underline{F}(\underline{r}, t, \hat{x}_i')\, dA = T_{ij}'(\underline{r}, t)\, \hat{x}_j'\, dA$
is the net force exerted by material on + side of dA on material on − side

But this stress can also be computed from $T_{kl}(\underline{r}, t)$
We write $\hat{n} = \hat{x}_i'$ as $\hat{n} = \cancel{n_l \hat{x}_l} = n_k \hat{x}_k$
Then the traction across dA is

$$\underline{F}(\underline{r}, t, \hat{n})\, dA = \hat{x}_l\, T_{kl}(\underline{r}, t)\, n_k \; dA$$

but $n_k = \hat{x}_k \cdot \hat{n} = (\hat{x}_k \cdot \hat{x}_i')$ so the traction is

$$\hat{x}_l\, T_{kl}(\underline{r}, t)\; (\hat{x}_k \cdot \hat{x}_i')\; dA$$

equate the two vectors $\quad T_{ij}'(\underline{r}, t)\, \hat{x}_j' = \hat{x}_l\, (\hat{x}_k \cdot \hat{x}_i')\, T_{kl}(\underline{r}, t)$

dot now by $\hat{x}_j'$

$$T_{ij}'(\underline{r}, t) = (x_i' \cdot \hat{x}_k)(\hat{x}_j' \cdot \hat{x}_l)\, T_{kl}(\underline{r}, t)$$

hence $T_{ij}(\underline{r}, t)$ and $T_{ij}'(\underline{r}, t)$ are the nine Cartesian components of a <u>Tensor field</u> $\underline{T}(\underline{r}, t)$ which we will call (now aha!)

the stress tensor

$$\underline{\underline{T}}(\underline{r},t) = T_{ij}(\underline{r},t)\,\hat{x}_i\hat{x}_j$$
$$T_{ij}(\underline{r},t) = \hat{x}_i \cdot \underline{\underline{T}}(\underline{r},t) \cdot \hat{x}_j$$
$$T_{ij}'(\underline{r},t) = (\hat{x}_i' \cdot \hat{x}_k)(\hat{x}_j' \cdot \hat{x}_\ell)\,T_{k\ell}(\underline{r},t)$$

$$\underline{F}(\underline{r},t,\hat{n}) = \underline{\underline{T}}^{T}(\underline{r},t)\cdot\hat{n} = \hat{n}\cdot\underline{\underline{T}}(\underline{r},t)$$

in the last eqn we see the nature of the stress tensor as a linear vector operator.

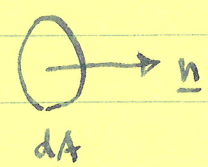
the stress tensor as a linear vector operator

$$\boxed{\begin{aligned} \underline{F}(\underline{r},t,\hat{n}(\underline{r})) &= \hat{n}\cdot\underline{\underline{T}}(\underline{r},t) \\ &= \underline{\underline{T}}^T(\underline{r},t)\cdot\hat{n} \end{aligned}}$$

$\underline{\underline{T}}^T(r,t)$ the transpose of the stress tensor, using our convention is the linear operator which assigns to any unit vector $\hat{n}$ the vector surface traction per unit area acting across the surface with normal $\hat{n}$.

Now lets assign dimensionful vectors to surface patches

let $\underline{n} = \hat{n}\,dA$



$|\underline{n}| = dA =$ area of patch

$\underline{F}dA$ = net traction on an infinitesimal patch dA

$= \underline{n}\cdot\underline{\underline{T}} = \underline{\underline{T}}^T\cdot\underline{n}$

What is physical meaning of linearity of stress tensor

1.a. $(a\underline{n})\cdot\underline{\underline{T}} = \quad a(\underline{n}\cdot\underline{\underline{T}})$

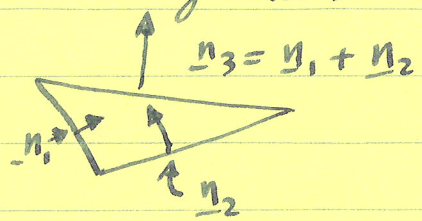
force acting on patch is $\propto$ area, a property stipulated in the beginning

1.b. $(-\underline{n})\cdot\underline{\underline{T}} = -(\underline{n}\cdot\underline{\underline{T}})$

is Newton #3 action = reaction

2. $(\underline{n}_1 + \underline{n}_2) \cdot \underline{\underline{T}} = \underline{n}_1 \cdot \underline{\underline{T}} + \underline{n}_2 \cdot \underline{\underline{T}}$

consider a cylinder with base



let stress be <u>homogeneous</u>

then linearity $\Rightarrow$ net force acting on cylinder is zero

more generally, consider a polyhedron with faces $\underline{n}_1, \ldots \underline{n}_{N-1}$, $\underline{n}_N = \underline{n}_1 + \cdots \underline{n}_{N-1}$

$$(\underline{n}_1 + \cdots + \underline{n}_{N-1}) \cdot \underline{\underline{T}} = \underline{n}_N \cdot \underline{\underline{T}}$$
$$= \underline{n}_1 \cdot \underline{\underline{T}} + \cdots + \underline{n}_{N-1} \cdot \underline{\underline{T}}$$

otherwise every infinitesimal volume (e.g. Cauchy tetrahedron) would suffer an infinite acceleration:

same physical interpretation: in a homogeneous stress field, the net force exerted on a closed region is zero

The case of the ∞-ly long cylinder is a somewhat special degenerate case

<u>Proof of polyhedron face theorem</u>: Let a polyhedron have N faces with areas $\underline{n}_1, \ldots \underline{n}_N$. Then $\sum \underline{n}_i = \sum A_i \hat{n}_i$

$$= \int_{\partial V} \hat{n} \, dA = \int_{\partial V} \hat{n} \cdot \underline{\underline{I}} \, dA = \int_V \nabla \cdot \underline{\underline{I}} \, dV$$

$$= \int_V \mathrm{tr}_{12} (\nabla \underline{\underline{I}}) \, dV = 0, \text{ of course}$$

This equation is a special case of a more general result.

Any standard conservation law may be written in two forms

$$\boxed{\partial\phi/\partial t + \nabla\cdot\underline{K} = k}$$

and, letting $\phi = \rho\psi$

$$\boxed{\rho\, D\psi/Dt + \nabla\cdot[\underline{K} - \rho\psi\underline{u}] = k}$$

$\phi \equiv$ density per unit volume of ϕ-stuff
$\psi \equiv$ density per unit mass of ϕ-stuff

$\underline{K} \equiv$ current density in space of ϕ-stuff
$\underline{K} - \rho\psi\underline{u} \equiv$ current density when moving with the material

e.g. continuity eqn $\underline{K} = \rho\underline{u}$ so $\psi = 1$ and
$\underline{K} - \rho\underline{u} = 0$

there can of course be no current density of mass relative to the material

e.g. momentum eqn $\rho\, D\underline{u}/Dt - \nabla\cdot\underline{\underline{T}} = \underline{f}$

$\underline{\underline{K}} - \rho\underline{u}\underline{u} = -\underline{\underline{T}}$ is the current density of momentum w.r.t. the material

9. A special kind of stress tensor. Pressure

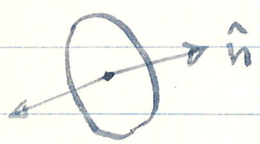
In a fluid (liquid or gas) the stress tensor may often but not always be approximated by the form

$$\boxed{\underline{\underline{T}}(\underline{r},t) = -\,p(\underline{r},t)\,\underline{\underline{I}}}$$

where $\underline{\underline{I}}$ is the identity tensor and $p(\underline{r},t)$ a scalar called the pressure at $\underline{r}$ at time t.

With such a stress tensor, the traction on any small area $\hat{n}\,dA$ is

$$dA\,\hat{n}\cdot\underline{\underline{T}} = -\,p\,dA\,\hat{n}\cdot\underline{\underline{I}} = -\,p\,dA\,\hat{n}$$



$\underline{F} = -\hat{n}p$ dA

the force exerted by the material in front of dA on the material behind dA is a push in the direction $-\hat{n}$.

If dA remains constant, the force $-p\,dA\,\hat{n}$ has the same magnitude for every orientation of $\hat{n}$. This is sometimes expressed roughly as "the pressure p at a point is the same in all directions.

Because of this sameness, a stress tensor of the form

$$\underline{\underline{T}} = -p\underline{\underline{I}} \quad \text{is called } \underline{\text{isotropic}}$$

if $\underline{\underline{T}} = -p\underline{\underline{I}}$ $\qquad (\nabla \cdot \underline{\underline{T}})_j = \partial_i T_{ij} = -\partial_i (p\delta_{ij})$

$$= -\partial_j p$$

$$\nabla \cdot \underline{\underline{T}} = -\nabla p$$

and the <u>momentum equation becomes</u>

$$\boxed{\rho D_t \underline{u} = -\nabla p + f}$$

<u>inviscid</u> fluid

10. <u>The stress deviator</u>

<u>Defn</u>: Let $\underline{\underline{T}}$ be any second order tensor. We define the <u>deviatoric part</u> of $\underline{\underline{T}}$ by the equation

$$\boxed{\underline{\underline{D}} = \underline{\underline{T}} - \left(\tfrac{1}{3} \operatorname{tr} \underline{\underline{T}}\right) \underline{\underline{I}}}$$

note that $\underline{\underline{D}}$, the deviator of any tensor $\underline{\underline{T}}$

has zero trace

$$\mathrm{tr}\,\underline{\underline{D}} = D_{ii} = T_{ii} - \left(\tfrac{1}{3}\mathrm{tr}\,T\right) I_{ii}$$

$$= \mathrm{tr}\,\underline{\underline{T}} - \mathrm{tr}\,\underline{\underline{T}} = 0$$

any tensor $\underline{\underline{T}}$ may be decomposed into its <u>deviatoric part</u> $\underline{\underline{D}}$ and its <u>isotropic part</u> $(\tfrac{1}{3}\mathrm{tr}\,\underline{\underline{T}})\,\underline{\underline{I}}$

$$\boxed{\underset{\substack{\uparrow \\ \text{zero trace}}}{\underline{\underline{T}} = \underline{\underline{D}}} + \underset{\substack{\uparrow \\ \text{isotropic part}}}{\left(\tfrac{1}{3}\mathrm{tr}\,\underline{\underline{T}}\right)\underline{\underline{I}}}}$$

if $\underline{\underline{T}}$ is the stress tensor $\underline{\underline{T}}(\underline{r},t)$ one often defines

$$\boxed{p(\underline{r},t) = -\tfrac{1}{3}\mathrm{tr}\,\underline{\underline{T}}(\underline{r},t)}$$

If the stress $\underline{\underline{T}}(\underline{r},t)$ is already a pressure this eqn is clearly correct. If $\underline{\underline{T}}(\underline{r},t)$ is not purely isotropic, then one defines the isotropic part

$$\left(\tfrac{1}{3}\mathrm{tr}\,\underline{\underline{T}}(\underline{r},t)\right)\underline{\underline{I}} = -p(\underline{r},t)\,\underline{\underline{I}}$$

and calls $p(\underline{r},t)$ the pressure.

The remaining / deviatoric part

$$\underline{\underline{D}}(\underline{r},t) = \underline{\underline{T}}(\underline{r},t) + p(\underline{r},t)\,\underline{\underline{I}}$$

is called the stress deviator or deviatoric stress

Deep in the Earth, the stress tensor is of course dominated by isotropic or pressure part. Deviatoric stresses are however much more interesting

1. they support density inhomogeneities
   if $\underline{\underline{T}}(\underline{r}) = -p(\underline{r})\,\underline{\underline{I}}$, the Earth would be an ellipsoid of revolution, a soln to Clairaut's eqn.
2. deviatoric stresses at plate margins are responsible for quakes.