Lecture # 24

IV. The Linearized Theory of Elasticity

1. Introductory remarks

We have thus far been discussing fluids, first perfect or non-viscous fluids and then viscous fluids, in particular non-viscous fluids.

Our discussion was carried out completely in the Eulerian domain. We utilized the Eulerian form of the conservation laws and we assumed constitutive eqns which related Eulerian variables.

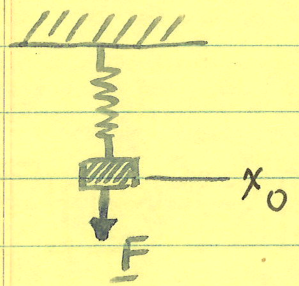
The Eulerian description is natural to the fluid. One naturally uses lab instruments to measure Eulerian quantities, e.g. anemometers $\underline{u}(\underline{r},t)$, temp $\theta(\underline{r},t)$, pressure $p(\underline{r},t)$

↑

pt. in space

On the contrary, the equations governing the motion of a <u>perfectly elastic solid</u> are most naturally and conveniently couched in the <u>Eulerian</u> framework.

Why? What is the main feature which characterizes a perfectly elastic solid? Consider the simplest possible example, a <u>mass-spring system</u>.



Before a force $\underline{F}$ is applied there is a certain reference position, say $x = 0$. The force stretches the spring but if the force is removed, the spring <u>returns to $x = x_0$</u>. This is the key aspect of a <u>perfectly elastic solid</u>:

1. there exists a natural reference configuration to which the material will return if all outside influences are removed.

Note that not all springs necessarily behave in this way. We might add too much weight and stretch the spring "beyond its elastic limit." In that case the ideal elastic constitutive theory is no longer valid. The existence of a natural reference configuration is a constitutive assumption about an ideal type of material (perfectly elastic).

The existence of a natural reference configuration makes the Lagrangian description a very useful one.

Lagrangian description : $\underline{r}(\underline{x}, t) \equiv$ posn at time t of macroparticle at posn $\underline{x}$ in the natural reference configuration

e.g. one clearly uses most naturally the L. descr. in discussing the mass-spring system.

2. second key aspect of a perfectly elastic solid. For mass-spring system. Force $\underline{F}(\underline{x})$ required to stretch spring to pt. $\underline{x}$ depends only on $\underline{x}$, not. e.g. on history of deformation.

Force $\underline{F}(t)$ depends only on instantaneous deformation of the spring $\underline{x}(t)$

Again this clearly only an <u>idealization</u> or a constitutive assumption.

3. linearization.

If force $\underline{F}(\underline{x})$ depends only on displacement $\underline{x}$, then for small displacements it can be written as a Taylor series

$$F = \underset{\uparrow}{F_0} + kx + O(x^2)$$

kx_0

if $O(x^2)$ terms are neglected, we can write as a <u>further</u> <u>approximation</u> $F = kx$, well-known <u>Hookes</u> <u>law</u>.

We will <u>generalize</u> 2. and 3. above to a general perfectly elastic solid. We will assume that a <u>perfectly</u> <u>elastic</u> <u>solid</u> (an idealization) is a material which possesses a <u>natural</u> <u>reference</u> <u>configuration</u>, and that the stresses necessary to deform it away from that configuration depend only on the <u>instantaneous</u> <u>deformation</u>. We will then make the further approximation that for small displacements away from equil. that a generalized form of Hooke's law will hold.

The assumption of a small displacement leads naturally to a completely linearized theory, in contrast to the theory of viscous fluids.
Since completely linear, the elasticity eqns are much more tractable.

Summary:

1. main difference from fluids ∃ a natural reference configuration this the first assumption of perfect elasticity
2. second assumption of perfect elasticity: stress depends only on inst. deformation, not on history of deformation.
3. assumption of small displacements away from equilibrium; generalized form of Hooke's law (a further assumption)
4. this leads to a completely linearized theory (in contrast with Navier-Stokes).

We begin with a discussion of the <u>kinematics</u> of elastic deformation, and we shall specialize to the linearized case immediately. The non-linear or finite-strain case is a completely developed theory, but I have decided in the interest of time to bypass and linearize from the start. Otherwise we would have to plunge into a discussion of the <u>kinematics</u> of <u>finite strain</u>.

<u>References:</u>

<u>Malvern</u>, Intro. to Mech. of a Cont. Medium
complete development of non-linear eqns from a stress-free initial state

<u>Love</u>, A Treatise on the Math. Theory of Elasticity
classical treatise on linearized theory, again from a stress-free initial state.

We are naturally interested in application to Earth, initial state <u>not</u> stress-free. I will develop here the linearized theory from a hydrostatic initial state.

The linearized theory from a non-hydrostatic initial state has been most extensively developed by

Biot, Mechanics of Incremental Deformations.

This work has also been reviewed and extended by

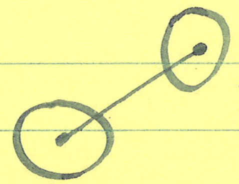
Dahlen (unpublished ms.) Elastic Dislocation Theory for a Self-Gravitating Elastic Configuration with an Initial Static Stress Field.
Landau and Lifshitz: very good on thermodynamics

2. Lagrangian Kinematics

see section III.7 on Eulerian kinematics for review. The discussion here follows that one very closely.
Recall that discussion of Eulerian kinematics

Given a continuum with Eulerian description $\underline{u}(\underline{r}, t)$. We asked: what happens to an inf. small sphere of fluid surrounding the particle $\underline{x}$ which is at $\underline{r}$ at time t.

let ρ', and ρ be *relative* posn vectors at times t, $t+\delta t$



1. a *translation* by $\underline{u}(\underline{r},t)\,\delta t$
2. a *deformation*

$$\underline{\rho}' = \underline{\rho}\cdot\left[\,\underline{\underline{I}} + \underline{\underline{\nabla u}}(\underline{r},t)\,\delta t\,\right]$$

$\underline{\underline{\nabla u}}(\underline{r},t) \equiv$ deformation rate tensor

if we consider a *small* time interval δt, then $\underline{\underline{I}} + \underline{\underline{\nabla u}}\,\delta t$ is a *linear operator* near the identity and may be decomposed readily

$$\underline{\rho}' = \left[\,\underline{\underline{I}} + \underline{\underline{\epsilon}}(\underline{r},t)\,\delta t\,\right]\cdot\left[\,\underline{\underline{I}} + \underline{\underline{\Omega}}(\underline{r},t)\,\delta t\,\right]\cdot\underline{\rho}$$

$\underline{\underline{\epsilon}}(\underline{r},t) \equiv$ strain rate tensor (symmetric)
$\underline{\underline{\Omega}}(\underline{r},t) \equiv$ rotation rate tensor (antisymmetric)

2a. an inf. *rotation* $\underline{\Omega} = -\frac{1}{2}\Lambda\,\underline{\underline{\Omega}} \equiv$ instantaneous angular velocity
2b. an inf. *symmetric squeeze* $\underline{\underline{I}} + \underline{\underline{\epsilon}}\,\delta t$

note this only an adequate descr. of the motion for a small enough sphere of material $O(\rho^2)$ and for a small enough interval of time $O(\delta t^2)$. This is a physical picture of the rate of deformation in terms of $\underline{u}(\underline{r},t)$ the rate of fluid motion. Now lets see if this type of physical picture may be extended to the Lagrangian description.

Consider a perfectly elastic medium in its natural reference configuration. We label macroparticles by their posns in this natural reference configuration $\underline{x}$

Let $\underline{r}(\underline{x},t)$ be the Lagrangian description of the motion of the material We will always write

$$\underline{r}(\underline{x},t) = \underline{x} + \underline{s}(\underline{x},t)$$

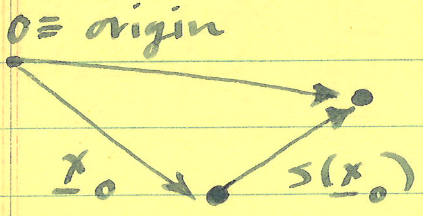
$\underline{s}(\underline{x},t) \equiv$ displacement away from equil. or reference position of particle $\underline{x}$ at time t

We fix attn. on a particular instant t
We will not be interested in the history of the deformation, only in the instantaneous deformation.

We will write $\underline{r}(\underline{x}) = \underline{x} + \underline{s}(\underline{x})$ for brevity.

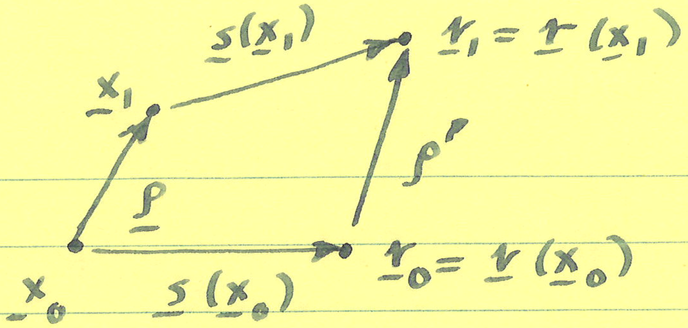
What happens to the material near a particular material particle $\underline{x}_0$?
Consider a small sphere of material.

1. First, it suffers a translation by an amount $\underline{r}(\underline{x}_0) = \underline{x}_0 + \underline{s}(\underline{x}_0)$



we must, as before, also examine the relative displacement of particles near $\underline{x}_0$

The vector $\underline{\rho}$ joining $\underline{x}_0$ to a nearby particle $\underline{x}_1$ is deformed into the vector $\underline{\rho}'$.



we have from the picture

$$\underline{\rho}' = \underline{r}_1 - \underline{r}_0 = \underline{r}(\underline{x}_1) - \underline{r}(\underline{x}_0)$$

$$= \underline{r}(\underline{x}_0 + \underline{\rho}) - \underline{r}(\underline{x}_0)$$

we look only at very nearby particles
For small $\underline{\rho}$

$$\underline{\rho}' = \underline{r}(\underline{x}_0) + \underline{\rho} \cdot \nabla_{\underline{x}} \underline{r}(\underline{x}_0) - \underline{r}(\underline{x}_0)$$

$$\boxed{\underline{\rho}' = \underline{\rho} \cdot \nabla_{\underline{x}} \underline{r}(\underline{x}_0)}$$

The tensor $\nabla_x \underline{r}$ is called the deformation gradient tensor

The deformation of any small sphere of material surrounding a particle $\underline{x}$ at time t may always be expressed in terms of the deformation gradient tensor $\nabla_{\underline{x}} \underline{r}(\underline{x},t)$.

We have not performed any linearization yet; the deformation gradient tensor is the most fundamental tensor even in non-linear or finite strain theories. All the various forms (Cauchy, Green, etc.) of finite strain tensors may all be expressed in terms of $\nabla_{\underline{x}} \underline{r}(\underline{x},t)$.

We however will perform the first linearization at this point.

$$\nabla_{\underline{x}} \underline{r}(\underline{x},t) = \nabla_{\underline{x}} \underline{x} + \nabla_{\underline{x}} \underline{s}(\underline{x},t)$$

$$= \underline{\underline{I}} + \nabla_{\underline{x}} \underline{s}(\underline{x},t)$$

we assume at this point the tensor $\nabla_{\underline{x}} \underline{s}(\underline{x},t)$ called the displacement gradient tensor is small, so that $\underline{\underline{I}} + \underline{\underline{\nabla s}}$ is near the identity.

we assume that the displacements $\underline{s}(\underline{x},t)$ of the particles are such that all displacement gradients are small

In terms of a particular Cartesian axis system

[redacted]

$$\underline{\rho}' = \underline{\rho} \cdot \left[\underline{\underline{I}} + \nabla_{\underline{x}} \underline{s} \right]$$

$$\rho_i' = \rho_j \left(\delta_{ij} + \frac{\partial s_i}{\partial x_j} \right)$$ ↖ write $\partial_j s_i$

Now $\| \nabla_{\underline{x}} \underline{s} \| \ll 1$ [redacted] if and only if $\left| \frac{\partial s_i}{\partial x_j} \right| \ll 1$ all nine components. Note smallness in any one Cart. axis system implies smallness in any Cart. axis system

$$(\partial_j s_i)' = \underset{\substack{\downarrow \\ \leq 1}}{(\hat{x}_j' \cdot \hat{x}_k)} \underset{\substack{\downarrow \\ \leq 1}}{(\hat{x}_i' \cdot \hat{x}_l)} \, \partial_k s_l$$

so if $\|\nabla_{\underline{x}}\underline{s}(\underline{x},t)\| \ll 1$ we can decompose the deformation into an inf. rotation and an inf. symmetric squeeze or strain just as we did in the Eulerian case. We must however make the assumption. In the Eulerian case no linearization was necessary since we were considering *rates* of deformation

Define $\underline{\underline{\sigma}}(\underline{x},t) = \frac{1}{2}\left[(\underline{\underline{\nabla s}})^T + \underline{\underline{\nabla s}}\right]$

$$\underline{\underline{\omega}}(\underline{x},t) = \frac{1}{2}\left[(\underline{\underline{\nabla s}})^T - \underline{\underline{\nabla s}}\right]$$

$\underline{\underline{\sigma}}$ and $\underline{\underline{\omega}}$ are the *symmetric and antisymmetric* parts of the *displacement gradient tensor*.

Now since $\underline{\underline{\nabla s}}$ is small $\ll 1$

$$\underline{\underline{I}} + \underline{\underline{\nabla s}} = \underline{\underline{I}} + \underline{\underline{\sigma}} - \underline{\underline{\omega}}$$

$$\cong (\underline{\underline{I}} + \underline{\underline{\sigma}}) \cdot (\underline{\underline{I}} - \underline{\underline{\omega}}) \approx (\underline{\underline{I}} - \underline{\underline{\omega}}) \cdot (\underline{\underline{I}} + \underline{\underline{\sigma}})$$

Cartesian decomposition → polar decomposition

$$\underline{\rho}' = \underline{\rho}\cdot(\underline{\underline{I}} + \underline{\underline{\nabla s}})$$

$$= \underline{\rho}\cdot[(\underline{\underline{I}} + \underline{\underline{\sigma}})\cdot(\underline{\underline{I}} - \underline{\underline{\omega}})]$$

$$\boxed{\underline{\rho}' = [(\underline{\underline{I}} + \underline{\underline{\omega}})\cdot(\underline{\underline{I}} + \underline{\underline{\sigma}})]\cdot\underline{\rho}}$$

1. $\underline{\underline{\sigma}}(\underline{x},t) \equiv$ infinitesimal strain tensor

$$\sigma_{ij} = \frac{1}{2}(\partial_j s_i + \partial_i s_j)$$

$\underline{\underline{I}} + \underline{\underline{\sigma}}$, a symmetric squeeze or
strain of the sphere of
material around $\underline{x}_0$.

2. $\underline{\underline{\omega}}(\underline{x},t) \equiv$ inf. rotation tensor

$$\omega_{ij} = \frac{1}{2}(\partial_j s_i - \partial_i s_j) \quad \text{antisymmetric}$$

define associated vector $\underline{\omega} = -\frac{1}{2}\Lambda\underline{\underline{\omega}}$

$\omega_i = -\frac{1}{2}\epsilon_{ijk}\omega_{jk}$ in rt. handed

an inf. rotation of the sphere thru
the angle $\underline{\omega}$ (axis $\underline{\omega}$, angle $|\underline{\omega}|$)

Material around any point $\underline{x}$ at any time t has been subjected to

1. a translation $\underline{\xi}(\underline{x},t)$
2. an inf. rotation $\underline{\omega}(\underline{x},t) = -\frac{1}{2}\Lambda\underline{\underline{\omega}}(\underline{x},t)$
3. a symmetric squeeze or <u>strain</u> $\sigma(\underline{x},t)$

$\underline{\underline{\sigma}}(\underline{x},t) \equiv$ strain tensor, is symmetric has three real eivs and three mutually orthogonal real eigenvectors, say $\sigma_1, \sigma_2, \sigma_3$

Then sphere of material around $\underline{x}$ is squeezed into an <u>ellipsoid</u> with principal axes ~~[illegible]~~

$1+\sigma_1,\ 1+\sigma_2,\ 1+\sigma_3$

I assume that you are already familiar enough with this that I do not have to discuss examples.

The general polar decomposition theorem may be applied even in the case of finite deformation (see Halmos or Malvern for a proof of this theorem), but it is usually not very useful since the resulting symmetric & orthogonal tensors are so complicated. More useful

strain → Tensors may be defined (see Malvern)

end here 10 Apr.

Lecture # 25 Review

Forgot Landau and Lifshitz
Began discussing the linearized theory of perfect elasticity.

Main differences from theories of fluids:

1. existence of a natural reference configuration
2. This makes the use of a Lagrangian formulation very convenient.

We seek a Lagrangian description of the motion

$$\underset{\substack{\uparrow \\ \text{posn. in} \\ \text{reference conf.}}}{\underline{r}(\underline{x},t)} = \underline{x} + \underset{\substack{\uparrow \\ \text{displacement of particle } \underline{x} \\ \text{at time } t.}}{\underline{s}(\underline{x},t)}$$

Local kinematics in Lagrangian formulation. Given a particle $\underline{x}$. At time t, displaced by $\underline{s}(\underline{x},t)$. What happens to an inf. small sphere of material surrounding $\underline{x}$.

1. translated by $\underline{s}(\underline{x}, t)$ (center of sphere)
2. deformed by

$$\underline{\rho}' = \underline{\rho} \cdot [\underline{\underline{I}} + \underset{\substack{\uparrow \\ \text{displacement gradient tensor}}}{\underline{\underline{\nabla s}}}(\underline{x}, t)]$$

~~deform~~

Only simple discussion of the nature of this deformation is for the case $\| \underline{\underline{\nabla s}} \| \ll 1$, small displacement gradients

In that case

$$\boxed{\underline{\rho}' = [\underline{\underline{I}} + \underline{\underline{\sigma}}(\underline{x}, t)] \cdot [\underline{\underline{I}} + \underline{\underline{\omega}}(\underline{x}, t)] \cdot \rho}$$

Material subjected to an infinitesimal rotation thru an angle $\underline{\omega} = -\frac{1}{2} \Lambda \underline{\underline{\omega}}$ followed by an infinitesimal strain

$$\underline{\underline{\sigma}} = \frac{1}{2} [(\underline{\underline{\nabla s}})^T + \underline{\underline{\nabla s}}]$$ strain tensor symmetric

We now begin our discussion of the constitution relations for a perfectly elastic material.

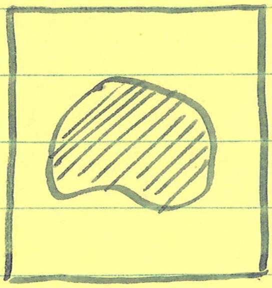
strain tensors may be defined (see Malvern)

end here 10 Apr 1972

3. Perfectly elastic constitutive equations

We consider a small piece of material. We are going to define what it means for this material to behave in a perfectly elastic fashion.

1. First, there exists a natural reference configuration. Let this be a state of complete thermodynamic equilibrium in which the static initial stress tensor is isotropic. We consider a tiny piece. We may suppose it is e.g. immersed in an oil bath whose pressure is p_0 and whose temp is θ_0



$$\boxed{\underline{\underline{T}}_0 = -p_0 \underline{\underline{I}}\ , \quad \theta = \theta_0}$$

We call this the initial or reference or equil configuration.

We wish to consider <u>small</u> <u>deformations</u> away from this initial state.
We adopt a <u>Lagrangian</u> <u>description</u>.

2. Fix attention on a particular particle $\underline{x}$. We assume that the temperature $\theta_L(\underline{x},t) = \theta(\underline{x},t)$ and the <u>displacement</u> <u>gradient tensor</u> $\nabla\underline{s}(\underline{x},t)$ represent a complete set of thermodynamical variables for the material immediately surrounding $\underline{x}$ and that that material is always in a state of <u>local</u> <u>thermodynamic equilibrium</u>.

1. θ and $\underline{\underline{\nabla s}}$ are a complete set of local <u>thermodynamic</u> <u>variables</u>.
2. material always and everywhere in a state of <u>local thermodynamic equilibrium</u>

Material, when deformed, not in <u>complete</u> thermo. equil since $\theta(\underline{x},t)$ may vary from particle to particle, hence from place to place, hence there will be heat flow, $\underline{H}$.

We will discuss the heat flow later

The only non-equilibrium variable is the heat flow $\underline{H}$.

All the other thermal and mechanical variables are given by the assumption of local thermodynamic equilibrium

e.g. the Lagrangian stress tensor at the particle $\underline{x}$ at time $\underline{t}$

$$\underline{\underline{T}}_L(\underline{x},t) = -p_0(\underline{x})\,\underline{\underline{I}} + \underline{\underline{E}}_L(\underline{x},t)$$

where $\underline{\underline{E}}_L(\underline{x},t) \equiv$ Lagrangian elastic stress

is a fcn only of $\theta_L(\underline{x},t)$ and $\underline{\underline{\nabla s}}(\underline{x},t)$

$$\underline{\underline{E}}_L = \underline{\underline{E}}_L(\theta, \underline{\underline{\nabla s}}, \underset{\uparrow}{\underline{x}})$$

to deal, e.g. with polycrystals

explicit fcn of particle $\underline{x}$

Likewise the Lagrangian entropy (per gram) $S_L(\underline{x},t)$ and internal energy $U_L(\underline{x},t)$

$$S_L = S_L(\theta, \underline{\underline{\nabla s}}, \underline{x})$$
$$U_L = U_L(\theta, \underline{\underline{\nabla s}}, \underline{x})$$

Just like a fluid except for $\underline{\underline{\nabla s}}$ rather than τ.

Note that $\underline{\underline{T}}_L$, S_L, U_L do not depend in any way on the history of the deformation, only the the <u>instantaneous</u> <u>deformation</u> and the instantaneous temperature.

Recall the mass-spring analogy.

<u>The above is really the key assumption of perfect elasticity</u>
The stress is said to perfectly or fully <u>recoverable</u>. If $\nabla s = 0$ and $\theta = \theta_0$, then $\underline{\underline{T}}_L = -p_0\underline{\underline{I}}$.

Let's see how this allows us to obtain a complete system of eqns for a perfectly elastic material. We will write these in a clumsy form, but all we wish to do is count variables

Eulerian eqns in terms of Eulerian quant.

$$D_t \rho + \rho \nabla \cdot \underline{u} = 0$$
$$\rho D_t \underline{u} = \nabla \cdot \underline{\underline{T}}_E + \underline{f}$$
$$\rho D_t U_E + \nabla \cdot \underline{H} = \mathrm{tr}\,(\underline{\underline{T}}_E \cdot \underline{\underline{\epsilon}}) + h$$

To convert to Lagrangian

$$\underline{u}_E(\underline{r}(\underline{x},t),t) = D_t \underline{s}(\underline{x},t)$$

Constitutive relations for l.t. equil. quant

$$\underline{\underline{T}}_L = -p_0 \underline{\underline{I}} + \underline{\underline{E}}_L(\theta, \underline{\underline{\nabla s}}, \underline{x})$$
$$U_L = U_L(\theta, \underline{\underline{\nabla s}}, \underline{x})$$

15 eqns
18 unknowns

$$\rho_L, U_L, \underline{s}, \theta_L, \underline{\underline{T}}_L, \underline{H}, \underline{u}_E$$

need 3 more eqns. Clearly need a heat flow eqn. Here one can take Fourier's law, or in some circumstances might require heat flow to depend on local deformation as well.

Above very clumsy form of relevant eqns. Usually one rewrites the Eulerian cons. eqns in Lagrangian form. This most conveniently done in terms of the Piola-Kirchoff stress tensor. We'll say no more about the general theory.

We just wanted to show that there is in fact a general theory (the above is fully non-linear). The key assumption is that of local thermodynamic equilibrium with θ, $\underline{\underline{\nabla s}}$ a complete set of variables.

4. Linearized theory

In the linearized theory the local deformation surrounding a particle $\underline{x}$ consists of a

1. translation
2. rotation
3. symmetric strain

Since $\underline{\underline{T}}_0$ is isotropic and in the absence of external magnetic or electric fields, the rotation will produce no changes but the stretching and the temp. change may.

$$\underline{\underline{T}}_0 = -p_0 \underline{\underline{I}}$$

$$\underline{\underline{T}}_L(\underline{x}, t) = -p_0(\underline{x}) \underline{\underline{I}} + \underline{\underline{E}}_L(\underline{x}, t)$$

$$\underline{\underline{E}}_L(\underline{x}, t) = \underline{\underline{E}}_L\left(\theta(\underline{x}, t), \underline{\underline{\nabla s}}(\underline{x}, t), \underline{x}\right)$$

the rotation part of $\underline{\underline{\nabla s}}$ can not give rise to any stresses.
$\underline{\underline{E}}_L$ depends on local deformation = local rigid body rotation + local strain. Local rigid body rotation does not alter stress.

Note: this *not* true if $\underline{\underline{T}}_0 \neq -p_0 \underline{\underline{I}}$. Then get additional term due to *rotated initial stress*.

$$\boxed{\underline{\underline{E}}_L = \underline{\underline{E}}_L(\theta, \underline{\underline{\sigma}}, \underline{x})}$$

Note: this result may be readily generalized to the non-linear case.

Recoverability of stress (perfect elasticity) requires

$$\boxed{\underline{\underline{E}}_L = 0 \quad \text{for} \quad \begin{array}{l} \theta = \theta_0 \\ \underline{\underline{\sigma}} = 0 \end{array}}$$

We write $\underline{\underline{E}}_L(\theta, \underline{\underline{\sigma}}, \underline{x})$ as a Taylor series and neglect higher order terms, already neglected in our discussion of kinematics.
i.e. we assume that dependence of $\underline{\underline{E}}_L$ on $\underline{\underline{\sigma}}$ and $\theta - \theta_0$ is *linear*

Very similar to discussion of N. viscosity

$$\underline{\underline{E}}_L = \underline{\underline{C}}_{\underline{x}}(\underline{\underline{\sigma}}) + \underline{\underline{W}}_{\underline{x}}(\Delta\theta)$$

$$\Delta\theta = \theta - \theta_0$$

The functions $\underline{\underline{C}}_{\underline{x}}$ and $\underline{\underline{W}}_{\underline{x}}$ depend explicitly on the particle $\underline{x}$ and are *linear* mappings

$$\underline{\underline{C}}_{\underline{x}} : \otimes^2 R^3 \to \otimes^2 R^3$$
$$\underline{\underline{W}}_{\underline{x}} : R \to \otimes^2 R^3$$

$$\underline{\underline{C}}_{\underline{x}}(a\underline{\underline{\sigma}}_1 + b\underline{\underline{\sigma}}_2) = a\underline{\underline{C}}_{\underline{x}}(\underline{\underline{\sigma}}_1) + b\underline{\underline{C}}_{\underline{x}}(\underline{\underline{\sigma}}_2)$$

etc.

This is simply a *linearization* of the const. relation which we hope will be valid for *small* strains $\underline{\underline{\sigma}}$ and small temp. differences $\Delta\theta$

We already know all about such tensor-valued fcns or linear mappings. We know that under the above assumptions.

There exists a fourth order tensor $\underline{\underline{C}}(\underline{x})$ and a second order tensor $\underline{\underline{W}}(\underline{x})$ such that

$$\underline{\underline{C}}\,(\underline{x}) \in \otimes^4 R^3$$

$$\underline{\underline{W}}\,(\underline{x}) \in \otimes^2 R^3$$

↑

explicit dependence on $\underline{x}$

In a particular $\hat{x}_1, \hat{x}_2, \hat{x}_3$

$C_{ijkl}(\underline{x})$ 81 comp.

$W_{ij}(x)$ 9 comp.

Note: it's $\underline{\underline{E}}^L$ not $\underline{\underline{E}}^E$ which is related to the strain by Hooke's law.

here is the explicit dependence on $\underline{x}$.

$$\boxed{\begin{aligned} E^L_{ij}(\underline{x},t) &= C_{ijkl}(\underline{x})\,\sigma_{kl}(\underline{x},t) \\ &\quad + W_{ij}(\underline{x})\,\Delta\Theta(\underline{x},t) \end{aligned}}$$

This in a part. Cart. axis system

Important: $C_{ijkl}(\underline{x})$ and $W_{ij}(\underline{x})$ are components of tensors $\underline{\underline{C}}(\underline{x})$ and $\underline{\underline{W}}(\underline{x})$

Recall we have assumed that a perfectly elastic material always in <u>local thermo. equil.</u> and that θ and $\underline{\underline{\nabla s}}$ serve as a complete set of thermodynamical variables

For the case $\underline{\underline{T}}_0 = -p_0 \underline{\underline{I}}$, and <u>linearized</u> only the inf. <u>strain</u> part of $\underline{\underline{\nabla s}}$ is required.

This means in particular that the <u>entropy per gram</u> $S_L(\underline{x},t)$ may be considered known if $\underline{\underline{\nabla s}}$ and θ are known

$$S_L(\underline{x},t) = S_L(\underline{\underline{\nabla s}}, \theta, \underline{x}) \qquad * .$$

we can always solve for $S_L(\underline{x},t)$ hence $\underline{\underline{\nabla s}}$, S serve equally well as a <u>complete set</u> of variables

$\underline{\underline{\nabla s}}, \theta$ or $\underline{\underline{\nabla s}}, S$ complete set of variables

Could equally well consider

$$\underline{\underline{E}}_L = \underline{\underline{E}}_L(\underline{\underline{\nabla s}}, S, \underline{x}) \quad \text{or}$$

$$= \underline{\underline{E}}_L(\underline{\underline{\sigma}}, S, \underline{x}) \quad \text{by same argument}$$

When this relationship is <u>linearized</u>

$$\underline{\underline{E}}_L = \underline{\underline{\tilde{C}}}_{\underline{x}}(\underline{\underline{\sigma}}) + \underline{\underline{\tilde{W}}}_{\underline{x}}(\Delta S)$$

$$\underline{\underline{\underline{\tilde{C}}}}_x : \otimes^2 R^3 \to \otimes^2 R^3 \quad \underline{\text{linear}}$$
$$\underline{\underline{\tilde{W}}}_x : R \to \otimes^2 R^3 \quad \underline{\text{linear}}$$

hence $\exists$ associated <u>tensors</u> $\underline{\underline{\underline{\tilde{C}}}}(\underline{x})$ and $\underline{\underline{\tilde{W}}}(\underline{x})$

$$\underline{\underline{\underline{\tilde{C}}}}(\underline{x}) \in \otimes^4 R^3$$
$$\underline{\underline{\tilde{W}}}(\underline{x}) \in \otimes^2 R^3$$

$$\boxed{E_{ij}{}^L(\underline{x},t) = \tilde{C}_{ijkl}(\underline{x})\,\sigma_{kl}(\underline{x},t) + \tilde{W}_{ij}(\underline{x})\,\Delta S(\underline{x},t)}$$

The tensor $C(\underline{x})$ is called the <u>isothermal</u> <u>elastic tensor</u>; if $\Delta\theta = 0$, this tensor gives the <u>stress</u> produced by a given strain.
In any $\hat{x}_1, \hat{x}_2, \hat{x}_3$, the coefficients $C_{ijkl}(\underline{x})$ are called <u>isothermal</u> elastic coefficients.

If $\Delta S = 0$, the tensor $\tilde{C}(\underline{x})$, called the isentropic elastic tensor, gives the stress.

$\tilde{C}_{ijkl}(\underline{x})$ isentropic elastic coefficients.

5. An example: an ideal gas

Consider as an example of a perfectly elastic "solid" an ideal gas.

Initially at temp θ_0 and specific volume τ_0

Subject to a deformation (small)

For all θ, τ (exact)

$$p = \frac{R\theta}{\tau} \qquad *$$

If τ_0 and θ_0 are changed slightly

$$\Delta p = \frac{R}{\tau_0}\Delta\theta - \frac{R\theta_0}{\tau_0^2}\Delta\tau$$

$$= p_0\frac{\Delta\theta}{\theta_0} - p_0\frac{\Delta\tau}{\tau_0}$$

But $\frac{\Delta\tau}{\tau_0} = \nabla\cdot\underline{s}$ and

$$\underline{\underline{E}}_L = -\Delta p\,\underline{\underline{I}}$$

$$E_{ij} = p_0\,\delta_{ij}\,(\nabla\cdot\underline{s}) - p_0\,\delta_{ij}\,\frac{\Delta\theta}{\theta_0}$$

$$\boxed{\begin{array}{l} E_{ij} = \left(p_0\,\delta_{ij}\,\delta_{kl}\right)\sigma_{kl} - \left(\frac{p_0}{\theta_0}\,\delta_{ij}\right)\Delta\theta \qquad *' \\ \text{hence} \quad C_{ijkl} = p_0\,\delta_{ij}\,\delta_{kl} \\ \qquad\qquad W_{ij} = -\frac{p_0}{\theta_0}\,\delta_{ij} \end{array}}$$

formalism works fine

$$p = \frac{R e^{\left(\frac{\gamma-1}{R}\right)S}}{\tau^{\gamma}} \quad (**)$$

eqn of state in terms of entropy per gram S

$$\Delta p = -\gamma\,\frac{R e^{\left(\frac{\gamma-1}{R}\right)S_0}}{\tau_0^{\gamma+1}}\,\Delta\tau + \frac{\gamma-1}{R}\,\frac{R e^{\left(\frac{\gamma-1}{R}\right)S_0}}{\tau_0^{\gamma}}\,\Delta S$$

$$\Delta p = -\gamma p_0\,\frac{\Delta\tau}{\tau_0} + \frac{\gamma-1}{R}\,p_0\,\Delta S$$

$$\boxed{E_{ij} = (\gamma P_0 \delta_{ij} \delta_{kl})\,\sigma_{kl} - \left(\frac{\gamma-1}{R} P_0 \delta_{ij}\right) \Delta S \qquad **'}$$

$$\tilde{C}_{ijkl} = \overset{\text{Laplace's error}}{\gamma} P_0 \delta_{ij} \delta_{kl}$$

$$\tilde{W}_{ij} = -\frac{\gamma-1}{R} P_0 \delta_{ij}$$

Remarks:

1. eqns * and ** are <u>exact</u> for ideal gases
   *' and **' are only valid for <u>small</u> changes away from θ_0 and τ_0

2. the <u>linearized</u> theory of elasticity may be perfectly well applied to <u>small</u> changes of <u>any</u> non-viscous fluid (not necessarily perfect gas)
   Usually this means sound (or P) wave propagation
   For a non-viscous fluid there are <u>fewer</u> thermodynamic variables
   Instead of θ and $\underline{\underline{\nabla s}}$ (or $\underline{\underline{\sigma}}$) ~~[illegible]~~ or S and $\underline{\underline{\nabla s}}$ (or $\underline{\underline{\sigma}}$) we need only θ (or S) and the density ρ or specific volume τ. The correspondence is always

Lecture # 26 Review

Linearized theory of perfect elasticity
Natural reference configuration in
complete thermo- and mechanical equil.
Particle posns $\underline{x}$
Use Lagrangian formulation $\underline{r}(\underline{x},t) = \underline{x} + \underset{\substack{\uparrow \\ \text{displ.}}}{\underline{s}}(\underline{x},t)$

When in motion, always and everywhere in a state of local thermodynamic equil. Complete set of thermo. variables is $\underline{\underline{\nabla s}}(\underline{x},t)$ and $\theta(\underline{x},t)$
or $\underline{\underline{\nabla s}}(\underline{x},t)$ and $S(\underline{x},t)$

~~Stress cannot depend on local rigid body rotation~~
Only non-equil. variable is heat flow $\underline{H}$

$\underline{\underline{T}}_L(\underline{x},t) \equiv$ Lagrangian descr. of Cauchy stress

$$\underline{\underline{T}}_L(\underline{x},t) = -p_0(\underline{x})\,\underline{\underline{I}} + \underset{\substack{\uparrow \\ \text{elastic stress}}}{\underline{\underline{E}}_L(\underline{x},t)}$$

$$\underline{\underline{E}}_L(\underline{x},t) = \underline{\underline{E}}_L(\theta, \underline{\underline{\nabla s}}, \underline{x}) \quad \underline{\underline{\text{or}}}$$

$$= \underline{\underline{\tilde{E}}}_L(S, \underline{\underline{\nabla s}}, \underline{x})$$

now <u>linearize</u> this stress – strain const. relation

incremental elastic stress $\underline{\underline{E}}_L$ <u>cannot</u> depend on local rigid rotation

<u>Recoverability</u> of stress (perfect elasticity) requires

[illegible]

$\underline{\underline{E}}_L(\underline{x},t) = 0$ when $\underline{\underline{\sigma}}(\underline{x},t) = 0$ and $\theta(\underline{x},t) = \theta_0$ <u>or</u> $S(\underline{x},t) = S_0(\underline{x}$

in a part. Cart. axis system

$$E_{ij}^{L}(\underline{x},t) = C_{ijkl}(\underline{x})\,\sigma_{kl}(\underline{x},t) + W_{ij}(\underline{x})\,\Delta\theta(\underline{x},t$$

$$E_{ij}^{L}(\underline{x},t) = \tilde{C}_{ijkl}(\underline{x})\,\sigma_{kl}(\underline{x},t) + \tilde{W}_{ij}(\underline{x})\,\Delta S(\underline{x},t)$$

$C_{ijkl}(\underline{x})$ comp. of $C(\underline{x}) \in \otimes^4 R^3$

$W_{ij}(\underline{x})$ comp of $\underline{\underline{W}}(\underline{x}) \in \otimes^2 R^3$

etc

$C(\underline{x})$ isothermal elastic tensor

$\tilde{C}(\underline{x})$ isentropic elastic tensor

We have in general 81 C_{ijkl}, $\tilde{C}_{ijkl}$, 9 W_{ij}, $\tilde{W}_{ij}$ <u>empirical</u> coeff. We must <u>measure</u>. How many are ind. We look for:

1. symmetries
2. relations between isothermal & isentropic

made by $\nabla \cdot \underline{s} = \mathrm{tr}\, \sigma_{kl} = \frac{\Delta \tau}{\tau}$

end here 12 Apr.

6. Elementary symmetries

C and $\tilde{C}$ have 81 components
W and $\tilde{W}$ have 9 components

81 isothermal elastic coeff. C_{ijkl}
81 isentropic " " $\tilde{C}_{ijkl}$

How many of these are independent

$$E_{ij} = C_{ijkl}\, \sigma_{kl} + W_{ij}\, \Delta\theta$$

first consider a change *only* of temp
$\underline{\sigma = 0}$

Since $E_{ij} = E_{ji}$

$$\boxed{W_{ij} = W_{ji} \quad \text{only six independent } W_{ij}}$$

now consider a change at *const* temp $\Delta\theta = 0$

$$E_{ij} = C_{ijkl}\, \sigma_{kl}$$

take all but $\sigma_{12} = 0$, then

$E_{ij} = C_{ij12} \sigma_{12}$ hence

$C_{ij12} = C_{ji12}$

similarly for any k, l

$C_{ijkl} = C_{jikl}$

second, note that $\sigma_{kl} = \sigma_{lk}$

($\underline{\underline{\sigma}}$ symmetric also)

$$E_{ij} = C_{ijkl} \sigma_{kl} = \frac{1}{2} C_{ijkl} (\sigma_{kl} + \sigma_{lk})$$

$$= \frac{1}{2} (C_{ijkl} + C_{ijlk}) \sigma_{kl}$$

C_{ijkl} always appears in the combination $\frac{1}{2}(C_{ijkl} + C_{ijlk})$ and neither can be measured separately. We may replace C_{ijkl} by $C_{ijkl}' = \frac{1}{2} (C_{ijkl} + C_{ijlk})$ in

$$E_{ij} = C_{ijkl}' \sigma_{kl} + W_{ij} \Delta\theta$$

but $C_{ijkl}' = C_{ijlk}'$ hence without loss of generality

$C_{ijkl} =$ [illegible] C_{ijlk}

Summary of findings:

$C_{ijkl} = C_{jikl} = C_{ijlk}$
$W_{ij} = W_{ji}$

36 ind. C_{ijkl} , 6 ind W_{ij}
Same true of isentropic coefficients

$\tilde{C}_{ijkl} = \tilde{C}_{jikl} = \tilde{C}_{ijlk}$
$\tilde{W}_{ij} = \tilde{W}_{ji}$

All others determined from symmetry relations.
We can also use the second law of thermo. to reduce the number of ind. elastic coefficients even further to 21.

7. Thermodynamic properties of the elastic tensors

Let us investigate the consequences of the second law of thermodynamics for a linearly elastic medium.

The law is valid in the following form because of the assumption of local thermodynamic equil. (Entropy / gm S is a state function, so is internal energy / gm U)

Recall for a simple system subjected to a reversible change (reversible means always in thermodynamic equil)

what is solid generalization of $d\mathcal{U} = \theta ds - pdv$

$$\theta\, dS = dU - \Delta W \qquad d\mathcal{U} = \theta\, dS + \Delta W$$

or if sample weighs m grams, $m\, d\mathcal{U} = \theta\, dS + \Delta W$

dS = change in entropy /gm

dU = change in internal energy /gm

ΔW = work done on system /gm

$\theta\, dS$ = heat added to the system /gm

This eqn requires modification if some of the external work is used

We usually write S = entropy/gm, U = energy/gm. If we want this, and if we consider a small mass m: $m\,dS \to dS$, $m\,dU \to dU$

to accelerate the c.o.m. of the whole system, imparting to it some k.e. of translation

ΔT = increase in k.e. of system

$$m\,\theta\,dS = m\,dU + \Delta T - \Delta W$$

↓ work done on a small mass m.

We now must find $\Delta W - \Delta T$

We are considering a lump of material moving with the material (always the same material)

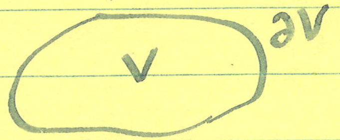
$\boxed{\text{go to top page 318 first}}$

Suppose there are no body forces

Suppose a change $\underline{\underline{\Delta\sigma}}$ is produced in the strain tensor by applying surface stresses in such a way that the stress tensor inside is $\underline{\underline{T}}_L$

$\underline{\underline{\sigma}} \to \underline{\underline{\sigma}} + \underline{\underline{\Delta\sigma}}$,

There may have some strain $\underline{\underline{\sigma}}$ present already so $\underline{\underline{T}}$ is not necessarily isotropic (it is the instantaneous $\underline{\underline{T}}$; also note $\underline{\underline{T}}$ is $\underline{\underline{T}}_L$, Lagrangian)



↑ apply stresses to produce $\underline{\underline{\Delta\sigma}}$

We are in the Lagrangian formulation

$$\underline{r}(\underline{x},t) = \underline{x} + \underline{s}(\underline{x},t)$$

$$\underline{u}_L(\underline{x},t) = D_t\,\underline{r}(\underline{x},t)$$

$\uparrow$ Lagrangian velocity

$$D_t\,\underline{r} = \underline{u}_L \qquad \text{integrate}$$

$$\underline{r} = \underline{x} + \underline{u}\,\Delta t + O(\Delta t)^2$$

or

$$\underline{s} = \underline{u}\,\Delta t + O(\Delta t)^2$$

during a small time Δt

The work done on the material during <u>small</u> time Δt by the applied stresses is

$$\Delta W = \Delta t \int_{\partial V(t)} u_i T_{ij} n_j \, dA =$$

$$\Delta t \int_{V(t)} \frac{\partial}{\partial x_j}(u_i T_{ij})\, dV \quad = (\text{to first order})$$

$$= \int_V \frac{\partial}{\partial x_j}(\Delta s_i T_{ij})\, dV =$$

$$= \int_V T_{ij} \frac{\partial}{\partial x_j}(\Delta s_i)\, dV + \int_V \Delta s_i \frac{\partial T_{ij}}{\partial x_j}\, dV$$

The change in kinetic energy of the material is

$$\Delta T = \Delta t \frac{d}{dt} \int_{V(t)} \tfrac{1}{2} \rho u_i u_i \, dV$$

$$= \Delta t \int_{V(t)} \rho u_i \frac{Du_i}{Dt} \, dV \quad =$$

by eqn of motion (no body force)

if there were we would have to include it in $\Delta W \rightarrow$ same answer

$$= \Delta t \int_{V(t)} u_i \frac{\partial T_{ij}}{\partial x_j} \, dV = \int_V \Delta s_i \frac{\partial T_{ij}}{\partial x_j} \, dV$$

Thus

$$\Delta W - \Delta T = \int_V T_{ij} \, \Delta\sigma_{ij} \, dV$$

$$= \int_V \mathrm{tr} \left(\underline{\underline{T}} \cdot \underline{\underline{\Delta\sigma}} \right) dV$$

For a very small mass M of material, this is

$$\Delta W - \Delta T = m\tau \, T_{ij} \Delta\sigma_{ij}$$

$$= m\tau \, \mathrm{tr} \left(\underline{\underline{T}} \cdot \underline{\underline{\Delta\sigma}} \right)$$

$$m\theta \, dS = m \, dU - m\tau \, \mathrm{tr} \left(\underline{\underline{T}} \cdot \underline{\underline{\sigma}} \right)$$

$$\boxed{\begin{aligned} dU &= \theta\, dS + \tau\, T_{ij}\, d\sigma_{ij} \\ &= \theta\, dS + \tau\, \mathrm{tr}\,(\underline{\underline{T}} \cdot d\underline{\underline{\sigma}}) \\ &= \theta\, dS + \tau\, \underline{\underline{T}} : d\underline{\underline{\sigma}} \end{aligned}}$$

This is <u>second law of thermodynamics</u> for a linearized perfectly elastic material

Note: recall that the linearized elasticity theory applies perfectly well for <u>small</u> displacements in non-viscous <u>fluids</u> as well as solids.

Fewer thermodynamic variables

In a <u>fluid</u> $\underline{\underline{T}} = -p\underline{\underline{I}}$

$$\begin{aligned} T_{ij}\, \Delta\sigma_{ij} &= -p\, \Delta\,(\nabla \cdot \underline{s}) \\ &= -p\, \frac{\Delta\tau}{\tau} \end{aligned}$$

$$\boxed{dU = \theta\, dS - p\, d\tau}$$

reduces to familiar form in a fluid

now U is a fcn of S and $\underline{\sigma}$
so

$$\boxed{\begin{aligned} \left(\frac{\partial U}{\partial \sigma_{ij}}\right)_S &= \tau T_{ij} \\ \left(\frac{\partial U}{\partial S}\right)_\sigma &= \theta \end{aligned}}$$

The stress is the partial derivative of the internal energy w.r.t. the strain. For Green this was the starting point of elasticity (see e.g. Landau & Lifshitz).

Clearly $\left(\frac{\partial U}{\partial \sigma_{11}}\right)_S$ means partial holding S and all other σ_{ij} fixed

If we define

$$\boxed{F = U - \theta S \quad \text{Helmholtz free energy / gm}}$$

$dU = dF + \theta dS + S d\theta$ so second law is

$$\boxed{dF = \tau T_{ij}\, d\sigma_{ij} - S d\theta}$$

$$\boxed{\begin{aligned} \left(\frac{\partial F}{\partial \sigma_{ij}}\right)_{\theta} &= \tau T_{ij} \\ \left(\frac{\partial F}{\partial \theta}\right)_{\sigma} &= -S \end{aligned}}$$

These equations may be used to give one further symmetry of the elastic coefficients.

For an <u>isothermal</u> change $\theta = \theta_0$ throughout

$$T_{ij} = -p_0 \delta_{ij} + C_{ijkl} \sigma_{kl}$$

so

$$\boxed{C_{ijkl} = \left(\frac{\partial T_{ij}}{\partial \sigma_{kl}}\right)_{\theta_0, p_0}}$$

derivatives evaluated in <u>initial state</u> $\underline{\underline{T}} = -p_0 \underline{\underline{I}}$, $\theta = \theta_0$ (a Taylor series)

For <u>isen</u>tropic changes (S kept const at $S = S_0$)

$$\boxed{\tilde{C}_{ijkl} = \left(\frac{\partial T_{ij}}{\partial \sigma_{kl}}\right)_{s_0, p_0}}$$

now from above

$$\left(\frac{\partial T_{ij}}{\partial \sigma_{kl}}\right)_\theta = \left[\frac{\partial}{\partial \sigma_{kl}}\left(\frac{1}{\tau}\frac{\partial F}{\partial \tau_{ij}}\right)\right]_\theta$$

$$= \frac{1}{\tau}\left(\frac{\partial^2 F}{\partial \sigma_{kl}\,\partial \sigma_{ij}}\right)_\theta - \frac{1}{\tau^2}\left(\frac{\partial F}{\partial \sigma_{ij}}\right)_\theta \left(\frac{\partial \tau}{\partial \sigma_{kl}}\right)_\theta$$

but $\dfrac{\Delta \tau}{\tau} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \operatorname{tr} \underline{\underline{\sigma}}$ so

$$\frac{\partial \tau}{\partial \sigma_{kl}} = \tau\,\delta_{kl}$$

$$\left(\frac{\partial T_{ij}}{\partial \sigma_{kl}}\right)_\theta = \frac{1}{\tau}\left(\frac{\partial^2 F}{\partial \sigma_{ij}\,\partial \sigma_{kl}}\right)_\theta - T_{ij}\,\delta_{kl}$$

now evaluating all terms in the initial state

$$C_{ijkl} = \frac{1}{\tau_0}\left(\frac{\partial^2 F}{\partial \sigma_{ij}\,\partial \sigma_{kl}}\right)_{\theta_0, p_0} + p_0\,\delta_{ij}\,\delta_{kl}$$

now we can show similarly that

$$\boxed{\tilde{c}_{ijkl} = \frac{1}{\tau_0}\left(\frac{\partial^2 U}{\partial\sigma_{ij}\,\partial\sigma_{kl}}\right)_{S_0, p_0} + p_0\,\delta_{ij}\,\delta_{kl}}$$

Since mixed partial derivatives are equal (continuity assumed)

$$\boxed{\begin{aligned} c_{ijkl} &= c_{klij} \\ \tilde{c}_{ijkl} &= c_{klij} \end{aligned}}$$

Hence there are at most 21 independent isothermal and isentropic elastic coefficients.

The assumption of <u>l.t.e.</u> so that U, S are fns of $\underline{\underline{\sigma}}$ and θ allows one to make this reduction Some modern textbooks call such a material <u>hyperelastic</u>.

We have performed a <u>standard</u>

thermodynamical argument. The proof hinges on the equality of mixed partial derivatives of various thermodynamic potentials w.r.t. the independent thermo. variables.

Such relations are called Maxwell relations in thermo. For a simple one comp. fluid, Maxwell relations are

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V$$

$$\left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P$$

$$\left(\frac{\partial V}{\partial T}\right)_P = -\left(\frac{\partial S}{\partial P}\right)_T$$

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T$$

familiar?
refer to Callen, Thermodynamics
very formal treatment
stresses Maxwell relations
Chapter 13 on linear elasticity.

stop here 14 April

Lecture # 27 Review

linearized theory of perfect elasticity
natural reference configuration
Lagrangian formulation $\underline{r}(\underline{x},t) = \underline{x} + \underline{s}(\underline{x},t)$

when in motion $\underline{s}(\underline{x},t)$, always and everywhere in a state of local thermodynamic equilibrium. Complete set of thermodynamical variables at any particle $\underline{x}$ is $\theta(\underline{x},t)$ or $S(\underline{x},t)$ and $\underline{\underline{\nabla s}}(\underline{x},t)$

$\underline{\underline{T}}_L(\underline{x},t) =$ Lagrangian descr. of Cauchy stress

$$\underline{\underline{T}}_L(\underline{x},t) = -\underset{\substack{\uparrow \\ \text{hydro. initial} \\ \text{stress}}}{p_0(\underline{x})}\,\underline{\underline{I}} + \underline{\underline{E}}_L(\underline{x},t)$$

$$\underline{\underline{E}}_L(\underline{x},t) = \underline{\underline{E}}_L(\theta, \underline{\underline{\nabla s}}, \underline{x})$$

$$= \underline{\underline{\tilde{E}}}_L(S, \underline{\underline{\nabla s}}, \underline{x})$$

elastic stress

now linearize this stress-deformation const. relation

$$E_{ij}^{L}(\underline{x},t) = \underset{\substack{\uparrow \\ \text{no dependence} \\ \text{on local rigid} \\ \text{body rotation}}}{C_{ijkl}(\underline{x})\,\sigma_{kl}(\underline{x},t)} + W_{ij}(\underline{x})\,\underset{\substack{\uparrow \\ \text{temp} \\ \underline{\text{differenc}} \\ \Delta\theta = \theta - \theta_0}}{\Delta\theta(\underline{x},t}$$

$$E_{ij}^{L}(\underline{x},t) = \tilde{C}_{ijkl}(\underline{x})\,\sigma_{kl}(\underline{x},t) + \tilde{W}_{ij}(\underline{x})\,\Delta S(\underline{x},t)$$

$C_{ijkl}(\underline{x})$ comp. of $\underline{\underline{\underline{C}}}(\underline{x}) \in \otimes^4 R^3$

$W_{ij}(\underline{x})$ comp. of $\underline{\underline{W}}(\underline{x}) \in \otimes^2 R^3$

etc.

$C(\underline{x})$ isothermal elastic tensor

$\tilde{C}(\underline{x})$ isentropic elastic tensor

This is most general form of linearized stress-strain constitutive relation for perfectly elastic materials.

$C(\underline{x})$ and $\tilde{C}(\underline{x})$ are empirical, must be measured for any given material.

Process made slightly easier because of certain symmetries

$$c_{ijkl} = c_{jikl} = c_{ijlk} \quad (81 \to 36)$$
$$\tilde{c}_{ijkl} = \tilde{c}_{jikl} = \tilde{c}_{ijlk} \quad (81 \to 36)$$
$$w_{ij} = w_{ji} \quad (9 \to 6)$$
$$\tilde{w}_{ij} = \tilde{w}_{ji} \quad (9 \to 6)$$

elementary symmetries due to symmetry of stress $\underline{\underline{E}}_L$ and $\underline{\underline{\sigma}}_L$

For a hyperelastic material ∃ one further symmetry which may be deduced from the second law of thermodynamics.

$U(\underline{x},t) \equiv U_L(\underline{x},t) \equiv$ internal energy per gram

$$\boxed{\begin{aligned} dU &= \theta\, dS + \tau T_{ij}\, d\sigma_{ij} \\ &= \theta\, dS + \tau \operatorname{tr}(\underline{\underline{T}} \cdot d\underline{\underline{\sigma}}) \end{aligned}}$$

Reduces to $dU = \theta\, dS - p\, d\tau$ for a fluid

$$\boxed{\begin{aligned} &\left(\frac{\partial U}{\partial \sigma_{ij}}\right)_S = \tau T_{ij} \\[2ex] &\left(\frac{\partial U}{\partial S}\right)_\sigma = \theta \end{aligned}}$$

if we define $F = U - \theta S$ Helmholtz free energy / gram

can rewrite second law

$$dF = \tau T_{ij}\, d\sigma_{ij} - S\, d\theta$$
$$= \tau \,\mathrm{tr}\,(\underline{\underline{T}} \cdot d\underline{\underline{\sigma}}) - S\, d\theta$$

$$\boxed{\begin{aligned} &\left(\frac{\partial F}{\partial \sigma_{ij}}\right)_\theta = \tau T_{ij} \\[2ex] &\left(\frac{\partial F}{\partial \theta}\right)_\sigma = -S \end{aligned}}$$

now $T_{ij} = -p_0 \delta_{ij} + C_{ijkl}\, \sigma_{kl} + W_{ij}\, \Delta\theta$

so

$C_{ijkl} = \left(\dfrac{\partial T_{ij}}{\partial \sigma_{kl}}\right)_\theta$, evaluated in initial state p_0, θ_0

likewise $\tilde{C}_{ijkl} = \left(\frac{\partial T_{ij}}{\partial \sigma_{kl}} \right)_S$, evaluated in initial state

equality of mixed partial derivatives (<u>Maxwell relations</u>) then yields

$$\boxed{\begin{array}{l} C_{ijkl} = C_{klij} \\ \tilde{C}_{ijkl} = \tilde{C}_{klij} \end{array}}$$

This reduces 36 → 21 independent elastic coefficients. In general this is the best we can do (<u>trigonal</u> <u>symmetry</u>)

A tensor $C \in \otimes^4 R^3$ such that $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$ is called an <u>elastic tensor</u>.
An excellent paper on the algebra of elastic tensor is that by Backus 1970 August <u>Rev</u> <u>Geophys</u>
"A geometrical picture of anisotropic elastic tensors."

~~thermodynamical argument. The proof hinges on the equality of mixed partial derivatives of various thermodynamic potentials w.r.t. the independent thermodynamical variables.~~

8. Adiabatic - Isothermal Relations

Let us derive the relationship between the isothermal C_{ijkl}, W_{ij} and isentropic or adiabatic coefficients $\tilde{C}_{ijkl}$, $\tilde{W}_{ij}$

$$dU = \theta\, dS + \tau\, T_{ij}\, d\sigma_{ij}$$

$$T_{ij} = -p_0\, \delta_{ij} + E_{ij}$$

$$dU = \theta\, dS - p_0\, d\tau + \tau_0\, E_{ij}\, d\sigma_{ij}$$

$$d(\underset{\substack{\uparrow \\ \text{call this } Q}}{U + p_0 \tau}) = \theta\, dS + \tau_0\, E_{ij}\, d\sigma_{ij}$$

$$dS = \frac{1}{\theta}\, dQ - \frac{\tau_0}{\theta}\, E_{ij}\, d\sigma_{ij}$$

$$= \frac{1}{\theta}\left[\left(\frac{\partial Q}{\partial \sigma_{ij}}\right)_\theta \underset{\substack{\uparrow \\ \text{must evaluate this}}}{-\,\tau_0\, E_{ij}}\right] d\sigma_{ij} + \frac{1}{\theta}\left(\frac{\partial Q}{\partial \theta}\right)_\sigma d\theta$$

thus

$$\left(\frac{\partial S}{\partial \sigma_{ij}}\right)_\theta = \frac{1}{\theta}\left[\left(\frac{\partial Q}{\partial \sigma_{ij}}\right)_\theta \bar{\bullet}\ \tau_0 E_{ij}\right]$$

$$\left(\frac{\partial S}{\partial \theta}\right)_\sigma = \frac{1}{\theta}\left(\frac{\partial Q}{\partial \theta}\right)_\sigma$$

then since $\frac{\partial}{\partial \theta}\frac{\partial S}{\partial \sigma_{ij}} = \frac{\partial}{\partial \sigma_{ij}}\frac{\partial S}{\partial \theta}$

$$-\frac{1}{\theta^2}\left[\frac{\partial Q}{\partial \sigma_{ij}} \bar{\bullet}\ \tau_0 E_{ij}\right] + \cancel{\frac{1}{\theta}\frac{\partial^2 Q}{\partial \theta\, \partial \sigma_{ij}}} \bar{\bullet}\ \frac{\tau_0}{\theta}\left(\frac{\partial E_{ij}}{\partial \theta}\right)_\sigma$$

$$= \cancel{\frac{1}{\theta}\frac{\partial^2 Q}{\partial \theta\, \partial \sigma_{ij}}} \nwarrow \text{interchanging}$$

but $\left(\frac{\partial E_{ij}}{\partial \theta}\right)_\sigma = W_{ij}$, so

$$\left(\frac{\partial Q}{\partial \sigma_{ij}}\right)_\theta \bar{\bullet}\ \tau_0 E_{ij} = -\theta \tau_0 W_{ij}$$

thus

$$dS = -\tau_0 W_{ij}\, d\sigma_{ij} + \frac{1}{\theta}\left(\frac{\partial Q}{\partial \theta}\right)_\sigma d\theta$$

now denote $c_\sigma = \left(\frac{\partial Q}{\partial \theta}\right)_\sigma$

specific heat at constant strain reduces to c_v for a fluid

$$\boxed{c_\sigma = \left(\frac{\partial Q}{\partial \theta}\right)_\sigma = \left(\frac{\partial U}{\partial \theta}\right)_\sigma \quad \text{since} \left(\frac{\partial \tau}{\partial \theta}\right)_\sigma = 0}$$

~~thus $E_{ij} = C_{ijkl}\sigma_{kl} + W_{ij}\Delta\theta$~~

~~and~~

~~$\Delta\theta = \frac{\theta}{c_\sigma}\Delta S + \frac{\theta\tau_0}{c_\sigma} W_{ij}\sigma_{ij}$~~

$$dS = -\tau_0 W_{ij}\, d\sigma_{ij} + \frac{c_\sigma}{\theta}\, d\theta$$

$$d\theta = \frac{\theta}{c_\sigma}\, dS + \frac{\theta\tau_0}{c_\sigma} W_{ij}\, d\sigma_{ij}$$

to firstorder

$$\Delta\theta = \underbrace{\frac{\theta_0}{c_\sigma}\Delta S + \frac{\theta_0\tau_0}{c_\sigma} W_{ij}\sigma_{ij}}$$

recall $E_{ij} = C_{ijkl}\sigma_{kl} + W_{ij}\,\Delta\theta$

$$E_{ij} = \left[c_{ijkl} + \frac{\theta_0 \tau_0}{c_\sigma} W_{ij} W_{kl} \right] \sigma_{kl}$$

$$+ \frac{\theta_0}{c_\sigma} W_{ij} \, \Delta S$$

but $E_{ij} = \tilde{c}_{ijkl} \sigma_{kl} + \tilde{W}_{ij} \, \Delta S$

hence

$$\boxed{\begin{aligned} \tilde{c}_{ijkl} &= c_{ijkl} + \frac{\theta_0 \tau_0}{c_\sigma} W_{ij} W_{kl} \\ \tilde{W}_{ij} &= \frac{\theta}{c_\sigma} W_{ij} \end{aligned}}$$

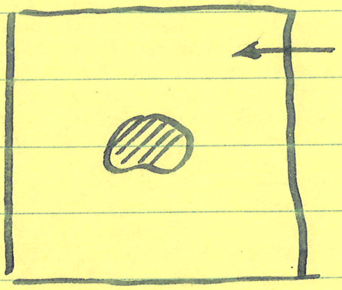
Again we have been pursuing a standard thermodynamic exercise. Seeking relations between various thermodynamical coefficients. Similar to seeking a relation between c_p and c_v.

The above relation will look more familiar when specialized to isotropic materials.

9. Mechanical potential energy

Consider a small lump V of perfectly elastic material. We will treat only the isothermal case. We can consider this lump immersed in an oil bath in a rigid container. The oil is at pressure p_0

no gravity p_0 = constant



temp throughout maintained at $\theta = \theta_0$ (isothermal)

initially, stress tensor in body $\underline{\underline{T}} = -p_0 \underline{\underline{I}}$ where

p_0 = pressure of gas bath.

We now deform the elastic body by the application of surface stresses. The rate at which these stresses do work on the material is

$$\frac{dW}{dt} = \int_{\partial V} u_i T_{ij} n_j \, dA =$$

$$\int_V \left[\frac{\partial u_i}{\partial r_j} T_{ij} + u_i \frac{\partial T_{ij}}{\partial r_j} \right] dV$$

(Eulerian formulation)

the rate at which the material gains <u>kinetic energy</u> is

$$\frac{dT}{dt} = \frac{d}{dt}\int_{V(t)} \frac{1}{2}\rho u^2 \, dV = \int_V \rho u_i \frac{Du_i}{Dt} \, dV$$

$$= \int_V u_i \frac{\partial T_{ij}}{\partial r_j} \, dV \qquad \text{(no body forces)}$$

now $$T_{ij} = -p_0 \delta_{ij} + \underset{\substack{\uparrow \\ \text{elastic stress}}}{E_{ij}}$$

$$\frac{dW}{dt} = \frac{dT}{dt} - p_0 \int_V \nabla \cdot \underline{u} \, dV + \int_V \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right) E_{ij} \, dV$$

and

$$\frac{dV(t)}{dt} = \frac{d}{dt}\int_{V(t)} dV = \int_{\partial V} \hat{n} \cdot \underline{u} \, dA = \int_{V(t)} \nabla \cdot \underline{u} \, dV$$

so

$$\frac{d}{dt}\left(T - W - p_0 V\right) + \int_V \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right) E_{ij}\, dV$$

$$= 0$$

now $\frac{\partial u_i}{\partial r_j} = \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial r_j} = \frac{\partial x_k}{\partial r_j}\frac{\partial}{\partial x_k}\frac{D r_i}{Dt}$

$= \frac{\partial x_k}{\partial r_j}\frac{D}{Dt}\frac{\partial r_i}{\partial x_k}$ but $r_i = x_i + s_i$

so to first order in $\frac{\partial s_i}{\partial x_k}$

$$\frac{\partial u_i}{\partial r_j} = \frac{D}{Dt}\frac{\partial s_i}{\partial x_j}$$

so correct to second order in $\underline{\underline{\sigma}}$ and $\Delta\theta$

$$\boxed{\begin{aligned}&\frac{d}{dt}\left(T - W - p_0 V\right) + \int_V E_{ij}\frac{D\sigma_{ij}}{Dt}\, dV = 0\\ &\frac{d}{dt}\left(T - W - p_0 V\right) + \int_V \mathrm{tr}\left(\underline{\underline{E}}\cdot\frac{D\underline{\underline{\sigma}}}{Dt}\right) dV = 0\end{aligned}}$$

the motion of the elastic body obeys this eqn. The motion of the gas will obey a similar eqn.

$$\frac{d}{dt}\underbrace{\left(W_G + P_0 V_G - T_G\right)}_{\text{signs wrong}} + \int_{V_G} E_{ij}^G \frac{D\sigma_{ij}^G}{Dt}\, dV = 0$$

$W \equiv$ work done on body by surface stresses

$W_G \equiv$ work done on gas by surface stresses

but

$W_G = -W$ or $W_G + W = 0$

also $V + V_G =$ const.

so

$$\frac{d}{dt}\left(T + T_G\right) + \int_V E_{ij} \frac{D\sigma_{ij}}{Dt}\, dV + \int_{V_G} E_{ij}^G \frac{D\sigma_{ij}^G}{Dt}\, dV$$

$$= 0$$

now $E_{ij} = C_{ijkl}\,\sigma_{kl}$ in an isothermal change, so

$$\int_V E_{ij} \frac{D\sigma_{ij}}{Dt}\, dV = \int_V C_{ijkl}\,\sigma_{kl}\, D_t \sigma_{ij}\, dV$$

since $C_{ijkl} = C_{klij}$ this is

$$\frac{d}{dt}\frac{1}{2}\int_V C_{ijkl}\,\sigma_{ij}\,\sigma_{kl}\,dV$$

and our eqn becomes

$$\frac{d}{dt}\left(\mathcal{J}+\mathcal{P}\right) = 0 \qquad *$$

$\mathcal{J} = T + T' =$ k.e. of body + gas

$$\mathcal{P} = \frac{1}{2}\int_V C_{ijkl}\,\sigma_{ij}\,\sigma_{kl}\,dV + \frac{1}{2}\int_{V_G} C_{ijkl}{}^G\,\sigma_{ij}{}^G\,\sigma_{kl}{}^G\,dV$$

= mechanical potential energy of the body + gas

$*$ is the <u>conservation</u> of <u>mechanical energy</u> in an <u>isothermal</u> change

In any state of deformation (<u>isothermal</u> of the elastic body the stored mechanical potential energy is

$$\boxed{P.E. = \frac{1}{2}\int_V C_{ijkl}\,\sigma_{ij}\,\sigma_{kl}\,dV}$$

It may similarly be shown that in any <u>isentropic</u> deformed state the mechanical potential energy of deformation is

$$\boxed{P.E. = \frac{1}{2}\int_V \tilde{C}_{ijkl}\,\sigma_{ij}\,\sigma_{kl}\,dV}$$

Another way of saying this:
In any <u>isothermal</u> ~~[illegible]~~ deformation of an elastic body completely specified by $\underline{\sigma}(\underline{x}, t)$

$\frac{1}{2} C_{ijkl}\,\sigma_{ij}\,\sigma_{kl}$ may be interpreted as the stored mechanical potential energy per unit volume (in the sense of eqn $*$)

In any <u>isentropic</u> change, $\frac{1}{2}\tilde{C}_{ijkl}\,\sigma_{ij}\,\sigma_{kl}$ may be so interpreted.

See remark bottom p. 335.3

Recall that $C_{ijkl}{}^G = P_0\,\delta_{ij}\delta_{kl}$ for a perfect gas

Thus

$$\mathcal{P} = \frac{1}{2}\int_V C_{ijkl}\,\sigma_{ij}\sigma_{kl}\,dV + \frac{1}{2}P_0\int_{V_G}\left(\sigma_{kk}{}^G\right)^2 dV$$

Now, end here 25 April 1972

Remark 1: If $\mathcal{P} > 0$ for every possible small deformation of the system away from equilibrium then by * the system is stable

$$\frac{d}{dt}\left(\mathcal{J} + \mathcal{P}\right) = 0$$

$$\Delta\mathcal{J} + \Delta\mathcal{P} = 0$$

$$\mathcal{P} = \Delta\mathcal{P} > 0 \Rightarrow \Delta\mathcal{J} < 0 \quad \text{stable}$$

If the elastic constants C_{ijkl} are such that for every symmetric tensor $\underline{\underline{\sigma}}$ we have

$$C_{ijkl}\,\sigma_{ij}\sigma_{kl} > 0$$

then the gas-lump system is stable (dynamical stability of a mech. system)

Lecture # 28 Review

Constitutive eqns for linearized theory of perfect elasticity.

$$T_{ij}(\underline{x},t) = -\underset{\substack{\uparrow \\ \text{initial hydro. pressure}}}{p_0(\underline{x})}\,\delta_{ij} + E_{ij}(\underline{x},t)$$

$$\begin{aligned} E_{ij}(\underline{x},t) &= C_{ijkl}(\underline{x})\,\sigma_{kl}(\underline{x},t) + W_{ij}(\underline{x})\,\Delta\theta(\underline{x},t) \\ &= \tilde{C}_{ijkl}(\underline{x})\,\sigma_{kl}(\underline{x},t) + \tilde{W}_{ij}(\underline{x})\,\Delta S(\underline{x},t) \end{aligned}$$

$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$, $W_{ij} = W_{ji}$

in an isothermal change:

$$E_{ij} = C_{ijkl}\,\sigma_{kl}$$

$C(\underline{x})$ isothermal elastic tensor

in an isentropic change

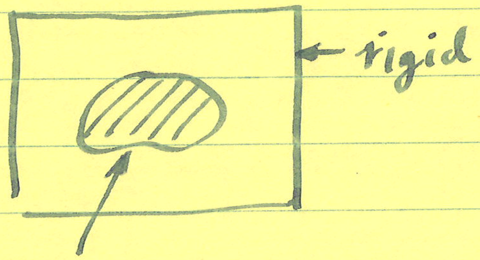
$$E_{ij} = \tilde{C}_{ijkl}\,\sigma_{kl}$$

$\tilde{C}(\underline{x})$ isentropic elastic tensor

For these two cases, thermodynamical variables may be essentially dispensed with

The energetics of the defn. may, in these two cases, be completely described purely in terms of a <u>mechanical</u> <u>potential</u> <u>energy</u>

Consider an <u>isothermal</u> change



$\theta = \theta_0$ throughout, all time

apply stresses, thereby deform system

$$\frac{d}{dt}\left(T + T_G\right) + \int_V E_{ij} \frac{D\sigma_{ij}}{Dt}\, dV + \int_{V_G} E_{ij}^G \frac{D\sigma_{ij}^G}{Dt}\, dV = 0$$

now in an <u>isothermal</u> change (here we first assume)

$E_{ij} = C_{ijkl}\, \sigma_{kl}$

$C_{ijkl} = C_{klij}$ so

$$\boxed{\frac{d}{dt}\left(\mathcal{J}+\mathcal{P}\right)=0 \qquad *}$$

$\mathcal{J} = T + T_G =$ total k.e. of lump + gas

$$\mathcal{P} = \frac{1}{2}\int_V C_{ijkl}\,\sigma_{ij}\,\sigma_{kl}\,dV + \frac{1}{2}\int_{V_G} C_{ijkl}^G\,\sigma_{ij}^G\,\sigma_{kl}^G\,dV$$

$\mathcal{P}$ may be interpreted as *mechanical potential* energy

$$\boxed{\text{* is cons. of mechanical energy}}$$

To note: the gas can be a vacuum $\boxed{\text{Reason for gas: so } p_0 \neq 0.}$

To note: $\frac{1}{2} C_{ijkl}\,\sigma_{ij}\,\sigma_{kl}$ may be interpreted as a mechanical pot. energy *density* (per unit volume)

$\frac{1}{2} C_{ijkl}\,\sigma_{ij}\,\sigma_{kl}$ = elastic energy density, defined for all $\underline{x}$.

There is however no *pointwise* energy

conservation law. Energy of whole system is conserved ★★

Recall for a perfect gas $C_{ijkl}^{G} = P_0 \delta_{ij} \delta_{kl}$

$$\mathcal{P} = \frac{1}{2} \int_V C_{ijkl}\, \sigma_{ij}\, \sigma_{kl}\, dV + \frac{1}{2} P_0 \int_{V_G} \left(\sigma_{kk}^{G}\right)^2 dV$$

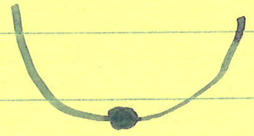
Remark 1:

if the elastic constants C_{ijkl} are $\ni$

$$C_{ijkl}\, \sigma_{ij}\, \sigma_{kl} > 0 \quad \text{for all } \underline{\underline{\sigma}}$$

then $\mathcal{P} > 0$ for all $\underline{\underline{\sigma}}$

stored P.E. of any deformation > 0 in that case system is mechanically stable.



$$\Delta \mathcal{T} + \Delta \mathcal{P} = 0$$
$$\Delta \mathcal{P} > 0$$
$$\Delta \mathcal{T} < 0$$

if any deformation away from equil increases the P.E. then system is stable

__Remark__ 2: we will now show that conversely if $\exists$ a tensor, say $\underline{\sigma}_0$, such that [redacted]

$$C_{ijkl}\,\sigma_{ij}\,\sigma_{kl} < 0 \qquad **$$

then the system may be __unstable__, at least for sufficiently large volumes of gas. The proof is in two parts.

1. if V_G is sufficiently large, then $\exists$ small deformations of the system which render $P < 0$.

   displace V so that its strain tensor is some $\underline{\sigma}_0$ satisfying ** /. This entails an __increase__ in the volume of V by $\Delta V = \epsilon V$. Then V_G must decrease by ϵV so, to first order in ϵ

$$\epsilon V \simeq V_G\,\sigma_{KK}{}^{G} \quad \text{or}$$

$$\sigma_{KK}{}^{G} = \epsilon\,\frac{V}{V_G}$$

$$P = \tfrac{1}{2} V C_{ijkl}\,\sigma_{ij}{}^{0}\,\sigma_{kl}{}^{0} + \tfrac{1}{2} P_0 V_G \left(\frac{V}{V_G}\right)^2 (\epsilon)^2$$

$$P = \tfrac{1}{2} V \left[C_{ijkl}\,\sigma_{ij}{}^{0}\,\sigma_{kl}{}^{0} + \tfrac{1}{2} P_0 (\epsilon)^2 \left(\frac{V}{V_G}\right) \right]$$

hence if $V_G \gg V$ and if $\underline{\sigma}_0$ is such that $C_{ijkl}\,\sigma_{ij}^0\,\sigma_{kl}^0 < 0$, then $P < 0$

2. if there are deformations $\ni P < 0$ and if there is even a small amount of velocity-dependent dissipation, then the system is unstable.
   In that case

   $$\frac{d}{dt}(J + P) = -\epsilon$$
   $$\epsilon > 0 \text{ for } J > 0$$

   apply to the body surface stresses which deform it so that [redacted] to a new state with $P = P_0 < 0$
   now release the stresses.
   the body, not being in equil will start to move and hence to dissipate energy. This will decrease $J + P$ below its initial value P_0 and the body will move even further from equilibrium ($P = J = 0$)

Such instability in the presence of friction is called <u>secular</u> instability. We discussed it before for rigid body rotation.

<u>Conclusion</u>: In any observable elastic material the isothermal elastic tensor has the property that <u>for every</u> symmetric tensor $\underline{\sigma}$

$$C_{ijkl}\,\sigma_{ij}\,\sigma_{kl} \geq 0$$

The same is true of the isentropic elastic tensor

$$\tilde{C}_{ijkl}\,\sigma_{ij}\,\sigma_{kl} \geq 0$$

Any material for which this is not true is <u>inherently</u> secularly unstable. This is an inherent internal instability. The presence of even a small amount of friction would give rise to a divergence away from any equilibrium state. Every <u>observable</u> perfectly elastic material must satisfy these eqns.

10. Isotropic perfectly elastic materials

lets now specialize to the case of isotropic perfectly elastic materials

An isotropic perfectly elastic solid has no characteristic directions or symmetry of any kind

In such solids the elastic tensors $C, \tilde{C}, W, \tilde{W}$ must be isotropic.

$$E_{ij} = C_{ijkl}\,\sigma_{kl} + W_{ij}\,\Delta\theta$$
$$= \tilde{C}_{ijkl}\,\sigma_{kl} + \tilde{W}_{ij}\,\Delta S$$

$C, \tilde{C} \in \otimes^4 R^3$ isotropic

$W, \tilde{W} \in \otimes^4 R^3$ isotropic

We have compiled a catalogue of isotropic tensors

$$C_{ijkl} = \alpha\delta_{ij} + \beta\,\delta_{ik}\,\delta_{jl} + \gamma\,\delta_{il}\,\delta_{jk}$$

α, β, γ constants

We also know that $C_{ijkl} = C_{jikl}$

this implies $\beta = \gamma$
We shall employ the customary notation

$$\boxed{c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})}$$

λ and μ are called the isothermal Lamé parameters of the substance
Similarly

$$\boxed{\tilde{c}_{ijkl} = \tilde{\lambda} \delta_{ij} \delta_{kl} + \tilde{\mu}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})}$$

$\tilde{\lambda}$ and $\tilde{\mu}$ are called the isentropic Lamé parameters

The tensors W and $\tilde{W}$ must also be isotropic

$$W_{ij} = \gamma \delta_{ij}$$
$$\tilde{W}_{ij} = \tilde{\gamma} \delta_{ij}$$

γ and $\tilde{\gamma}$ have no common name. We shall see later that they can be expressed in terms of the linear coefficient of thermal expansion α.

The stress strain relations for an isotropic substance are

$$E_{ij} = \lambda \delta_{ij} \sigma_{kk} + 2\mu \sigma_{ij} + \gamma \Delta\theta \delta_{ij}$$

$$\underline{\underline{E}} = \lambda (\mathrm{tr}\, \underline{\underline{\sigma}}) \underline{\underline{I}} + 2\mu \underline{\underline{\sigma}} + \gamma \Delta\theta \underline{\underline{I}}$$

$$\underline{\underline{E}} = \tilde{\lambda} (\mathrm{tr}\, \underline{\underline{\sigma}}) \underline{\underline{I}} + 2\tilde{\mu} \underline{\underline{\sigma}} + \tilde{\gamma} \Delta S \underline{\underline{I}}$$

and in terms of $\underline{s}(\underline{x}, t)$

$$\underline{\underline{E}} = \lambda (\nabla \cdot \underline{s}) \underline{\underline{I}} + \mu \left[\underline{\underline{\nabla s}} + (\underline{\underline{\nabla s}})^T \right] + \gamma \Delta\theta \underline{\underline{I}}$$

$$\underline{\underline{E}} = \tilde{\lambda} (\nabla \cdot \underline{s}) \underline{\underline{I}} + \tilde{\mu} \left[\underline{\underline{\Delta s}} + (\underline{\underline{\nabla s}})^T \right] + \tilde{\gamma} \Delta S \underline{\underline{I}}$$

The coefficients λ, μ, γ and $\tilde{\lambda}, \tilde{\mu}, \tilde{\gamma}$ are fcns of the particle posns $\underline{x}$ in the initial reference configuration

$$\lambda(\underline{x}), \mu(\underline{x}), \gamma(\underline{x}), \tilde{\lambda}(\underline{x}), \tilde{\mu}(\underline{x}), \tilde{\gamma}(\underline{x})$$

The assumption of isotropy of the material (good maybe for an aggregate, not for a single xtl) reduces the number of independent C_{ijkl} from 21 to 2 (called the Lamé parameters) and the number of ind. W_{ij} from 6 to 1.

Let us now investigate mechanical potential energy for an <u>isotropic</u> medium, <u>but first</u>.

11. Adiabatic - Isothermal relations for isotropic

$$\tilde{C}_{ijkl} = C_{ijkl} + \frac{\theta\tau}{c_\sigma} W_{ij} W_{kl}$$

$$\tilde{W}_{ij} = \frac{\theta}{c_\sigma} W_{ij}$$

c_σ = specific heat at const. strain

$$\tilde{\lambda}\delta_{ij}\delta_{kl} + \tilde{\mu}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\theta\tau}{c_\sigma}\gamma^2\delta_{ij}\delta_{kl}$$

$$\tilde{\gamma}\delta_{ij} = \gamma\delta_{ij}\frac{\theta}{c_\sigma}$$

$$\boxed{\begin{aligned} \tilde{\lambda} &= \lambda + \frac{\theta\tau}{c_\sigma}\gamma^2 \\ \tilde{\mu} &= \mu \\ \tilde{\gamma} &= \frac{\theta}{c_\sigma}\gamma \end{aligned}}$$

We'll see later that $\gamma = -(3\lambda+2\mu)\alpha$ where α is linear thermal exp. coeff.

isothermal → adiabatic & vice-versa

we come back to interpret after interp. of λ, μ, γ

12. Elastic Potential Energy for an Isotropic Material

The stored elastic potential energy in an isothermally deformed volume V of isotropic perfectly elastic material is

$$\frac{1}{2}\int_V C_{ijkl}\,\sigma_{ij}\sigma_{kl}\,dV = \mathcal{P}_V$$

The elastic potential energy density is the integrand

$$\frac{1}{2} C_{ijkl}\,\sigma_{ij}\sigma_{kl}$$

$$= \left[\frac{1}{2}\lambda\,\delta_{ij}\delta_{kl} + \frac{1}{2}\mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right]\sigma_{ij}\sigma_{kl}$$

$$= \frac{1}{2}\left[\lambda\,\sigma_{ii}\sigma_{kk} + 2\mu\,\sigma_{ij}\sigma_{ij}\right]$$

$$\boxed{\begin{aligned}\frac{1}{2} C_{ijkl}\,\sigma_{ij}\sigma_{kl} &= \frac{1}{2}\left[\lambda\left(\sigma_{ii}\right)^2 + 2\mu\,\sigma_{ij}\sigma_{ij}\right]\\ &= \frac{1}{2}\left[\lambda\left(\mathrm{tr}\,\underline{\underline{\sigma}}\right)^2 + 2\mu\,\mathrm{tr}\left(\underline{\underline{\sigma}}\cdot\underline{\underline{\sigma}}\right)\right]\end{aligned}}$$

It is convenient to separate the <u>strain</u> $\underline{\underline{\sigma}}$ into its isotropic and deviatoric parts

$$\underline{\underline{\sigma}} = \underset{\substack{\uparrow \\ \text{isotropic} \\ \text{part}}}{\left(\tfrac{1}{3}\operatorname{tr}\underline{\underline{\sigma}}\right)\underline{\underline{I}}} + \underset{\substack{\nwarrow \\ \text{deviatoric} \\ \text{part}}}{\underline{\underline{D}}_\sigma}$$

we <u>interpret</u> this as before

$$\tfrac{1}{3}\operatorname{tr}\underline{\underline{\sigma}} = \tfrac{1}{3}\nabla\cdot\underline{s} = \tfrac{1}{3}\,(\text{dilatation})$$

$$\underline{\underline{\sigma}} = \underset{\substack{\uparrow \\ \text{dilatation or} \\ \text{local change of} \\ \text{volume}}}{\left(\tfrac{1}{3}\operatorname{tr}\underline{\underline{\sigma}}\right)}\underline{\underline{I}} + \underset{\substack{\nwarrow \\ \text{local volume-} \\ \text{preserving } \underline{\text{pure}} \\ \underline{\text{shear}} \text{ symmetric} \\ \text{squeeze}}}{\underline{\underline{D}}_\sigma}$$

Recall interpretation from theory of Newtonian viscosity.

The stress-strain law now takes the form for an iso<u>ther</u>mal change

$$\underline{\underline{E}} = \lambda(\mathrm{tr}\,\underline{\underline{\sigma}})\underline{\underline{I}} + 2\mu\,\underline{\underline{\sigma}}$$

$$= \lambda(\mathrm{tr}\,\underline{\underline{\sigma}})\underline{\underline{I}} + 2\mu\left[\left(\tfrac{1}{3}\mathrm{tr}\,\underline{\underline{\sigma}}\right)\underline{\underline{I}} + \underline{\underline{\mathcal{D}}}_\sigma\right]$$

$$\underline{\underline{E}} = \underbrace{\left(\lambda + \tfrac{2}{3}\mu\right)(\mathrm{tr}\,\underline{\underline{\sigma}})\underline{\underline{I}}}_{\substack{\text{isotropic}\\ \text{part}}} + \underbrace{2\mu\,\underline{\underline{\mathcal{D}}}_\sigma}_{\substack{\text{deviatoric}\\ \text{part}}}$$

natural division of $\underline{\underline{E}}$ into isotropic and deviatoric parts

comb. $\lambda + \frac{2}{3}\mu$ → purely isotropic incremental stress

μ → pure <u>shear</u> stress or deviatoric stress.

Same true for an <u>isentropic</u> change

$$\underline{\underline{E}} = \left(\tilde{\lambda} + \tfrac{2}{3}\tilde{\mu}\right)(\mathrm{tr}\,\underline{\underline{\sigma}})\underline{\underline{I}} + 2\tilde{\mu}\,\underline{\underline{\mathcal{D}}}_\sigma$$

Now for the elastic potential energy in an iso<u>ther</u>mal change

$$\lambda(\mathrm{tr}\,\underline{\underline{\sigma}})^2 + 2\mu\,\mathrm{tr}(\underline{\underline{\sigma}}\cdot\underline{\underline{\sigma}})$$

$$2\mu\,\sigma_{ij}\sigma_{ij} = 2\mu\left[\left(\tfrac{1}{3}\mathrm{tr}\,\underline{\underline{\sigma}}\right)\delta_{ij} + \mathcal{D}_{ij}\right]\left[\left(\tfrac{1}{3}\mathrm{tr}\,\underline{\underline{\sigma}}\right)\delta_{ij} + \mathcal{D}_{ij}\right]$$

$$= 2\mu \left[\left(\tfrac{1}{3} \operatorname{tr} \underline{\underline{\sigma}}\right)^2 \delta_{ij}\delta_{ij} + \left(\tfrac{2}{3} \operatorname{tr} \underline{\underline{\sigma}}\right) \overset{\nearrow \text{zero}}{D_{ii}} + D_{ij} D_{ij} \right]$$

$$= \mu \left[\left(\tfrac{2}{9} \cdot 3\right) \left(\operatorname{tr} \underline{\underline{\sigma}}\right)^2 + 2 D_{ij} D_{ij} \right]$$

$$\boxed{\tfrac{1}{2}\, C_{ijkl}\, \sigma_{ij} \sigma_{kl} = \tfrac{1}{2} \left[\left(\lambda + \tfrac{2}{3}\mu\right) \left(\operatorname{tr} \underline{\underline{\sigma}}\right)^2 + 2\mu \operatorname{tr} \left(\underline{\underline{D}}_{\sigma} \cdot \underline{\underline{D}}_{\sigma} \right) \right]}$$

comb $\lambda + \frac{2}{3}\mu$ stored energy due to dilation
μ stored energy due to <u>pure</u> shear strain
Same for <u>isentropic</u>

13. <u>Stability of an isotropic material</u>

What constraints are imposed on an isotropic elastic material's elastic coefficients λ, μ by its existence.

$$C_{ijkl}\, \sigma_{ij} \sigma_{kl} \geq 0 \quad \text{\underline{for any}} \ \underline{\underline{\sigma}}$$

$$\lambda (\sigma_{ii})^2 + 2\mu \sigma_{ij}\sigma_{ij} \geqslant 0$$

$$\lambda (\sigma_{11} + \sigma_{22} + \sigma_{33})^2 + 2\mu (\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{12}^2 + 2\sigma_{13}^2 + 2\sigma_{23}^2)$$

$$(\lambda + 2\mu)(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2) + 2\lambda(\sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33}) + 4\mu(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2) \geqslant 0$$

must be true for all $\underline{\sigma}$, i.e. all $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}$ independently

take $\sigma_{12} = \sigma_{21} = 1$
all other $\sigma_{ij} = 0$

$$4\mu\sigma_{12}^2 = 4\mu \geqslant 0$$

$$\boxed{\mu \geqslant 0}$$

now taking $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$ (off diag. terms)

$$(\lambda + 2\mu)(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2) + 2\lambda(\sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33}) \geqslant 0$$

for *every* choice of $\sigma_{11}, \sigma_{22}, \sigma_{33}$

write $x_1 = \sigma_{11}$
$x_2 = \sigma_{22}$
$x_3 = \sigma_{33}$

$Q_{11} = \lambda + 2\mu \qquad Q_{12} = \lambda \qquad Q_{13} = \lambda$
$Q_{21} = \lambda \qquad Q_{22} = \lambda + 2\mu \qquad Q_{23} = \lambda$
$Q_{31} = \lambda \qquad Q_{32} = \lambda \qquad Q_{33} = \lambda + 2\mu$

Then the above eqn can be written

$$x_i Q_{ij} x_j \geqslant 0 \qquad \text{or}$$

$$\underline{x} \cdot \underline{\underline{Q}} \cdot \underline{x} \geqslant 0$$

$\underline{\underline{Q}}$ is a symmetric matrix belonging to a symmetric linear operator $\underline{\underline{Q}}$.

$\underline{x} \cdot \underline{\underline{Q}} \cdot \underline{x} \geqslant 0$ for every vector $\underline{x}$
i.e. $\underline{\underline{Q}}$ must be a <u>positive definite</u> symmetric linear operator

It is well known that a linear operator is <u>positive definite</u> iff its eigen<u>values</u> are all $\geqslant 0$

hence $\underline{x} \cdot \underline{\underline{Q}} \cdot \underline{x} \geq 0$ for all $\underline{x}$ iff

$\det(\underline{\underline{Q}} - \underline{q}\,\underline{\underline{I}})$ has no negative roots $\underline{q}$

$$\det \begin{vmatrix} \lambda + 2\mu - \underline{q} & \lambda & \lambda \\ \lambda & \lambda + 2\mu - \underline{q} & \lambda \\ \lambda & \lambda & \lambda + 2\mu - \underline{q} \end{vmatrix} = 0$$

$$\det = (2\mu - \underline{q})^2 (3\lambda + 2\mu - \underline{q})$$

conditions are $\mu \geq 0$ which we already knew and

$$3\lambda + 2\mu \geq 0 \quad \text{or}$$

$$\boxed{\lambda + \tfrac{2}{3}\mu \geq 0}$$

An isotropic elastic medium for which either $\mu < 0$ or $\lambda + \frac{2}{3}\mu < 0$ will be unstable to small disturbances and will not be able to exist.

end here 26 April '72

Lecture # 29 review Final lecture of year.

We finally specialized to the case of isotropic linearized perfectly elastic materials.

Suitable for aggregates of various minerals, also for small displacements in fluids.

Isotropic: no symmetry or preferred directions

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

$$\tilde{C}_{ijkl} = \tilde{\lambda} \delta_{ij} \delta_{kl} + \tilde{\mu} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

isothermal and isentropic

$$W_{ij} = \gamma \delta_{ij}$$
$$\tilde{W}_{ij} = \tilde{\gamma} \delta_{ij}$$

stress-strain relations thus take the form

$$\underline{\underline{T}}(\underline{x},t) = - \underset{\substack{\uparrow \\ \text{initial}}}{p_0}(\underline{x}) \underline{\underline{I}} + \underline{\underline{E}}(\underline{x},t)$$

where

$$\underline{\underline{E}} = \lambda(\mathrm{tr}\,\underline{\underline{\sigma}})\,\underline{\underline{I}} + 2\mu\,\underline{\underline{\sigma}} + \gamma\,\Delta\theta\,\underline{\underline{I}}$$

$$= \tilde{\lambda}(\mathrm{tr}\,\underline{\underline{\sigma}})\,\underline{\underline{I}} + 2\tilde{\mu}\,\underline{\underline{\sigma}} + \tilde{\gamma}\,\Delta S\,\underline{\underline{I}}$$

introducing the <u>strain deviator</u> $\underline{\underline{D}}_\sigma$

$$\underline{\underline{\sigma}} = \underset{\substack{\uparrow \\ \text{dilatation}}}{\left(\tfrac{1}{3}\mathrm{tr}\,\underline{\underline{\sigma}}\right)\underline{\underline{I}}} + \underset{\substack{\uparrow \\ \text{volume-preserving} \\ \text{shear strain}}}{\underline{\underline{D}}_\sigma}$$

~~[illegible]~~

$$\underline{\underline{E}} = \underbrace{\left(\lambda + \tfrac{2}{3}\mu\right)(\mathrm{tr}\,\underline{\underline{\sigma}})\,\underline{\underline{I}}}_{\substack{\text{isotropic} \\ \text{part} \\ \text{(pressure)}}} + \underbrace{2\mu\,\underline{\underline{D}}_\sigma}_{\substack{\text{deviatoric} \\ \text{part} \\ \text{(shear} \\ \text{stress)}}} + \gamma\,\Delta\theta\,\underline{\underline{I}}$$

also $\underline{\underline{E}} = \left(\tilde{\lambda} + \tfrac{2}{3}\tilde{\mu}\right)(\mathrm{tr}\,\underline{\underline{\sigma}})\,\underline{\underline{I}} + 2\tilde{\mu}\,\underline{\underline{D}}_\sigma + \tilde{\gamma}\,\Delta S\,\underline{\underline{I}}$

Elastic potential energy stored in

a deformation of the system.

<u>isothermal deformation</u>:

$$\frac{1}{2} C_{ijkl}\sigma_{ij}\sigma_{kl} = \frac{1}{2}\Big[(\lambda + \tfrac{2}{3}\mu)(\mathrm{tr}\,\underline{\underline{\sigma}})^2 + 2\mu\,\mathrm{tr}\,(\underline{\underline{D}}_\sigma \cdot \underline{\underline{D}}_\sigma)\Big]$$

compressional energy due to compression
shear energy due to shear

<u>isentropic deformation</u>

$$\frac{1}{2} C_{ijkl}\sigma_{ij}\sigma_{kl} = \frac{1}{2}\Big[(\tilde{\lambda} + \tfrac{2}{3}\tilde{\mu})(\mathrm{tr}\,\underline{\underline{\sigma}})^2 + 2\tilde{\mu}\,\mathrm{tr}\,(\underline{\underline{D}}_\sigma \cdot \underline{\underline{D}}_\sigma)\Big]$$

Relation between isothermal and isentropic coefficients

$$\tilde{\lambda} = \lambda + \frac{\theta\tau}{c_\sigma}\gamma^2$$
$$\tilde{\mu} = \mu$$
$$\tilde{\gamma} = \frac{\theta}{c_\sigma}\gamma$$

stability of an isotropic elastic material

general criterion for inherent or intrinsic stability

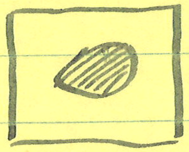
$$C_{ijkl}\sigma_{ij}\sigma_{kl} \geq 0 \quad \forall \underline{\underline{\sigma}}$$

$$\boxed{\begin{array}{l}\mu \geq 0 \\ \lambda + \frac{2}{3}\mu \geq 0\end{array}}$$ criteriafor stability

$\mu = 0$ a fluid

14. Experimental determination of the elastic parameters for an isotropic body.

We consider a stable isotropic body in thermo. equil near a pressure state



$$\underline{T} = -p_0\underline{\underline{I}} + \lambda(\mathrm{tr}\,\underline{\underline{\sigma}})\underline{\underline{I}} + 2\mu\underline{\underline{\sigma}} + \gamma'\underline{\underline{I}}\,\Delta\theta$$

stability ⇒ $\mu \geq 0, \quad \lambda + \frac{2}{3}\mu \geq 0$

1. consider a purely dilatational deformation $\underline{s}(\underline{x}) = \epsilon\underline{x}$

■ resulting elastic stress

$$\underline{\underline{E}} = (3\lambda + 2\mu)\epsilon\,\underline{\underline{I}} + \gamma'\underline{\underline{I}}\,\Delta\theta$$

stress at surface

$$[(3\lambda + 2\mu)\epsilon + \gamma'\Delta\theta]\,\hat{n}$$

exactly that produced by raising pressure in oil bath by an amount

$$\Delta p = -\gamma'\Delta\theta - (3\lambda + 2\mu)\epsilon$$

we achieve the desired dilation $\underline{s}(\underline{x}) = \epsilon \underline{x}$ by raising oil bath pressure by Δp and temp by $\Delta\theta$

But $\quad \mathrm{tr}\,\underline{\underline{\sigma}} = \nabla\cdot\underline{s} = 3\epsilon = \frac{\Delta\tau}{\tau} \quad$ so

$$\boxed{\frac{\Delta\tau}{\tau} = -\frac{3}{3\lambda+2\mu}\Delta p - \frac{3\gamma}{3\lambda+2\mu}\Delta\theta}$$

hence

$\frac{3}{3\lambda+2\mu}$ = isothermal compressibility $-\frac{1}{\tau}\left(\frac{\partial\tau}{\partial p}\right)_\theta$

$\frac{3\lambda+2\mu}{3} = \lambda + \frac{2}{3}\mu$ = isothermal bulk modulus $-\tau\left(\frac{\partial p}{\partial\tau}\right)_\theta$

usually denoted κ, B by engineers

$-\frac{3\gamma}{3\lambda+2\mu}$ = isobaric coefficient of bulk thermal expansion $-\frac{1}{\tau}\left(\frac{\partial\tau}{\partial\theta}\right)_p$

$-\frac{\gamma}{3\lambda+2\mu} = \alpha$ = isobaric coefficient of linear thermal expansion

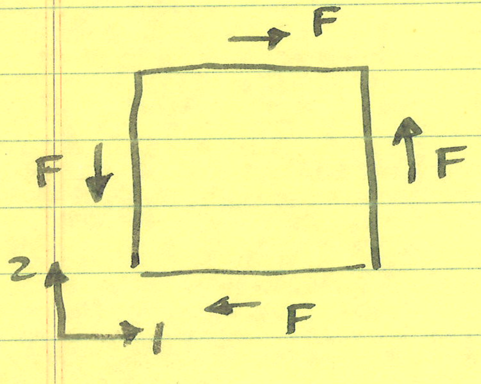
κ and α are readily measured hence $\lambda + \frac{2}{3}\mu$ and γ can be measured ; note $\gamma = -3\alpha\kappa$

note: our stability criterion is

$\kappa \geq 0$

Go to page 355 from here

2. suppose we have a cube initially in hydro. pressure and we deform by application of surface stresses

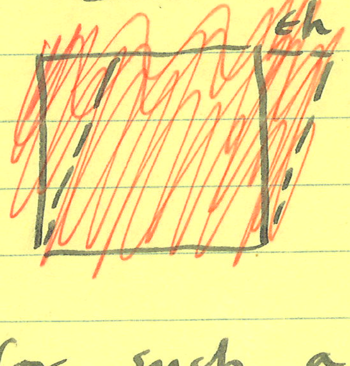


F tangential stress to top and bottom and both sides

no stress to $\hat{x}_3$ face

$\Delta\theta = 0$

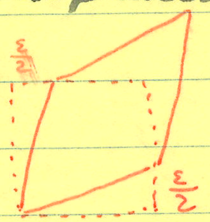
we conjecture that resulting displacement is



$s_1 = \frac{1}{2}\epsilon x_2$

$s_2 = \cancel{0}\ \frac{1}{2}\epsilon x_1$

$s_3 = 0$



for such a displacement

$\sigma_{12} = \sigma_{21} = \frac{\epsilon}{2}$, $\sigma_{ij} = 0$ otherwise

then

$$T_{ij} = 0 \text{ except } T_{12} = T_{21} = \mu\epsilon$$

$$\boxed{\epsilon = \frac{F}{\mu}}$$

thus the displacement

$$s_1 = \frac{1}{2}\frac{F}{\mu}x_2 , \quad s_2 = \frac{1}{2}\frac{F}{\mu}x_1 , \quad s_3 = 0$$

does give the required surface tractions and does make $\partial T_{ij} / \partial r_j = 0$ inside

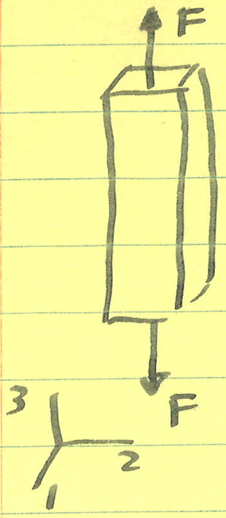
by uniqueness, this *is* the displacement produced by the stresses

rigidity

μ is the shear modulus

note: $\mu = 0$ for a fluid

3. long thin cylinder, initially under pressure



no stress to sides

again we conjecture displ. of form

$$s_1 = \alpha x_1 , \quad s_2 = \beta x_2 , \quad s_3 = \gamma x_3$$

then if it is $\quad T_{12} = T_{23} = T_{13} = 0$

$$T_{11} = \lambda(\alpha+\beta+\gamma) + 2\mu\alpha$$
$$T_{22} = \lambda(\alpha+\beta+\gamma) + 2\mu\beta$$
$$T_{33} = \lambda(\alpha+\beta+\gamma) + 2\mu\gamma$$

$$T_{11} = T_{22} = 0$$
$$T_{33} = F$$

we must take $\quad \alpha = \beta = -\dfrac{\lambda}{2(\lambda+\mu)}\gamma \quad$ and

$$\lambda\left(-\frac{\lambda}{\lambda+\mu}+1\right)\gamma + 2\mu\gamma = F \qquad \text{or}$$

$$\gamma = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} F$$

Thus $\quad Y = \dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu} \quad$ Young's modulus

$$\frac{\lambda}{2(\lambda+\mu)} = \sigma \qquad \text{Poisson's ratio}$$

Poisson's ratio gives the amt. of contraction of the cross-section upon lengthening.

rewrite: $\underline{\underline{I}} = -p_0 \underline{\underline{I}} + \kappa(\nabla\cdot\underline{s})\underline{\underline{I}} + 2\mu \underline{\underline{D}}_\varepsilon - 3\alpha\kappa\Delta\theta\, \underline{\underline{I}}$

These static experiments measure isothermal constants; for isentropic constants, could e.g. use our formulae.

note when $\underline{\underline{I}} \to -p_0\underline{\underline{I}}$, $\nabla\cdot\underline{s} = \frac{\Delta\tau}{\tau} = 3\alpha\Delta\theta$ unconstrained expansion

$$\tilde{\mu} = \mu$$
$$\tilde{\lambda} = \lambda + \frac{\theta\tau}{c_\sigma}\gamma^2 = \lambda + \frac{\theta\tau}{c_\sigma} 9\kappa^2\alpha^2$$

$$\tilde{\mu} = \mu$$
$$\tilde{\kappa} = \kappa\left(1 + 9\frac{\theta\tau}{c_\sigma}\kappa\alpha^2\right)$$

actually for most solids $\tilde{\kappa} \approx \kappa$; for highly compressible fluids $\tilde{\kappa} \not\approx \kappa$ (e.g. air — see Newton)

recall formula for energy

$$\frac{1}{2}\Big[\underbrace{(\lambda + \tfrac{2}{3}\mu)}_{\kappa}(\mathrm{tr}\,\underline{\underline{\sigma}})^2 + 2\mu\,\mathrm{tr}(\underline{\underline{D}}_\sigma \cdot \underline{\underline{D}}_\sigma)\Big]$$

$$\boxed{\frac{1}{2}\Big[\underbrace{\kappa(\mathrm{tr}\,\underline{\underline{\sigma}})^2}_{\text{compr. energy}} + \underbrace{2\mu\,\mathrm{tr}(\underline{\underline{D}}_\sigma \cdot \underline{\underline{D}}_\sigma)}_{\text{shear energy}}\Big]}$$

Recall stability $\kappa \geq 0$

$\mu \geq 0$

Recall $\underline{\underline{E}} = \underset{\substack{\uparrow \\ \text{isotropic} \\ \text{part}}}{\kappa(\nabla\cdot\underline{s})\,\underline{\underline{I}}} + \underset{\substack{\uparrow \\ \text{dev. part}}}{2\mu\,\underline{\underline{D}}\sigma}$

everything looks nicer in terms of κ and μ, not λ and μ.

Lastly we note the following values of σ

fluid: $\mu = 0$ so $\sigma = \frac{1}{2}$

incompressible: $\kappa = \infty$ so $\sigma = \frac{1}{2}$

stable: $-1 < \sigma < \frac{1}{2}$

most real: $\sigma \simeq \frac{1}{4} \Rightarrow \lambda = \mu$

$$\boxed{\sigma = \frac{1}{2} - \frac{3}{2}\,\frac{\mu}{3\kappa + \mu}}$$

no material ever discovered has $\sigma < 0$.

15. Linearized equations of motion (the normal mode eqns)

Long ago I promised that one of the main differences between the linearized theory of elasticity and the theory of Newtonian viscosity was that we will be led naturally to a completely linear system of eqns.

How do we do this?

Our eqn of motion is an Eulerian one and we have been working with a Lagrangian description

How do we get the Eulerian description of the stress field which we use in the momentum eqn.

We assume (now but not before) that not only $\underline{\underline{\nabla s}}(\underline{x},t)$ but also $\underline{s}(\underline{x},t)$ is small

We have an expression for $\underline{\underline{T}}_L(\underline{x},t)$

$$\underline{\underline{T}}_L(\underline{x},t) = \underline{\underline{T}}_E(\underline{r}(\underline{x},t),t) =$$

$$-p_0(\underline{x},0)\underline{\underline{I}} + \underline{\underline{E}}_L(\underline{x},t)$$

$$\underline{\underline{E}}_L(\underline{x},t) = \underline{\underline{E}}_L(\underline{r}-\underline{s},t) =$$

$$\underline{\underline{E}}_L(\underline{r},t) - \underline{s}\cdot\nabla\underline{\underline{E}}_L(\underline{r},t) + O(s^2)$$

(Taylor series)
assuming $|\underline{s}|$ small

but $\underline{\underline{E}}_L$ is already $O(\underline{s})$ so if we keep only terms linear in $\underline{s}$

$$\underline{\underline{E}}_L(\underline{x},t) = \underline{\underline{E}}_L(\underline{r},t) + O(s^2)$$

Proceeding in like manner with $p_0(\underline{x},0)$

$$p_0(\underline{x},0) = p_0(\underline{r}-\underline{s},0) = p_0(\underline{r},0) - \underline{s}\cdot\nabla p_0(\underline{r},0) + O(s^2)$$

combining

$$\underline{\underline{T}}_E(\underline{r},t) = -p_0(\underline{r})\,\underline{\underline{I}} + \underline{s}\cdot\nabla p_0\,\underline{\underline{I}}$$

$$+ \underline{\underline{E}}_L(\underline{r},t) + O(\underline{s}^2)$$

using our stress-strain relation (say the deformation occurs isentropically

in the Earth, for both seismic waves and longer period free modes

$$\frac{L^2}{T} \gg \underset{\uparrow}{K}$$ thermal diffusivity

L ~ length scale
T ~ time scale

this is condition that negligible heat flow occur
For typical rocks $K \sim 0.1$ cm^2/sec

fundamental mode ${}_0S_2$ 54 min
period ~ $3 \cdot 10^3$ sec
L of ${}_0S_2$ ~ 1 Earth radius ~ $6 \cdot 10^8$ cm

$$\frac{(6\cdot 10^8)^2}{3\cdot 10^3} \gg 0.1$$

for any seismological application
one uses <u>isen</u>tropic coefficients

From now on we drop the $\sim$ on $\tilde{\kappa}, \tilde{\mu}$

$$\boxed{\begin{aligned}\underline{\underline{T}}_E(\underline{r},t) = {} & -p_0(\underline{r})\,\underline{\underline{I}} + [\underline{s}\cdot\nabla p_0(\underline{r})]\,\underline{\underline{I}} \\ & + \lambda\,\nabla\cdot\underline{s}\,\underline{\underline{I}} + \mu[(\underline{\underline{\nabla s}})^T + \underline{\underline{\nabla s}}] + O(\underline{s})^2\end{aligned}}$$

all ∇ are w.r.t. $\underline{r}$ now
This is a description at pts $\underline{r}$ in space

Lets look now at the momentum
equation to first order in $\underline{s}(\underline{r},t)$

$$\rho D_t \underline{u} = \nabla\cdot\underline{\underline{T}}_E + \underline{f}$$

Our application is to the Earth
(or any planet)

$\underline{f}$ is <u>gravitational</u> force

$$\underline{f} = -\rho\underline{g} = -\rho\nabla\phi$$

$\underline{f} = -\rho\underline{g} = -\rho\nabla\phi$ grav. force per unit volume

now

$$\underline{u} = D_t\underline{s} = \partial_t\underline{s} + \underline{u}\cdot\underline{\underline{\nabla s}} = \partial_t\underline{s} + O(\underline{s})^2$$

$$D_t\underline{u} = \partial_t\underline{u} + \underline{u}\cdot\underline{\underline{\nabla u}} = \partial_t\underline{u} + O(\underline{s}^2)$$

$$= \partial_t^2\underline{s} + O(\underline{s})^2$$

$$\rho\partial_t^2\underline{s} = -\nabla p_0 + \nabla[\underline{s}\cdot\nabla p_0] + \nabla[\lambda(\nabla\cdot\underline{s})]$$

$$+ \nabla\cdot[\mu(\underline{\underline{\nabla s}}^T + \underline{\underline{\nabla s}})] - \rho\underline{\nabla}\phi$$

now the continuity eqn is

[illegible]

[illegible]

[illegible]

[illegible]

$$\partial_t \rho + \nabla_{\underline{r}} \cdot (\rho\, \underline{u}) = 0$$

$\underline{u} = D_t \underline{s} = \partial_t \underline{s}$ to first order

$$\boxed{\text{let } \rho = \underset{\substack{\uparrow \\ \text{initial} \\ \text{density}}}{\rho_0} + \overset{\swarrow\ \text{small change in density}}{\rho_1}}$$

$$\partial_t (\rho_0 + \rho_1) + \nabla_{\underline{r}} \cdot \left[(\rho_0 + \rho_1)\, \partial_t \underline{s} \right] = 0$$

$$\partial_t \rho_1 + \nabla_{\underline{r}} \cdot (\rho_0\, \partial_t \underline{s}) = 0$$

$$\partial_t \rho_1 + \partial_t\, \nabla_{\underline{r}} \cdot (\rho_0 \underline{s}) = 0$$

$$\partial_t \left(\rho_1 + \nabla \cdot (\rho_0 \underline{s}) \right) = 0$$

$$\rho_1 = -\nabla \cdot (\rho_0 \underline{s}) + \underset{\substack{\downarrow \\ \text{zero}}}{\text{const}}$$

$$\boxed{\rho_1 = -\nabla \cdot (\rho_0 \underline{s}) \qquad \begin{array}{l}\text{linearized} \\ \text{cont eqn}\end{array}}$$

the change ρ_1 in the mass distr. will give rise to a change in the grav. pot. and grav. field

$$\boxed{\begin{aligned} \phi &= \phi_0 + \phi_1 \quad \swarrow \text{ field caused by } \rho_1 \\ \underline{g} &= \underline{g}_0 + \underline{g}_1 \end{aligned}}$$

$$\begin{aligned} \rho \underline{g} = \rho \nabla \phi &= (\rho_0 + \rho_1)\nabla(\phi_0 + \phi_1) \\ &= \rho_0 \nabla\phi_0 + \rho_0 \nabla \phi_1 + \rho_1 \nabla\phi_0 + O(\underline{s})^2 \\ &= \rho_0 \nabla\phi_0 + \rho_0 \nabla\phi_1 - \\ &\qquad - \nabla\cdot(\rho_0 \underline{s})\nabla\phi_0 \end{aligned}$$

substitute into momentum eqn.
neglect all quadratic terms

$$\begin{aligned} \rho_0 \partial_t^2 \underline{s} = &-\nabla p_0 + \nabla[\underline{s}\cdot\nabla p_0] + \nabla[\lambda(\nabla\cdot\underline{s})] \\ &+ \nabla\cdot[\mu(\underline{\underline{\nabla s}}^T + \underline{\underline{\nabla s}})] - \rho_0 \nabla\phi_0 \\ &- \rho_0 \nabla\phi_1 + \nabla\cdot(\rho_0 \underline{s})\nabla\phi_0 \end{aligned}$$

before deformation in the equil. state the momentum eqn is

$$\boxed{0 = -\nabla p_0 + \rho_0 \nabla \phi_0}$$

we use this above and get finally

$$\boxed{\begin{aligned}\rho_0 \partial_t^2 \underline{s} &= -\rho_0 \nabla \phi_1 + \nabla \cdot (\rho_0 \underline{s}) \underline{g}_0 \\ &\quad - \nabla [\rho_0 \underline{g}_0 \cdot \underline{s}] + \nabla [\lambda (\nabla \cdot \underline{s})] \\ &\quad + \nabla \cdot [\mu (\underline{\underline{\nabla s}}^T + \underline{\underline{\nabla s}})]\end{aligned}}$$

linearized eqn of motion. Given $\rho_0(\underline{r})$, $\lambda(\underline{r})$, $\mu(\underline{r})$

$\rho_0(\underline{r}) \to g_0(\underline{r})$ from Poisson's eqn

$$\nabla^2 \phi = 4\pi G \rho$$

$$\nabla^2 \phi_0 = 4\pi G \rho_0$$

ϕ_1 determined from linearization

$$\nabla^2 (\phi_0 + \phi_1) = 4\pi G (\rho_0 + \rho_1)$$

$$\boxed{\nabla^2 \phi_1 = -4\pi G \, \nabla \cdot (\rho_0 \underline{s})}$$

four linear eqns for four unknowns $\underline{s}(\underline{r},t)$, $\phi_1(\underline{r},t)$
$\rho_1(\underline{r},t)$ determined from

$$\rho_1 = -\nabla\cdot(\rho_0\underline{s})$$

complete system of four eqns in four unknowns for a self-gravitating perfectly elastic system

This is the starting point for a discussion of the free oscillations of the Earth.

16. Seismic waves

Let us consider the case:

$$\vec{g}_0 = \vec{g}_1 = 0 \quad \text{(no gravity)}$$

$$\rho_0 = \lambda = \mu = \text{constant (homogeneous material)}$$

$$\rho_0 \partial_t^2 \underline{s} = \lambda \nabla(\nabla\cdot\underline{s}) + \mu \nabla\cdot\left[\underline{\underline{\nabla s}}^T + \underline{\underline{\nabla s}}\right]$$

now $\{\nabla\cdot[(\underline{\underline{\nabla s}})^T]\}_i = \partial_j(\partial_i s_j) = \partial_i \partial_j s_j$

$$= [\nabla(\nabla\cdot\underline{s})]_i$$

and $\nabla\cdot\underline{\underline{\nabla s}} = \nabla^2 \underline{s}$

[redacted] $\nabla\cdot[\underline{\underline{\nabla s}}^T + \underline{\underline{\nabla s}}] = \nabla(\nabla\cdot\underline{s}) + \nabla^2\underline{s}$

$$\boxed{\rho_0 \partial_t^2 \underline{s} = (\lambda+\mu)\nabla(\nabla\cdot\underline{s}) + \mu\nabla^2\underline{s}}$$

This called the Navier eqn of linearized perfect elasticity.

now $\nabla^2\underline{s} = \nabla(\nabla\cdot\underline{s}) - \nabla\times\nabla\times\underline{s}$ so

$$\boxed{\rho_0 \partial_t^2 \underline{s} = (\lambda+2\mu)\nabla(\nabla\cdot\underline{s}) - \mu\nabla\times\nabla\times\underline{s}}$$

another form of Navier eqn

Take the divergence

$$\boxed{\rho_0 \, \partial_t^2 (\nabla\cdot\underline{s}) = (\lambda + 2\mu)\, \nabla^2 (\nabla\cdot\underline{s})}$$

$\nabla\cdot\underline{s}$ obeys a wave eqn with phase velocity

$$c_p = \sqrt{\frac{\lambda+2\mu}{\rho}} = \sqrt{\frac{\kappa+\frac{4}{3}\mu}{\rho}}$$

waves in $\nabla\cdot\underline{s}$ (dilatation) are called compressional waves or p waves
(P = primary)

now if we take the curl

$$\nabla\times\nabla\times\nabla\times\underline{s} = -\nabla^2\nabla\times\underline{s} + \nabla(\nabla\cdot\nabla\times\underline{s})$$
$$= -\nabla^2\,\nabla\times\underline{s}$$

$$\boxed{\rho_0 \, \partial_t^2 (\nabla\times\underline{s}) = \mu\, \nabla^2 (\nabla\times\underline{s})}$$

$\nabla\times\underline{s}$ obeys a wave eqn with phase velocity

$$c_s = \sqrt{\mu/\rho}$$

waves in $\nabla \times \underline{s}$ called shear waves or S waves.
(S = secondary)

S waves are <u>transverse</u>, cannot be transmitted through a <u>fluid</u> $\mu = 0$

in <u>stable</u> materials

$$c_P^2 = \frac{\lambda + 2\mu}{\rho} = \frac{1}{3\rho}(3\lambda + 2\mu + 4\mu)$$

$$= \frac{K}{3\rho} + \frac{4}{3}\frac{\mu}{\rho}$$

$$c_P^2 = \frac{K}{3\rho} + \frac{4}{3}c_S^2 \geq \frac{4}{3}c_S^2$$

since $K \geq 0$ (stable)

$$\boxed{\text{in } \underline{\text{every}} \text{ stable material} \quad c_P \geq \sqrt{\tfrac{4}{3}}\, c_S}$$

P (primary) waves <u>always</u> at least 15% faster than S (secondary) waves, regardless of chemical composition or physical state of material.