11. angular momentum density

We begin now our study of angular momentum in a continuum, and we shall derive the angular momentum conservation law.

$\mathcal{O}$ = origin of coordinate system
all torques and all angular momenta will be taken about $\mathcal{O}$ (fixed in space)

The angular momentum about $\mathcal{O}$ of a point mass is $\underline{r} \times \underline{p}$, $\underline{p} \equiv$ linear momentum $\equiv m\underline{u}$

In any infinitesimally small volume dV (large enough to contain many atoms but small enough so that ρ and $\underline{u}$ are nearly constant inside)
The total linear momentum of dV is $\rho\underline{u}\, dV$ and the total angular momentum about $\mathcal{O}$ is $\underline{r} \times \rho\underline{u}\, dV$

But such an infinitesimally small volume (a macroparticle) is not a point mass. It contains many atoms ~~and~~ which can spin and which contain spinning electrons and spinning nuclei. Therefore the material in dV may have additional internal angular momentum.

The amount of this additional internal angular momentum is also $\propto dV$
The total angular momentum of the infinitesimally small region dV will be

$$\underline{r} \times \rho \underline{u}\, dV + \rho \underline{l}\, dV$$

$\rho \underline{l} \equiv$ internal angular momentum per unit volume
$\underline{l} \equiv$ internal angular momentum per unit mass.

After this motivation, we now postulate that in our idealized continuum $\exists$ an internal angular momentum density $\rho \underline{l}$ such that for any region V the total angular momentum of all the material in V is

$$\boxed{\int_V \rho(\underline{r} \times \underline{u} + \underline{l})\, dV}$$

If V is a volume $V(t)$ moving with the material, and if $\underline{M}$ is the total torque exerted on the material in $V(t)$ by the matter outside, we have by a basic law of physics.

$$\boxed{\frac{d}{dt}\int_{V(t)} \rho(\underline{r}\times\underline{u}+\underline{l})\, dV = \underline{M}}$$

we want to use this law of physics to deduce a conservation law for angular momentum. We use the theorem for differ. integrals over moving volumes to convert l.h.s. into a volume integral. We must then consider the term $\underline{M} \equiv$ net external torque on $V(t)$

We need to find out what kinds of things can give rise to a net torque and then put them all into a volume integral to use the vanishing integral theorem. Beginning to look familiar?

12. The torque exerted by stresses and body forces.

If the body force density $\underline{f}(\underline{r},t)$ acts on an infinitesimal volume dV it produces on that matter a torque $\underline{r}\times\underline{f}\, dV$. Hence the total torque exerted by body forces on a large region V is

$$\boxed{\underline{M}_f = \int_V \underline{r}\times\underline{f}\, dV}$$

torque due to body forces

If the surface traction $\hat{n}\cdot\underline{\underline{T}}(\underline{r},t)$ acts on an infinit. small element $\hat{n}dA$ of surface area dA it produces on that small patch a torque

$$\underline{r}\times[dA\,\hat{n}\cdot\underline{\underline{T}}] = -dA(\hat{n}\cdot\underline{\underline{T}})\times\underline{r}$$

The total torque exerted on the matter in a large region V by the surface traction on ∂V is

$$\boxed{\underline{M}_T = -\int_{\partial V} dA\,(\hat{n}\cdot\underline{\underline{T}})\times\underline{r}}$$

$\hat{n}$ is the unit <u>outward</u> normal

now $(\hat{n}\cdot\underline{\underline{T}})\times\underline{r} = \hat{n}\cdot(\underline{\underline{T}}\times\underline{r})$

since

$$[(\hat{n}\cdot\underline{\underline{T}})\times\underline{r}]_i = \epsilon_{ijk}(n_\ell T_{\ell j})r_k$$

$$= n_\ell(\epsilon_{ijk}T_{\ell j}r_k) \quad \text{just rearrange terms}$$

$$= [\hat{n}\cdot(\underline{\underline{T}}\times\underline{r})]_i \qquad (\underline{\underline{T}}\times\underline{r})_{\ell i} = T_{\ell j}r_k\epsilon_{ijk}$$

note: for a <u>dyad</u>, $\underline{\underline{fg}}\times\underline{r} = \underline{f}(\underline{g}\times\underline{r})$. This motivates the notation $\underline{\underline{T}}\times\underline{r}$

so

$$\underline{M}_T = -\int_{\partial V} dA\,\hat{n}\cdot(\underline{\underline{T}}\times\underline{r})$$

now use Gauss' theorem to convert to a volume integral

here $(\underline{\underline{T}}\times\underline{r})_{ij} = T_{ik}r_\ell\varepsilon_{jk\ell}$

$$\boxed{\underline{M}_T = -\int_V \nabla\cdot[\underline{\underline{T}}\times\underline{r}]\,dV}$$

$$\boxed{\text{What are } \underline{\underline{T}} \times \underline{u} \text{ and } \underline{u} \times \underline{\underline{T}}}$$

Look in a particular Cart. axis system

$$\underline{\underline{T}} = T_{ij}\,\hat{x}_i\hat{x}_j \quad \text{so}$$

$$\begin{aligned}
\underline{\underline{T}} \times \underline{u} &= T_{il}\,\hat{x}_i\,(\hat{x}_l \times \underline{u}) \\
&= T_{il}\,\hat{x}_i\,(\epsilon_{kjm}\,\delta_{jl}\,u_m\,\hat{x}_k) \\
&= [T_{il}\,u_m\,\epsilon_{klm}]\,\hat{x}_i\hat{x}_k
\end{aligned}$$

$$\boxed{\underline{\underline{T}} \times \underline{u} = [T_{il}\,u_m\,\epsilon_{klm}]\,\hat{x}_i\hat{x}_k}$$

$$\begin{aligned}
\underline{u} \times \underline{\underline{T}} &= \underline{u} \times T_{il}\,\hat{x}_i\hat{x}_l \\
&= T_{il}\,(\underline{u} \times \hat{x}_i)\,\hat{x}_l \\
&= T_{il}\,(\epsilon_{kjm}\,u_j\,\delta_{im}\,\hat{x}_k)\,\hat{x}_l \\
&= [T_{il}\,u_j\,\epsilon_{kji}]\,\hat{x}_k\hat{x}_l \\
&= [T_{li}\,u_j\,\epsilon_{kjl}]\,\hat{x}_k\hat{x}_i
\end{aligned}$$

$$\boxed{\text{for } \underline{\underline{T}} \text{ symmetric} \quad \underline{u} \times \underline{\underline{T}} = -(\underline{\underline{T}} \times \underline{u})^T}$$

$$\underline{u} \times \underline{\underline{T}} = [u_m\,T_{li}\,\epsilon_{kml}]\,\hat{x}_k\hat{x}_i = [-T_{li}\,u_m\,\epsilon_{klm}]\,\hat{x}_k\hat{x}_i$$

$$\boxed{\underline{u} \times \underline{\underline{T}} = -(\underline{\underline{T}}^T \times \underline{u})^T} \quad \text{wierd}$$

in the same way one can make sense out of expressions like $\underline{\underline{T}} \times \underline{\underline{M}}$

e.g. for $\underline{\underline{T}}, \underline{\underline{M}}$ second order

$$\begin{aligned}
\underline{\underline{T}} \times \underline{\underline{M}} &= (T_{ij}\,\hat{x}_i\hat{x}_j) \times (M_{kl}\,\hat{x}_k x_l) \\
&= T_{ij}\,M_{kl}\,\hat{x}_i\,(\hat{x}_j \times \hat{x}_k)\,\hat{x}_l
\end{aligned}$$

third order

here

13. body torques

It is possible for external fields to exert torques on atoms and yet exert no forces on them. A homogeneous magnetic field exerts a torque and no force on the unpaired electron in an atom of iron
In general a homogeneous magnetic field will exert a net torque but no force on any atom with a net magnetic moment.

Our infinitesimal macroparticles are not point masses, they contain many atoms, and in general we must admit the existence of an

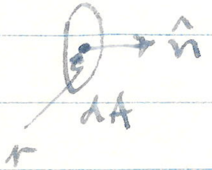
external body torque density $\underline{m}$ which produces on the matter in a volume V a net torque

$$\boxed{\underline{M}_m = \int_V \underline{m}\, dV}$$

14. Surface torques ~~[illegible]~~

The atoms just outside V may exert very large torques across ∂V just as they exert very large forces at short range. Since macroparticles contain many atoms there is no reason to expect that the interaction between two nearby particles across a surface ∂V will consist only of a force between them. In general they can exert torques on each other as well. We must admit the existence of a surface torque which involves no transmission of force.

If dA is a small plane area with unit normal $\hat{n}$ at posn $\underline{r}$ at time t, the material just in front of dA will exert on the material just behind dA a torque proportional to dA (because of the short range nature of the interaction) and depending on $\hat{n}$, $\hat{r}$, and t.



We can write this torque $\underline{M}(\underline{r}, t, \hat{n})\, dA$ where $\underline{M}$ is the torque per unit area. The total surface torque exerted on the matter in a volume V by the matter just outside is then

$$\boxed{\underline{\mathcal{M}}_S = \int_{\partial V} dA\ \underline{M}(\underline{r}, t, \hat{n}(\underline{r}))}$$

15. The torque tensor

Combining all the sources of torque on the matter in a volume $V(t)$ we have for the

<u>total torque</u>

$$\underline{\mathcal{M}} = \underline{\mathcal{M}}_f + \underline{\mathcal{M}}_T + \underline{\mathcal{M}}_m + \mathcal{M}_S$$

due to body force density | due to stress on ∂V | due to body torque density | due to surface torques

$$\underline{\mathcal{M}} = \int_V dV \left[\underline{r} \times \underline{f} - \nabla \cdot (\underline{\underline{T}} \times \underline{r}) + m \right] + \int_{\partial V} dA\, \underline{M}(\underline{r}, t, \hat{n}(\underline{r}))$$

true for any volume V

recall the rate of change of angular momentum of a <u>volume</u> $V(t)$ moving with the material

$$\frac{d}{dt} \int_{V(t)} \rho \underline{L}\, dV$$

where $\underline{L} = \underline{r} \times \underline{u} + \underline{\ell} \equiv$ total angular momentum density / unit mass

$$= \int_{V(t)} \partial_t (\rho \underline{L})\, dV + \int_{\partial V(t)} dA\, \hat{n} \cdot \underline{u} \rho \underline{L}$$

$$= \int_{V(t)} \left[\partial_t (\rho \underline{L}) + \nabla \cdot (\rho \underline{u} \underline{L}) \right] dV$$

we can thus write our law of Newtonian physics in the form

$$\int_{V(t)} \left[\partial_t(\rho \underline{L}) + \nabla\cdot(\rho \underline{u}\,\underline{L}) + \nabla\cdot(\underline{\underline{T}} \times \underline{r}) - \underline{r}\times\underline{f} - \underline{m} \right] dV$$

$$= \int_{\partial V(t)} \underline{M}(\underline{r}, t, \hat{n}(\underline{r}))\, dA$$

Now this equation has the same <u>paradoxical property</u> as did our form of the momentum equation at this point.
The l.h.s. $\to 0$ like ℓ^3 while ordinarily one would expect the r.h.s. to $\to 0$ like ℓ^2 as one integrates over increasingly smaller volumes V_ℓ of linear dimension ℓ.
We can utilize this apparent paradox exactly as we did before.
There must be some <u>extra cancellation</u> going on which makes the surface integral go to zero faster than one would ordinarily expect.
We have seen before that the exact nature of the cancellation is that

The dependence of $\underline{M}(\underline{r}, t, \hat{n})$ on $\hat{n}$ is <u>linear</u>. i.e. $\exists$ a second order tensor $\underline{\underline{M}}(\underline{r}, t)$ such that $$\underline{M}(\hat{r}, t, \hat{n}(\underline{r})) = \hat{n}(\underline{r}) \cdot \underline{\underline{M}}(\underline{r}, t)$$

The tensor $\underline{M}(\underline{r},t)$ will be called the torque-stress tensor

Many writers call it the couple-stress tensor

$\underline{M}(\underline{r},t)$ is the second order tensor (linear operator which assigns to an infinitesimal area element $\hat{n}dA$ the net torque exerted by the material in front of of $\hat{n}dA$ on the material behind

Now we can write the net torque on $V(t)$ due to torque-stresses or couple-stresses as

$$\underline{M}_S = \int_{\partial V(t)} M(\underline{r},t,\hat{n}(\underline{r}))\,dA = \int_{\partial V(t)} \hat{n}(\underline{r})\cdot\underline{M}(\underline{r},t)\,dA$$

$$= \int_{V(t)} \nabla\cdot\underline{M}(\underline{r},t)\,dV \qquad \text{by Gauss}$$

16. The angular momentum equation

We now have everything inside the volume integral

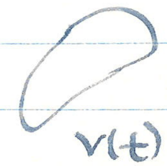
$$\int_{V(t)} \left\{\partial_t(\rho\underline{L}) + \nabla\cdot[\rho\underline{u}\underline{L} + \underline{\underline{T}}\times\underline{r} - \underline{M}]\right\} dV$$

$$= \int_{V(t)} [\,\underline{r}\times\underline{f} + \underline{m}\,]\,dV$$

Lecture #7 Review

Discussing the third law of conservation: conservation of angular momentum

Law of physics $\quad \dfrac{d}{dt}\displaystyle\int_{V(t)} \rho \underline{L}\, dV = \underline{M}$



$V(t)$ moving with material

$\underline{L} = \underline{r} \times \underline{u} + \underset{\substack{\uparrow \\ \text{intrinsic or internal}}}{\underline{l}}$ angular momentum density / unit mass

$\underline{M}$ = sum of all external torques acting on $V(t)$

$= \underset{\substack{\uparrow \\ \text{body} \\ \text{forces}}}{\underline{M}_{\underline{f}}} + \underset{\substack{\uparrow \\ \text{surface} \\ \text{stress}}}{\underline{M}_{\underline{\underline{T}}}} + \underset{\substack{\uparrow \\ \text{body} \\ \text{torques}}}{\underline{M}_{\underline{m}}} + \underset{\substack{\uparrow \\ \text{surface torques}}}{\underline{M}_{s}}$

here I mean

$(\underline{\underline{T}} \times \underline{r})_{ik} = T_{il} r_m \epsilon_{klm}$

$[(\hat{n} \cdot \underline{\underline{T}}) \times \underline{r}]_k = n_i T_{il} r_m \epsilon_{klm}$

$$\int_{V(t)} [\, \partial_t(\rho \underline{L}) + \nabla \cdot (\rho \underline{u}\, \underline{L}) + \nabla \cdot (\underline{\underline{T}} \times \underline{r}) - \underline{r} \times \underline{f} - \underline{m} \,]\, dV$$

$$= \int_{\partial V(t)} \underset{\substack{\uparrow \\ \text{surface torque density} \\ \text{(per unit area)}}}{\underline{M}(\underline{r}, t, \hat{n}(\underline{r}))}\, dA$$

$\boxed{\text{over one page}}$

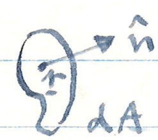
paradoxical property - This property tells us that $M(\underline{r}, t, \hat{n})$ must depend on $\hat{n}$ in a particular manner, namely <u>linearly</u>

i.e. because of the above eqn $\exists$ a <u>tensor</u> $\underline{\underline{M}}(\underline{r},t) \ni$

$$\underline{M}(\underline{r},t,\hat{n}) = \hat{n}\cdot\underline{\underline{M}}(\underline{r},t)$$

the tensor (<u>linear operator</u>) field (defined for all $\hat{r}, t$) called the <u>torque-stress tensor</u> or <u>couple-stress tensor</u>.

$\underline{\underline{M}}(\underline{r},t)$ is that <u>linear operator</u> which assigns to an infin. patch $dA\,\hat{n}$ centered at $\underline{r}$ at time t the net torque exerted by the material in front of $\hat{n}\,dA$ on the material behind $\hat{n}\,dA$



net torque exerted on $V(t)$ by torque- or couple stresses is

$$\int_{\partial V(t)} \hat{n}\cdot\underline{\underline{M}}(\underline{r},t)\,dA = \int_{V(t)} \nabla\cdot\underline{\underline{M}}(\underline{r},t)\,dV$$

now everything inside the volume integral now once again we appeal to <u>vanishing integral theorem</u>. Since $V(t)$ is arbitrary, the integrand must vanish for all $\underline{r}$ and t.

$$\boxed{\begin{array}{l} \underline{L} = \underline{r} \times \underline{u} + \underline{L} = \text{angular momentum density per unit mass} \\[2ex] \partial_t(\rho \underline{L}) + \nabla \cdot [\rho \underline{v} \underline{L} + \underline{\underline{T}} \times \underline{r} - \underline{\underline{M}}] = \underline{r} \times \underline{f} + \underline{m} \end{array}}$$

14 Feb, 1972 and here ♡ day

This is the <u>angular momentum conservation law</u>
Has the form of a general conservation equation

ϕ stuff is angular momentum
$\phi \equiv$ angular momentum density (per unit vol.)
$k \equiv \underline{r} \times \underline{f} + \underline{m} \equiv$ rate of production of ϕ-stuff per unit vol. per second.
$\underline{K} = \rho \underline{v} \underline{L} + \underline{\underline{T}} \times \underline{r} - \underline{\underline{M}} =$ current density of ϕ-stuff in space.

Now we can simplify the above expression
Recall that we simplified the momentum eqn by extracting the buried copy of the cont. eqn.
It turns out that the above eqn has buried copies of both the cont. eqn <u>and</u> the

momentum equation.
First we remove the continuity eqn.

We need to know that

$$\boxed{\text{If } \underline{f}, \underline{g} \text{ are vector fields} \quad \nabla\cdot(\underline{f}\,\underline{g}) = (\nabla\cdot \underline{f})\underline{g} + \underline{f}\cdot\nabla \underline{g}}$$

proof: write out components w.r.t. some $\hat{x}_1, \hat{x}_2, \hat{x}_3$

$$\partial_i (f_i g_j) = (\partial_i f_i) g_j + f_i \partial_i g_j$$

in this form it is obvious

Now

$$\partial_t (\rho \underline{L}) + \nabla\cdot(\rho \underline{u}\,\underline{L}) = (\partial_t \rho)\underline{L} + \rho \partial_t \underline{L} + (\nabla\cdot \rho\underline{u})\underline{L} + \rho\underline{u}\cdot\nabla\underline{L}$$

$$= \underbrace{\cancel{[\partial_t \rho + \nabla\cdot(\rho\underline{u})]}}_{0}\underline{L} + \rho D_t \underline{L}$$

so the angular momentum eqn may be written

$$\rho D_t \underline{L} + \nabla\cdot(\underline{\underline{T}} \times \underline{r}) = \nabla\cdot\underline{M} + \underline{r}\times\underline{f} + m$$

now substitute $\underline{L} = \underline{r} \times \underline{u} + \underline{l}$

$$\rho D_t \underline{l} + \rho D_t (\underline{r} \times \underline{u}) + \nabla \cdot (\underline{\underline{T}} \times \underline{r}) - \underline{r} \times \underline{f} = \nabla \cdot \underline{\underline{M}} + \underline{m}$$

now to remove the momentum eqn

allow me to remind you about something introduced last term p. 94 of my notes

<u>The Wedge Operator Λ</u> on second order tensors

$$\Lambda : R^3 \otimes R^3 \to R^3$$

Λ is a linear operator, acts on a second order tensor to produce a vector

defn: $\forall \; \underline{\underline{T}} \in R^3 \otimes R^3$

third order alternating tensor

$$\Lambda \underline{\underline{T}} \equiv tr_{23} \left[tr_{35} \left(\underline{\underline{\underline{A}}} \otimes T \right) \right] , \text{ a vector}$$

components in an arbitrary Cartesian axis system $\hat{x}_1, \hat{x}_2, \hat{x}_3$:

$$\boxed{\begin{array}{l} \qquad\qquad (\Lambda \underline{\underline{T}})_i = A_{ijk} T_{jk} \\ \text{if } \hat{x}_1, \hat{x}_2, \hat{x}_3 \text{ right handed} \qquad (\Lambda T)_i = \epsilon_{ijk} T_{jk} \\ \text{if } \hat{x}_1, \hat{x}_2, \hat{x}_3 \text{ left-handed} \qquad (\Lambda T)_i = -\epsilon_{ijk} T_{jk} \end{array}}$$

effect on a dyad: $\underline{\underline{T}} = \underline{f}\,\underline{g}$, then

$$\wedge \underline{\underline{T}} = \wedge \underline{f}\,\underline{g} = \underline{f} \times \underline{g}$$ ordinary cross product

The wedge operator is in this sense a generalization of the cross product.

We shall now prove the following

Theorem: Given $\underline{\underline{T}}(\underline{r})$ a second order tensor field in R^3. Then

$$\boxed{\nabla \cdot (\underline{\underline{T}} \times \underline{r}) = (\nabla \cdot \underline{\underline{T}}) \times \underline{r} - \wedge \underline{\underline{T}}}$$

proof: do the computation in $\hat{x}_1, \hat{x}_2, \hat{x}_3$

$$(\underline{\underline{T}} \times \underline{r})_{ik} = T_{il}\, r_m\, \epsilon_{klm} \quad \text{and}$$

$$[\nabla \cdot (\underline{\underline{T}} \times \underline{r})]_k = \partial_i (T_{il}\, r_m\, \epsilon_{klm})$$

$$= \epsilon_{klm} [r_m \partial_i T_{il} + T_{il} \partial_i r_m]$$

$$= (\nabla \cdot \underline{\underline{T}})_l\, r_m\, \epsilon_{klm} + \epsilon_{klm} T_{il} \delta_{im}$$

$$= \epsilon_{klm} (\nabla \cdot \underline{\underline{T}})_l\, r_m + \epsilon_{kli} T_{il}$$

$$= \epsilon_{klm} (\nabla \cdot \underline{\underline{T}})_l\, r_m - \epsilon_{kil} T_{il}$$

$$= [(\nabla \cdot \underline{\underline{T}}) \times \underline{r}]_k - (\wedge \underline{\underline{T}})_k \qquad \text{q.e.d.}$$

now back to the task at hand
first we note

$$D_t(\underline{r} \times \underline{u}) = D_t \underline{r} \times \underline{u} + \underline{r} \times D_t \underline{u}$$

$$= \underline{u} \times \underline{u} + \underline{r} \times D_t \underline{u}$$

$$= \underline{r} \times D_t \underline{u}$$

substitute this and the result of our theorem back into ang. mom. eqn.

$$\rho D_t \underline{l} + \rho \underline{r} \times D_t \underline{v} + (\nabla \cdot \underline{\underline{T}}) \times \underline{r} - \Lambda \underline{\underline{T}} - \underline{r} \times \underline{f} = \nabla \cdot \underline{\underline{M}} + \underline{m}$$

$$\underline{r} \times \underbrace{[\rho D_t \underline{v} - \nabla \cdot \underline{\underline{T}} - \underline{f}]}_{\text{the momentum eqn} = 0} + \rho D_t \underline{l} = \nabla \cdot \underline{\underline{M}} + \underline{m} + \Lambda \underline{\underline{T}}$$

hence the angular momentum equation reduces to

$$\boxed{\rho D_t \underline{l} = \nabla \cdot \underline{\underline{M}} + \underline{m} + \Lambda \underline{\underline{T}}}$$

call this the internal ang. mom. eqn.

$$\boxed{\partial_t (\rho \underline{L}) + \nabla \cdot [\rho \underline{u} \underline{L} + \underline{\underline{T}} \times \underline{r} - \underline{\underline{M}}] = \underline{r} \times \underline{f} + \underline{m}}$$

call this (the original form) the ang. mom. eqn.

In almost all materials, and certainly in geophysically interesting materials, it is a very good approximation to assume that

$\underline{l} = 0$, $\underline{m} = 0$, $\underline{\underline{M}} = 0$ } examples where these may be important: organic substances, fluids with a suspension of particles, any small λ phenomenon, cosmology (galaxies), plastic flow of metals; mostly engineering applications.

no internal angular momentum in our macroparticles
no body torques or torque-stresses (couple-stresses)
There are a few peculiar continua for which these assumptions are not justified.

We henceforth assume $\underline{l} = 0$, $\underline{m} = 0$, $\underline{\underline{M}} = \underline{0}$.
In that case the ang. mom. eqn. reduces to

$$\boxed{\Lambda \underline{\underline{T}} = 0}$$

Theorem: $\underline{\underline{T}}$ is symmetric iff $\Lambda \underline{\underline{T}} = 0$

proof:

look in a part. Cart. axis system $\hat{x}_1, \hat{x}_2, \hat{x}_3$

$$(\Lambda T)_i = \epsilon_{ijk} T_{jk}$$

if $\underline{\underline{T}} = \underline{\underline{T}}^T$, then $T_{jk} = T_{kj}$ so

$$\epsilon_{ijk} T_{jk} = \epsilon_{ijk} T_{kj} = \epsilon_{ikj} T_{jk} = - \epsilon_{ijk} T_{jk}$$

hence $= 0$

conversely if $(\Lambda \underline{\underline{T}})_i = 0$

$(\Lambda T)_i = \epsilon_{ijk} T_{jk}$ mult. by ϵ_{imn}

$\epsilon_{imn} (\Lambda T)_i = \epsilon_{imn} \epsilon_{ijk} T_{jk}$

$= (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) T_{jk} = T_{mn} - T_{nm}$ so

$T_{mn} - T_{nm} = \epsilon_{imn} (\Lambda T)_i = \epsilon_{imn} (0)_i = 0$

$T_{mn} = T_{nm}$ hence $\underline{\underline{T}} = \underline{\underline{T}}^T$ q.e.d.

$$\boxed{\begin{array}{l}\text{if } \underline{l} = 0,\ \underline{m} = 0,\ \underline{M} = 0 \text{, then the (Eulerian)} \\ \text{stress tensor } \underline{\underline{T}}(\underline{r}, t) \text{ is symmetric} \\ \qquad \underline{\underline{T}}(\underline{r}, t) = \underline{\underline{T}}^T(\underline{r}, t) \\ \qquad T_{ij}(\underline{r}, t) = T_{ji}(\underline{r}, t)\end{array}}$$

in such a medium the stress tensor $\underline{\underline{T}}$ only has six independent components.

note that if $\underline{\underline{T}}^T(\underline{r}, t) = \underline{\underline{T}}(\underline{r}, t)$ then
$\hat{n} \cdot \underline{\underline{T}} = \underline{\underline{T}} \cdot \hat{n}$

Give physical interpretation of $\underline{\underline{T}} = \underline{\underline{T}}^T$.
Consider a small cube.

end here
16 Feb 1972

Lecture #8

18. The law of conservation of energy

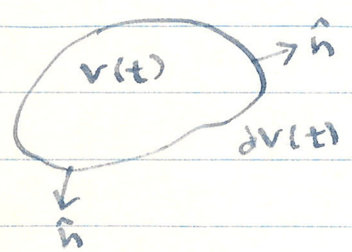
We assume henceforth that body torques, torque stresses, and intrinsic internal angular momentum can be neglected.

Our equation for the conservation of energy is **not** however for this case only.

~~Hence we shall consider only materials for which the stress tensor is symmetric, $\underline{\underline{T}}^T = \underline{\underline{T}}$.~~

Consider an open set $V(t)$ moving with the material in a continuum

$\hat{n}$ = unit outward normal to boundary $\partial V(t)$



The rate at which the external body force does work on a small volume dV is $\underline{u} \cdot \underline{f}\, dV$, where $\underline{u}$ = Eulerian velocity, so

The total rate at which the body force $\underline{f}(\underline{r}, t)$ does work on $V(t)$ is

$$\underline{W}_f = \int_{V(t)} \underline{u} \cdot \underline{f}\, dV$$

The rate at which the surface forces $\underline{F}(\underline{r}, t, \hat{n})$ acting across a small patch $\hat{n}\,dA$ of the surface $\partial V(t)$ is $\underline{u} \cdot \underline{F}(\underline{r}, t, \hat{n})\,dA$ or $\underline{u} \cdot (\hat{n} \cdot \underline{\underline{T}}(\underline{r}, t))\,dA$ or $\hat{n} \cdot T(\underline{r}, t) \cdot \underline{u}\,dA$

The total rate at which mechanical work is done on the matter in $V(t)$ by the surface stresses acting across $\partial V(t)$ is

$$\boxed{\underline{W}_S = \int_{\partial V(t)} dA\, \hat{n} \cdot \underline{\underline{T}}(\underline{r}, t) \cdot u(\underline{r}, t)}$$

by Gauss this is

$$\underline{W}_S = \int_{V(t)} \nabla \cdot [\underline{\underline{T}} \cdot \underline{u}]\, dV$$

The total rate at which mechanical work is done on the matter in $V(t)$ by both body and surface forces is

$$\boxed{\underline{W} = \int_{V(t)} [\underline{f} \cdot \underline{u} + \nabla \cdot (\underline{\underline{T}} \cdot \underline{u})]\, dV}$$

The kinetic energy of the macroscopic motion attributable to an infin. small volume dV of matter is $\frac{1}{2}(\rho\, dV)u^2 \quad (u^2 = \underline{u} \cdot \underline{u})$

Thus the total kinetic energy attributable to macroscopic motion in $V(t)$ is

$$\boxed{\mathbb{K} = \int_{V(t)} \tfrac{1}{2}\rho u^2 \, dV}$$

It is an experimentally observed fact that in many continua (in fact almost all continua — all but incompressible perfect fluids)

$$d\mathbb{K}/dt \neq \mathbb{W}$$

Energy is apparently not conserved. To salvage the law of conservation of energy we must convince ourselves that $\mathbb{K}$ is not the only energy in $V(t)$ ~~and~~ or that $\mathbb{W}$ is not the only way for $V(t)$ to acquire energy, or both.

To do this we must introduce thermodynamical concepts because in general a continuum must be treated as a thermodynamical system. Joule in 1843 was the first to show that the law of conservation of energy could be applied to thermodynamical systems; in that theory, the law of conservation of energy is called the first law of thermodynamics. The above treatment considers only macroscopic purely mechanical energy.

consider the first point. K is not the only energy in $V(t)$. (a macroparticle)
Consider an infin. volume dV, and run along beside it so that it seems to have velocity $\underline{u} = 0$. Then from your point of view, the macroscopic k.e. $\frac{1}{2} \rho dV u^2$ is zero. This does not mean from your point of view it has no energy.
Each such macroparticle contains many atoms, each of which has k.e. due to motion within the macroparticle, and the atoms exert forces on one another by means of which atomic potential energy can be stored. These extra stores of energy in an ~~macrovolume dV~~ infin. volume dV, over and above the macroscopic k.e., is called the internal energy in dV. It is the energy seen to belong to dV by an observer in a reference frame where dV is at rest. (so the mean motion of the atoms and molecules comprising dV vanishes, but not the relative internal motion).

The internal energy of a ~~the~~ system with no k.e. is a well known thermodynamical concept. In our idealized continuum, we try to imitate the above physical situation by postulating

There exists in the continuum an <u>internal energy density per gram</u> $\mathbf{U}(\underline{r},t)$. There are $\rho \mathbf{U}$ ergs / cm^3 of internal energy at a point $\underline{r}$ in the material so that the total internal energy in $V(t)$ is

$$\mathbb{U} = \int_{V(t)} \rho \mathbf{U} \, dV$$

Consider now the second point ($V(t)$ has other ways of acquiring energy besides having mechanical work performed on it.

When sunlight shines through water, some of the sunlight is absorbed by the water and is converted to internal energy, detectable as a warming of the water. We may say that internal energy appears in the water at a rate $h(\underline{r},t)$ ergs per sec per cm^3 due to this absorption of sunlight. The rate of the appearance of this internal energy h will be smaller at great depths where little sunlight penetrates, and will often be negative after sunset when the water radiates back into space. Another example of such a non-mechanical production of energy per unit volume is <u>radioactivity</u> which if present (e.g. dissolved U salts) heats the material.

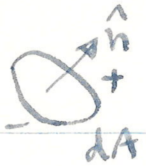
We conclude that in general there may be internal energy production at a rate $h(\underline{r},t)$ ergs / cm^3-sec. The total rate of appearance of energy in $V(t)$ will be

$$\mathbb{P} = \int_{V(t)} h(\underline{r},t)\, dV$$

There is another way in which $V(t)$ may gain or lose energy. It may gain or lose energy by <u>heat flow</u>. The concepts that heat is a form of energy and of the <u>internal energy</u> of a system were of course were what was needed to extend the law of cons. of energy to thermodynamic situations.

During a short time δt, when mechanical work $\mathbb{W}\,\delta t$ and internal energy $\mathbb{P}\,\delta t$ are added to $V(t)$, some of the energy in $V(t)$ may simply leak or flow out through the surface $\partial V(t)$. In the atomic picture this energy flow results from atomic collisions and the crossing of $\partial V(t)$ by atoms, and amounts to what is usually called heat conduction.

Consider a small nearly plane patch dA on $\partial V(t)$



If atoms move only a short distance between collisions (short compared with the dimensions of dA) then an heuristic argument much like the one leading to the idea of surface forces leads us to assume that:

> In our idealized continuum, the amount of energy which in time δt leaks across the small area dA with unit normal $\hat{n}$ from the back of dA to its front is proportional to dA and to δt: it is
>
> $$H(\underline{r}, t, \hat{n})\, dA\, \delta t$$
>
> note sign convention

The proportionality to dA is a result of the shortness (compared to dA, but long compared to δ = mean atomic size) of the mean free path between collisions.

The proportionality constant $H(\underline{r}, t, \hat{n})$ may depend on the orientation of dA as expressed by its unit normal $\hat{n}$.

> The total rate at which energy leaks out of $V(t)$ across $\partial V(t)$ by heat conduction is
>
> $$\mathbb{H} = \int_{\partial V(t)} H(\underline{r}, t, \hat{n}(\underline{r}))\, dA \quad \text{ergs/sec.}$$

The fact that energy, while leaking across boundaries in the above manner is called <u>heat</u> is of course historical.

The <u>first law of thermodynamics</u> (or the law of conservation of energy) for the matter in $V(t)$ can now be written as

$$\frac{d}{dt}(\mathbb{K}+\mathbb{U}) = \mathbb{W} + \mathbb{P} - \mathbb{H} \qquad (*)$$

or $\qquad \delta(\mathbb{K}+\mathbb{U}) = (\mathbb{W} + \mathbb{P} - \mathbb{H})\,\delta t$ in an infinitesimally short time δt

One often says that $\exists$ a function $\mathbb{K}+\mathbb{E}$ which is an <u>exact</u> or <u>perfect differential</u>.

$(*)$ is a law of physics, but in the above form and for complicated kinds of continua, it is almost a tautology. If we perform an experiment in which $(*)$ fails then we must repair it by "discovering" some new of storing internal energy or some "overlooked" source of energy which contributes to $\mathbb{P}$ or $\mathbb{H}$. It is a useful law (<u>for prediction purposes</u>) only insofar as we can really describe beforehand, in a concrete experiment, what $\mathbb{U}$, $\mathbb{P}$, and $\mathbb{H}$ will be.

19. The total energy equation

We now reduce (*) to a p.d.e. valid at each point in the continuum.

$$\boxed{\text{Let } E = \tfrac{1}{2}u^2 + U \text{ so } E \text{ is the total energy per gram of material}}$$

Then (*) can be written

$$\frac{d}{dt}\int_{V(t)} \rho E \, dV = \int_{V(t)} \left[h + \underline{f}\cdot\underline{u} + \nabla\cdot(\underline{\underline{T}}\cdot\underline{u}) \right] dV$$

$$- \int_{\partial V(t)} dA \; H(\underline{r}, t, \hat{n}(\underline{r}))$$

Since $V(t)$ moves with the material, we have as usual

$$\frac{d}{dt}\int_{V(t)} \rho E \, dV = \int_{V(t)} \partial_t (\rho E) \, dV + \int_{\partial V(t)} dA \, (\hat{n}\cdot\underline{u}\rho E)$$

$$= \int_{V(t)} \left[\partial_t (\rho E) + \nabla\cdot(\rho E \underline{u}) \right] dV$$

hence $(\star)$ becomes

$$\boxed{\int_{V(t)} dV \left[\partial_t(\rho E) + \nabla\cdot(\rho E \underline{u}) - h - \underline{f}\cdot\underline{u} - \nabla\cdot(\underline{\underline{T}}\cdot\underline{u}) \right] = - \int_{\partial V(t)} dA \; H(\underline{r}, t, \hat{n}(\underline{r}))}$$

The above eqn is in the standard form for producing an apparent paradox.
As we consider volumes V_ℓ which shrink to a point as $\ell \to 0$

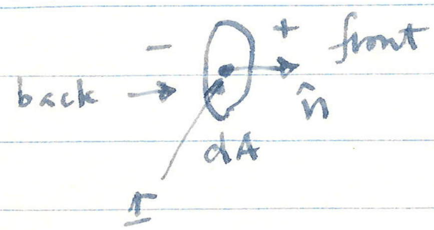
$$\text{l.h.s.} \to 0 \quad \text{like } \ell^3$$
$$\text{r.h.s.} \to 0 \quad \text{like } \ell^2,$$

or at least one would ordinarily think so.

We proceed just as before. There must be some extra cancellation. We have seen that the nature of this extra cancellation is that

> $H(\underline{r}, t, \hat{n})$ must depend linearly on $\hat{n}$
> i.e. $\exists$ a linear functional (or a vector) $\underline{H}(\underline{r}, t)$ such that
>
> $$H(\underline{r}, t, \hat{n}) = \hat{n}\cdot\underline{H}(\underline{r}, t)$$

$\underline{H}(\underline{r},t)$ is called the <u>heat flux vector</u>
the linear functional associated with the heat flux vector is that linear functional which assigns to an infin. area element $\hat{n}\,dA$ ~~the net heat~~ centered on $\hat{r}$ the net heat flux (a scalar) transported across that surface (per unit time)



$$H(\underline{r},t,\hat{n})\,dA = \hat{n}\,dA\cdot\underline{H}(\underline{r},t)$$

We have essentially <u>deduced</u> the existence of a heat flux vector from our simple and reasonable hypotheses about the way energy is stored and transmitted in a continuum.
Our eqn now becomes, using Gauss

$$\int_{V(t)} dV\left[\partial_t(\rho E) + \nabla\cdot(\rho E\underline{u} - \underline{\underline{T}}\cdot\underline{u} + \underline{H}) - \underline{f}\cdot\underline{u} - h\right] = 0$$

Now by the vanishing integral theorem

$$\boxed{\begin{gathered}\partial_t(\rho E) + \nabla\cdot(\rho E\underline{u} - \underline{\underline{T}}\cdot\underline{u} + \underline{H}) = \underline{f}\cdot\underline{u} + h \\ E = \tfrac{1}{2}u^2 + U \equiv \text{internal energy per gram}\end{gathered}}$$

This is called the <u>total energy equation</u>
It has the usual form of a conservation law for ϕ-stuff

ϕ-stuff $\equiv$ total energy (kinetic + internal)
$\phi \equiv \rho E \equiv$ density (per cm^3) of total energy
$k \equiv$ rate of production of total energy per unit volume per second
$\equiv \underline{f} \cdot \underline{u} + h$ (not surprising)
$\underline{K} \equiv$ current density of total energy in space
$\equiv \rho E \underline{u} - \underline{\underline{T}} \cdot \underline{u} + \underline{H}$

$\underline{H} =$ heat flux (already interpreted)
$\rho E \underline{u}$ obviously just energy carried along by the material as it moves (<u>advected</u> energy)
$-\underline{\underline{T}} \cdot \underline{u}$ indicates that when we push (from inside $V(t)$) with force $\hat{n} \cdot \underline{\underline{T}}$ per unit area, we pump energy across the surface at a <u>rate</u> $-\hat{n} \cdot \underline{\underline{T}} \cdot \underline{u}$ if the surface moves with velocity $\underline{u}$
This is an energy flux due to the action of surface forces.

this is problem 2.

Note: $\phi \equiv \rho E = \frac{1}{2}\rho v^2 + \rho U$ includes only kinetic and intrinsic internal energy. We have <u>not</u> (yet) introduced <u>mechanical potential energy</u> (only a useful concept if $\underline{f}$ (or part of $\underline{f}$) is derivable from a scalar potential.

end here 21 Feb.

Lecture # 9 Review

Reduced the form taken by the first law of thermodynamics, or the law of conservation of energy for our continuum

For the case of no couple-stresses and no body torques

kinetic energy, internal energy

let $\phi = \rho E = \rho\left(\frac{1}{2}u^2 + U\right)$

$=$ total energy density per unit vol.

$$\boxed{\partial_t(\rho E) + \nabla\cdot(\rho E \underline{u} - \underline{\underline{T}}\cdot\underline{u} + \underline{H}) = \underline{f}\cdot\underline{u} + h}$$

production rate of total energy per unit vol. is not surprisingly $\underline{f}\cdot\underline{u} + h$

the current density of total energy in space is

$$\underbrace{\rho E \underline{u}}_{\text{advected}} - \underbrace{\underline{\underline{T}}\cdot\underline{u}}_{\text{energy flux due to action of surface forces.}} + \underbrace{\underline{H}}_{\text{heat flux}}$$

20. The Internal Energy Equation

Once again this equation has buried in it copies of the continuity equation and the momentum eqn. We can extract these and obtain a conservation eqn for the internal energy.

To remove the cont. eqn

$$\partial_t(\rho E) + \nabla\cdot(\rho \underline{u} E) = (\partial_t \rho)E + (\partial_t E)\rho + \nabla\cdot(\rho\underline{u})E + \rho\underline{u}\cdot\nabla E$$

$$= \underbrace{[\partial_t\rho + \nabla\cdot(\rho\underline{u})]}_{0}E + \rho(\partial_t E + \underline{u}\cdot\nabla E)$$

$$= \rho D_t E.$$

Thus our eqn is

$$\rho D_t\left(\tfrac{1}{2}u^2 + U\right) + \nabla\cdot\underline{H} - \nabla\cdot(\underline{\underline{T}}\cdot\underline{u}) = \underline{f}\cdot\underline{u} + h$$

Now we need to establish an identity

Identity: $\forall$ tensor field $\underline{\underline{T}}$ and any vector field $\underline{u}$

$$\boxed{\nabla\cdot(\underline{\underline{T}}\cdot\underline{u}) = (\nabla\cdot\underline{\underline{T}})\cdot\underline{u} + \mathrm{tr}(\underline{\underline{T}}^T\cdot\nabla\underline{u})}$$

proof: Consider components in an arbitrary Cartesian axis system $\hat{x}_1, \hat{x}_2, \hat{x}_3$

$$
\begin{aligned}
\nabla\cdot(\underline{\underline{T}}\cdot\underline{u}) &= \partial_i (T_{ij} u_j) \\
&= \partial_i T_{ij} u_j + T_{ij}\partial_i u_j \\
&= (\nabla\cdot\underline{\underline{T}})_j u_j + T^T{}_{ji}\partial_i u_j \\
&= (\nabla\cdot\underline{\underline{T}})\cdot\underline{u} + \mathrm{tr}(\underline{\underline{T}}^T\cdot\nabla\underline{u})
\end{aligned}
$$

q.e.d.

in addition we observe that

$$D_t(\tfrac{1}{2}u^2) = \tfrac{1}{2}(D_t\underline{u})\cdot\underline{u} + \tfrac{1}{2}\underline{u}\cdot(D_t\underline{u}) = \underline{u}\cdot D_t\underline{u}$$

Thus the energy eqn reduces to

$$\cancelto{0}{[\rho D_t\underline{u} - \nabla\cdot\underline{\underline{T}} - \underline{f}]}\cdot\underline{u} + \rho D_t U + \nabla\cdot\underline{H} = h + \mathrm{tr}[\underline{\underline{T}}^T\cdot\nabla\underline{u}]$$

We thus obtain by removal of the momentum eqn

$$\boxed{\rho D_t U + \nabla\cdot\underline{H} = h + \mathrm{tr}[\underline{\underline{T}}^T\cdot\nabla\underline{u}]}$$

The internal energy equation

in terms of components w.r.t. some Cartesian axis system $\hat{x}_1, \hat{x}_2, \hat{x}_3$ this may be written

$$\boxed{\begin{aligned} &\rho D_t U + \partial_j H_j = h + T_{ij}\, \partial_i u_j \qquad \text{or} \\ &\rho(\partial_t U + u_j \partial_j U) + \partial_j H_j = h + T_{ij}\, \partial_i u_j \end{aligned}}$$

Note: in the foregoing discussion we have <u>nowhere</u> utilized the symmetry of $\underline{\underline{T}}$. The above eqn is valid for the general case of a medium in which there may $\exists$ intrinsic angular momentum $\underline{l}$, body torques $\underline{m}$, and couple-stresses $\underline{\underline{M}}$.

21. <u>The Internal energy equation; the case $\underline{\underline{T}} = \underline{\underline{T}}^T$</u>

If $\underline{\underline{T}}$ is symmetric we can replace $\underline{\underline{T}}^T$ by $\underline{\underline{T}}$ in the above. But we can do more. If $T_{ij} = T_{ji}$, then

$$\begin{aligned} T_{ij}\,\partial_i u_j &= \tfrac{1}{2} T_{ij}\,\partial_i u_j + \tfrac{1}{2} T_{ji}\,\partial_i u_j \\ &= \tfrac{1}{2} T_{ij}\,\partial_i u_j + \tfrac{1}{2} T_{ij}\,\partial_j u_i \\ &= T_{ij}\left[\tfrac{1}{2}(\partial_i u_j + \partial_j u_i)\right] \\ &= T_{ij}\,\epsilon_{ji} \qquad \text{where} \end{aligned}$$

$$\epsilon_{ji} = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i)$$

Now if we define a second order tensor $\underline{\underline{\epsilon}}$ by

$$\underline{\underline{\epsilon}} = \epsilon_{ij}\,\hat{x}_i\hat{x}_j \quad , \text{ then}$$

$$\begin{aligned}\underline{\underline{\epsilon}} &= \tfrac{1}{2}\,\partial_i u_j\,\hat{x}_i\hat{x}_j + \tfrac{1}{2}\,\partial_j u_i\,\hat{x}_i\hat{x}_j \\ &= \tfrac{1}{2}(\nabla \underline{u}) + \tfrac{1}{2}(\partial_j u_i\,\hat{x}_j\hat{x}_i)^T \\ &= \tfrac{1}{2}(\nabla \underline{u}) + \tfrac{1}{2}(\nabla \underline{u})^T\end{aligned}$$

$$\boxed{\underline{\underline{\epsilon}} = \tfrac{1}{2}\left[\nabla \underline{u} + (\nabla \underline{u})^T\right]}$$

The tensor field, $\nabla \underline{u}(\underline{r},t) = \partial_i u_j(\underline{r},t)\,\hat{x}_i\hat{x}_j$ is called the <u>deformation rate tensor</u>

The tensor field

$$\underline{\underline{\epsilon}}(\underline{r},t) = \tfrac{1}{2}\left[\nabla \underline{u}(\underline{r},t) + (\nabla \underline{u}(\underline{r},t))^T\right],$$

the <u>symmetric</u> part of the deformation rate tensor, is called the <u>strain rate tensor</u>

It is symmetric clearly.

Note we have <u>not</u> defined <u>strain</u> or a <u>strain tensor</u>, only the local strain <u>rate</u> tensor. In general there is no unique way to define a <u>strain</u> tensor

Continuing $\quad T_{ij}\,\epsilon_{ji} = \mathrm{tr}\,(\underline{\underline{T}}\cdot\underline{\underline{\epsilon}})$

we conclude that if $\underline{\underline{T}}$ is symmetric

$$\boxed{\rho D_t U + \nabla\cdot \underline{H} = h + \mathrm{tr}(\underline{\underline{T}}\cdot\underline{\underline{\epsilon}})}$$

The internal energy equation for the usual case $\underline{\underline{T}} = \underline{\underline{T}}^T$.

Let's convert this to the usual form of a conservation law. We can do this by adding the cont. eqn.

$$\rho\,\partial_t U + \rho\underline{u}\cdot\nabla U + \nabla\cdot\underline{H} = h + \mathrm{tr}(\underline{\underline{T}}\cdot\underline{\underline{\epsilon}})$$

add $\quad [\,\partial_t\rho + \nabla\cdot(\rho\underline{u})\,] = 0$

$$\boxed{\partial_t(\rho U) + \nabla\cdot[\rho\underline{u}U + \underline{H}] = h + \mathrm{tr}(\underline{\underline{T}}\cdot\underline{\underline{\epsilon}})}$$

This is the usual form of a conservation law. This form good for interpretation.

<u>ϕ- stuff is internal energy</u>

$\phi = \rho U$ internal energy per unit volume

$\underline{K} = \rho U\underline{u} + \underline{H}$ current density or flux density of internal energy in space.

$\rho U\underline{u} + \underline{H}$

<u>advection</u> of internal energy

diffusion of internal energy w.r.t. the material is just <u>heat</u> <u>flux</u>.

$k = h + \mathrm{tr}(\underline{\underline{T}} \cdot \underline{\underline{\epsilon}})$ rate of production of internal energy per unit vol. per second

h is obvious, internal production

the other term $\mathrm{tr}(\underline{\underline{T}} \cdot \underline{\underline{\epsilon}})$ represents a production of internal energy due to the motion of the material.

Even in frictionless (zero viscosity) materials this term need not vanish. Can represent a recoverable storage of energy as internal energy

As the simplest example, consider a fluid with a homogeneous, isotropic stress tensor, a so-called perfect fluid.

We consider a homogeneous parcel of perfect fluid.

$$\underline{\underline{T}}(\underline{r}, t) = -p(t)\, I$$

↑ no dependence on $\underline{r}$

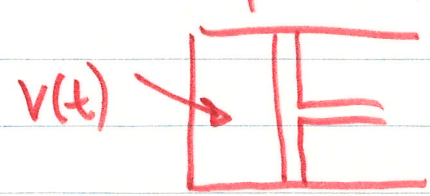
$$\underline{\underline{T}} \cdot \underline{\underline{\epsilon}} = -p\, \underline{\underline{I}} \cdot \underline{\underline{\epsilon}} = -p\, \underline{\underline{\epsilon}} \quad \text{and}$$

$$\mathrm{tr}\, \underline{\underline{\epsilon}} = \tfrac{1}{2} \mathrm{tr}\, \underline{\nabla u} + \tfrac{1}{2} \mathrm{tr}(\underline{\nabla u})^T = \mathrm{tr}(\underline{\nabla u}) = \nabla \cdot \underline{u}$$

so

$$\mathrm{tr}(\underline{\underline{T}} \cdot \underline{\underline{\epsilon}}) = -p\, \nabla \cdot \underline{u}$$

Example: a piston (in free fall)



$p(\underline{r}, t) = p(t)$ only

Now consider a volume $V(t)$ <u>moving with the material</u>.

The rate at $\mathrm{tr}\,(\underline{\underline{T}} \cdot \underline{\underline{\epsilon}})$ adds internal energy to $V(t)$ is

$$-\int_{V(t)} p(t)\, \nabla \cdot \underline{u}(\underline{r}, t)\, dV = -p \int_{V(t)} \nabla \cdot \underline{u}\, dV$$

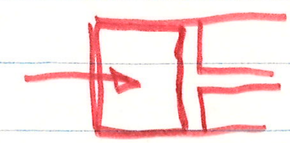
$$= -p \int_{\partial V(t)} \hat{n} \cdot \underline{u}\, dA = -p \frac{d}{dt} \int_{V(t)} dV$$

↑ by thm on differ. $\int_{V(t)}$ over moving volumes.

$$= -p \frac{d}{dt} (\text{volume of } V)$$

example: a piston

V(t)



The amount of energy added to the material in $V(t)$ during time δt is

$$-p \frac{d}{dt} (\text{volume of } V)\, \delta t = -p\, \delta (\text{volume of } V)$$

$$= -p(t)\, \delta V(t)$$

This way of storing internal energy in a gas by compressing it is familiar from elementary thermodynamics

Now consider a volume $V(t)$ moving with the material

the internal energy eqn is

$$\rho D_t U + \nabla\cdot\underline{H} = h + \mathrm{tr}(\underline{\underline{T}}\cdot\underline{\underline{\epsilon}})$$

take h to be zero and integrate over $V(t)$

$$\int_{V(t)} \rho D_t U \, dV = \frac{d}{dt}\int_{V(t)} \rho U \, dV \quad \text{(by first homework set)}$$

$$= \frac{d}{dt}\,(\text{amt. of internal energy in } V)$$

$$\int_{V(t)} \nabla\cdot\underline{H}\, dV = \int_{\partial V(t)} \hat{n}\cdot\underline{H}\, dV$$

~~now consider the rate at which $\mathrm{tr}(\underline{\underline{T}}\cdot\underline{\underline{\epsilon}})$ adds internal energy to $V(t)$.~~

The law thus takes the familiar form. During an infin. small time δt

$$\delta U(t) = -\delta t \int_{\partial V(t)} \hat{n}\cdot\underline{H}\, dV - p(t)\,\delta V(t)$$

$$\delta U(t) = \overbrace{-\Delta Q(t)}^{\text{heat added to system}} - p(t)\,\delta V(t)$$

the familiar form of the first law for a homogeneous non-viscous fluid substance

end here 23 Feb.