Summarizing Eulerian conservation laws

1. mass (continuity) : $\partial_t \rho + \nabla \cdot (\rho \underline{u}) = 0$ or
$$D_t \rho + \rho \nabla \cdot \underline{u} = 0$$

2. momentum : $\rho D_t \underline{u} = \nabla \cdot \underline{\underline{T}} + \underline{f}$

3. angular momentum : $\underline{\underline{T}}^T = \underline{\underline{T}}$

4. internal energy : $\rho D_t \upsilon + \nabla \cdot \underline{H} = h + \underline{\underline{T}} : \underline{\underline{\varepsilon}}$

5. Clausius-Duhem : $\rho D_t s \geq \frac{h}{\theta} - \nabla \cdot \left(\underline{H} / \theta \right)$

We've also obtained the Lagrangian form of the mass conservation law.

Let $\underline{r}(\underline{x}, t)$ be the Lagrangian description of the motion.

Comoving volume $V(t)$:

$$M(t) = \int_{V(t)} \rho_E(\underline{r}, t)\, dV = \int_{V(0)} \rho_L(\underline{x}, t)\, J(\underline{x}, t)\, dV_0$$

where $J(\underline{x}, t) = \det \begin{vmatrix} \partial r_1 / \partial x_1 & \partial r_1 / \partial x_2 \\ & \text{etc.} \\ & \end{vmatrix}$

$$= M(0) = \int_{V(0)} \rho_L(\underline{x}, 0)\, dV_0$$

$$\boxed{\rho_L(\underline{x}, t)\, J(\underline{x}, t) = \rho_L(\underline{x}, 0) \equiv \rho_0(\underline{x})}$$

initial density of particle $\underline{x}$

To obtain the Lagrangian version of the momentum law we rewrite the Eulerian form slightly

Let $\quad \underline{f}_E(\underline{r}, t) = \rho_E(\underline{r}, t)\, \underline{g}_E(\underline{r}, t)$

body force per unit mass.

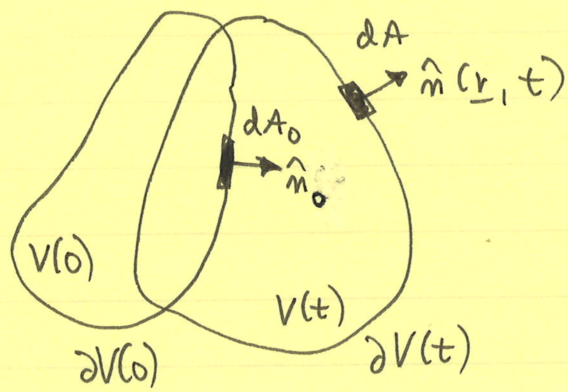
$$\rho_E D_t \underline{u}_E = \nabla \cdot \underline{\underline{T}}_E + \rho_E \underline{g}_E$$

$\underline{\underline{T}}_E(\underline{r}, t) = \underline{\underline{T}}_E^T(\underline{r}, t)$ is called the <u>Cauchy stress</u>

Regarded as a linear operator:

$$\underbrace{d\underline{F}(\underline{r}, t)}_{\text{force acting on deformed area element}} = \underbrace{\hat{n}(\underline{r}, t)\, dA}_{\text{deformed area element}} \cdot \underline{\underline{T}}(\underline{r}, t)$$

<u>Define</u> first Piola – Kirchoff stress tensor by



Note : we do <u>not</u> define

$$\underline{\underline{T}}_L(\underline{x},t) = \underline{\underline{T}}_E(\underline{r}(\underline{x},t),t)$$

"as usual"

$$\cancel{d\underline{F} = \hat{n}(\underline{r},t)\, dA \cdot \underline{\underline{T}}(\underline{r},t)}$$

$$\cancel{\hat{n}_0\, dA_0 \cdot \underline{\underline{T}}}$$

$$\underline{dF} = \hat{n}\, dA \cdot \underline{\underline{T}} = \hat{n}_0\, dA_0 \cdot \overset{\bullet}{\underline{\underline{T}}}_L, \quad \text{or written out}$$

$$\cancel{\hat{n}(\underline{r}(\underline{x},t),t) \cdot \underline{\underline{T}}(\underline{r}(\underline{x},t),t)}$$

$$\boxed{\begin{aligned} &\hat{n}(\underline{r}(\underline{x},t),t)\, dA(\underline{r}(\underline{x},t),t) \cdot \underline{\underline{T}}_E(\underline{r}(\underline{x},t),t) \\ &\qquad = \hat{n}_0(\underline{x})\, dA_0(\underline{x}) \cdot \overset{\bullet}{\underline{\underline{T}}}_L(\underline{x},t) \end{aligned}}$$

Viewed as a linear operator, $\overset{\bullet}{\underline{\underline{T}}}_L$ gives the incremental surface force acting across the <u>deformed</u> surface element in terms of the initial $\hat{n}_0\, dA_0$ of that element.

$$\underbrace{\underline{dF}(\text{on } \hat{n}\, dA)}_{\substack{\text{result is} \\ \text{force on} \\ \text{deformed} \\ \text{element}}} = \underbrace{\hat{n}_0\, dA_0}_{\substack{\text{initial} \\ \text{element goes in slot}}} \cdot \overset{\bullet}{\underline{\underline{T}}}_L$$

For this reason, Malvern refers to $\underline{\underline{T}}_L$ as a "two-point" tensor.

Go back to $\rho_E D_t \underline{u}_E = \nabla \cdot \underline{\underline{T}}_E + \rho_E g_E$

Integrate over a comoving volume $V(t)$:

$$\int_{V(t)} \rho_E D_t \underline{u}_E \, dV = \int_{V(t)} \nabla \cdot \underline{\underline{T}}_E \, dV + \int_{V(t)} \rho_E \underline{g}_E \, dV$$

$$= \int_{\partial V(t)} \hat{n}(\underline{r}, t) \cdot \underline{\underline{T}}_E(\underline{r}, t) \, dA + \int_{V(t)} \rho_E \underline{g}_E \, dV$$

Transform variable of integration from $\underline{r}$ to $\underline{x}$. Domain $V(t) \rightarrow V(0)$.

$$\int_{V(t)} \rho_E D_t \underline{u}_E \, dV = \int_{V(0)} \rho_E D_t \underline{u}_L \, J \, dV_0$$

$$= \int_{V(0)} \rho_0 D_t \underline{u}_L \, dV_0 = \int_{V(0)} \rho_0 D_t^2 \underline{r}(\underline{x}, t) \, dV_0$$

Likewise: $\displaystyle\int_{V(t)} \rho_E \underline{g}_E \, dV = \int_{V(0)} \rho_0 g_L(\underline{x}, t) \, dV_0$

(This is why we use $\underline{g}$ instead of $\underline{f}$)

Finally: $\displaystyle\int_{\partial V(t)} \hat{n} \cdot \underline{\underline{T}}_E \, dA = \int_{\partial V(0)} \hat{n}_0 \cdot \underline{\underline{T}}_L \, dA_0$

$$= \int_{V(0)} \nabla_{\underline{x}} \cdot \underline{\underline{T}}_L(\underline{x}, t)\, dV$$

(This is why the PK stress is defined as it is)

Now $V(0)$ is arbitrary, so:

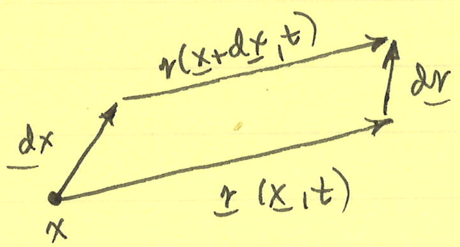
$$\boxed{\rho_0(\underline{x})\, D_t^2 \underline{r}(\underline{x}, t) = \nabla_{\underline{x}} \cdot \underline{\underline{T}}_L(\underline{x}, t) + \rho_0(x)\, \underline{g}_L(x, t)}$$

Note appearance of $\rho_0(\underline{x})$, not $\rho_L(\underline{x}, t)$.
Note D_t^2 is not nonlinear as in $\rho_0 D_t \underline{u}$

How is $\underline{\underline{T}}_L(\underline{x}, t)$ related to $\underline{\underline{T}}_E(\underline{r}, t)$?

We must find out how $\hat{n}\, dA$ is related to $\hat{n}_0\, dA_0$. This is a purely kinematic question.

Given Lagrangian description $\underline{r}(\underline{x}, t)$.
Consider a small ball of material around $\underline{x}$ at $t=0$.



How is $\underline{dr}$ related to $\underline{dx}$?

$$\underline{dr} = \underline{r}(\underline{x} + \underline{dx}, t) - \underline{r}(\underline{x}, t)$$
$$= \underline{r}(\underline{x}, t) + \underline{dx} \cdot \nabla_{\underline{x}} \underline{r}(\underline{x}, t) + \cdots - \underline{r}(\underline{x}, t)$$

$$\underline{dr} = \underline{dx} \cdot \nabla_{\underline{x}} \underline{r}(\underline{x}, t)$$

Customary to define deformation gradient $\underline{\underline{F}}(\underline{x}, t)$ by

$$\boxed{\underline{\underline{F}}^T(\underline{x}, t) = \nabla_{\underline{x}} \underline{r}(\underline{x}, t) \quad \text{so that} \quad \underline{dr} = \underline{\underline{F}} \cdot \underline{dx} = \underline{dx} \cdot \underline{\underline{F}}^T}$$

$\underline{\underline{F}}(\underline{x}, t)$ is the linear operator that gives $\underline{dr}$ in terms of $\underline{dx}$

Note that $\boxed{J(\underline{x}, t) = \det \underline{\underline{F}}(\underline{x}, t)}$

By definition $\underline{r}(\underline{x}, 0) = \underline{x}$ $\quad \underline{\underline{F}}(\underline{x}, 0) = \underline{\underline{I}}$, $J(\underline{x}, 0) = 1$

If the deformation is continuous, then $J(\underline{x}, t) > 0$ for all t.

$x(\underline{r}, t)$ exists

As a result $\underline{\underline{F}}$ is always invertible.

The inverse of a tensor $\underline{\underline{F}}$ (if it $\exists$) is defined by

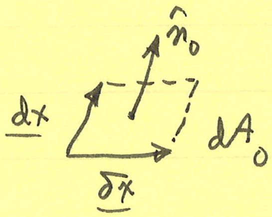
$$\underline{u} = \underline{\underline{F}} \cdot \underline{v} \iff \underline{v} = \underline{\underline{F}}^{-1} \cdot \underline{u}$$

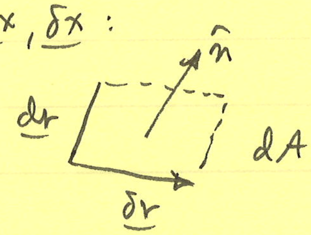
In this case $\boxed{\underline{dx} = \underline{\underline{F}}^{-1} \cdot \underline{dr} = \underline{dr} \cdot \underline{\underline{F}}^{-T}}$

$$\boxed{\underline{\underline{F}}^{-T}(\underline{r}, t) = \nabla_{\underline{r}} \underline{x}(\underline{r}, t)}$$

Now to relate $\hat{n}\, dA$ to $\hat{n}_0\, dA_0$

Let the initial dA_0 be a parallelogram bounded by $\underline{dx}, \underline{\delta x}$:





Then $$\hat{n}_0\, dA_0 = \underline{dx} \times \underline{\delta x}$$

$$\hat{n}\, dA = \underline{dr} \times \underline{\delta r}$$

i.e. $$n_{0i}\, dA_0 = \varepsilon_{ijk}\, dx_j\, \delta x_k$$

$$n_i\, dA = \varepsilon_{ijk}\, dr_j\, \delta x_k \qquad [\; n_r\, dA = \varepsilon_{rpq}\, dr_r\, \delta r_q \;]$$

But $$dx_j = F^{-1}_{jp}\, dr_p$$

$$\delta x_k = F^{-1}_{\underline{kq}}\, \delta r_{\underline{q}}$$

So $$n_{0i}\, dA_0 = \varepsilon_{ijk}\, F^{-1}_{jp}\, F^{-1}_{\underline{kq}}\, dr_p\, \delta r_{\underline{q}}$$

Multiply both sides by F^{-1}_{ir} :

$$F^{-1}_{ir}\, n_{0i}\, dA_0 = \varepsilon_{ijk}\, F^{-1}_{ir}\, F^{-1}_{jp}\, F^{-1}_{\underline{kq}}\, dr_p\, \delta r_{\underline{q}}$$

We use a general formula for the determinant:

$$\varepsilon_{rpq}\left(\det M\right) = \varepsilon_{ijk}\, M_{ir}\, M_{jp}\, M_{kq}$$

$$\varepsilon_{rpq}\left(\det F^{-1}\right) = \varepsilon_{rpq}\, J^{-1} = \varepsilon_{ijk}\, F^{-1}_{ir}\, F^{-1}_{jp}\, F^{-1}_{kq}$$

$$F^{-1}_{ir}\, n_{0i}\, dA_0 = J^{-1}\varepsilon_{rpq}\, dr_p\, \delta r_q$$

$$= J^{-1}\, n_r\, dA$$

$$n_r\, dA = J\, n_{0i}\, dA_0\, F^{-1}_{ir}$$ or

$$\boxed{\hat{n}\, dA = J\, \hat{n}_0 \cdot \underline{\underline{F}}^{-1}\, dA_0}$$

$$\hat{n}\cdot \underline{\underline{T}}_E\, dA = J\,\hat{n}_0 \cdot \underline{\underline{F}}^{-1}\cdot \underline{\underline{T}}_E\, dA_0 = \hat{n}_0 \cdot \underline{\underline{T}}_L\, dA_0$$

$$\boxed{\underline{\underline{T}}_L = J\, \underline{\underline{F}}^{-1}\cdot \underline{\underline{T}}_E}$$

$$\underline{\underline{T}}_L(\underline{x},t) = J(\underline{x},t)\, \underline{\underline{F}}^{-1}(\underline{r}(\underline{x},t),t)\cdot \underline{\underline{T}}_E(\underline{r}(\underline{x},t),t)$$

rather than $\underline{\underline{T}}_L(\underline{x},t) = \underline{\underline{T}}_E(\underline{r}(\underline{x},t),t)$

The inverse relationship is

$$\boxed{\underline{\underline{T}}_E = J^{-1}\, \underline{\underline{F}}\cdot \underline{\underline{T}}_L}$$

Note that $\underline{\underline{T}}_L^T = J\, \underline{\underline{T}}_E^T \cdot \underline{\underline{F}}^{-T} = J\, \underline{\underline{T}}_E \cdot \underline{\underline{F}}^{-T}$

$$\boxed{\underline{\underline{T}}_L^T \neq \underline{\underline{T}}_L}$$

The (first) Piola–Kirchoff stress tensor is not symmetric even though $\underline{\underline{T}}_E$ is.

$\boxed{\mathbf{T}_L^T = \mathbf{F} \cdot \mathbf{T}_L \cdot \mathbf{F}^{-T}}$ can be regarded as ang. mom. cons. law

Now consider the conservation of energy law

$$\rho_E D_t u_E + \nabla \cdot \underline{H}_E = h_E + \underline{\underline{T}}_E : \underline{\underline{\varepsilon}}$$

Write $h_E = \rho_E \gamma_E$ $\quad \gamma_E(r,t)$ = rate of external heating / unit mass

Integrate over $V(t)$ as before.

Define $\underline{H}_L(\underline{x},t)$ by

$$\boxed{\hat{n}\, dA \cdot \underline{H}_E = \hat{n}_0\, dA_0 \cdot \underline{H}_L}$$

$$\boxed{\begin{aligned} \underline{H}_L &= J\, \underline{\underline{F}}^{-1} \cdot \underline{H}_E \\ \underline{H}_E &= J^{-1}\, \underline{\underline{F}} \cdot \underline{H}_L \end{aligned}}$$

Viewed as a linear functional $\underline{H}_L(\underline{x},t)$ gives the heat flux through a deformed patch in terms of the corresponding undeformed patch $\hat{n}_0\, dA_0$.

$$\int_{V(0)} [\rho_0 D_t u_L - \nabla_x \cdot \underline{H}_L - \rho_0 \gamma_L]\, dV_0 = \underbrace{\int_{V(t)} \underline{\underline{T}}_E : \underline{\underline{\varepsilon}}\, dV}_{\text{need to transform this.}}$$

Consider $D_t \underline{\underline{F}}^T = D_t \nabla_{\underline{x}}\, \underline{r}(\underline{x},t) = \nabla_{\underline{x}}\, D_t \underline{r}(\underline{x},t)$

$$= \nabla_{\underline{x}}\, \underline{u}_E(\underline{r}(\underline{x},t),t)$$

$$= \nabla_x r(x,t) \cdot \nabla_r u_E(r(x,t),t)$$

$$= \nabla_r u_E(\underline{r}(\underline{x},t),t)) \cdot \underline{\underline{F}}(\underline{x},t)$$

$$\nabla_r u_E(r(x,t),t) = D_t F^T(x,t) \cdot F^{-1}(x,t)$$

$$\boxed{D_t \underline{\underline{F}}^T(\underline{x},t) = \nabla_r u_E(\underline{r}(\underline{x},t),t) \cdot \underline{\underline{F}}(\underline{x},t)}$$

$$\underline{\underline{T}}_E : \underline{\underline{\varepsilon}} = T^E_{ij}\, \partial_j u_i = T^E_{ij} (D_t F^T_{jk}) F^{-1}_{ki}$$

$$\int_{V(t)} \underline{\underline{T}}_E : \partial_j u_i\, dV = \int_{V(0)} J F^{-1}_{ki} T^E_{ij}\, D_t F^T_{jk}\, dV_0$$

$$= \int_{V(0)} T^L_{kj}\, D_t F_{kj}\, dV_0$$

$$= \int_{V(0)} \underline{\underline{T}}_L \cdot\cdot D_t \underline{\underline{F}}\, dV_0 = \int_{V(0)} \underline{\underline{T}}_L^T : D_t \underline{\underline{F}}\, dV_0$$

$$\boxed{\begin{aligned}\rho_0(\underline{x})\, D_t\, u_L(\underline{x},t) + \nabla_{\underline{x}}\cdot \underline{H}_L(\underline{x},t) &= \rho_0(\underline{x})\,\gamma_L(\underline{x},t) \\ &\quad + \underline{\underline{T}}_L(\underline{x},t) \bullet\bullet D_t \underline{\underline{F}}(\underline{x},t)\end{aligned}}$$

By similar reasoning the Lagrangian form of the Clausius – Duhem ≠ is:

$$\boxed{\begin{aligned}\rho_0(\underline{x})\, D_t\, S_L(\underline{x},t) &\geq - \nabla_{\underline{x}}\cdot\left[\theta_L^{-1}(\underline{x},t)\,\underline{H}_L(\underline{x},t)\right] \\ &\quad + \rho_0(\underline{x})\,\theta_L(\underline{x},t)^{-1}\,\gamma_L(\underline{x},t)\end{aligned}}$$

Some do not regard $\rho_0 D_t^2 \underline{r} = \nabla_{\underline{x}}\cdot \underline{\underline{T}}_L + \rho_0 \underline{g}_L$ as <u>strictly</u> <u>Lagrangian</u> because of the two-point nature of $\underline{\underline{T}}_L(\underline{x},t)$

$\hat{n}_0\, dA_0 \cdot \underline{\underline{T}}_L(\underline{x},t) = \underline{dF}$ is the force on the deformed patch $\hat{n}\, dA$

Note that $\hat{n}_0 \cdot \underline{\underline{T}}_L$ is the force on the deformed patch measured <u>per unit undeformed area</u>.

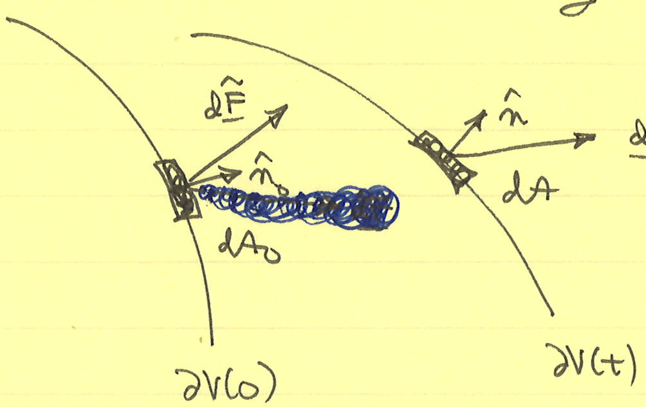
This has led to the introduction of the second Piola Kirchoff stress tensor, defined by

$$\boxed{\hat{n}_0\, dA_0 \cdot \underline{\underline{\tilde{T}}}_L = d\underline{\tilde{F}} = \underline{\underline{F}}^{-1} \cdot d\underline{F} = \underline{\underline{F}}^{-1} \cdot (\hat{n}\, dA \cdot \underline{\underline{T}}_E)}$$

The force $d\underline{\tilde{F}}$ is related to $d\underline{F}$ in the same way that an undeformed vector differential $d\underline{x}$ is related to the deformed $d\underline{r}$:

$$d\underline{x} = \underline{\underline{F}}^{-1} \cdot d\underline{r}$$

The picture is (note that $\hat{n}_0$ is rotated to $\hat{n}$ by same amount $d\underline{\tilde{F}}$ is rotated to $d\underline{F}$)



$d\underline{\tilde{F}}$ is attached to $\underline{x}$ whereas $d\underline{F}$ is attached to $\underline{r}(\underline{x}, t)$

How is $\underline{\underline{\tilde{T}}}_L$ related to $\underline{\underline{T}}_L$?

$$\hat{n}_0\, dA_0 \cdot \underline{\underline{\tilde{T}}}_L = \underline{\underline{F}}^{-1} \cdot \left[J\, \hat{n}_0\, dA_0 \cdot \underline{\underline{F}}^{-1} \cdot \underline{\underline{T}}_E \right]$$

$$= J\, \hat{n}_0\, dA_0 \cdot \underline{\underline{F}}^{-1} \cdot \underline{\underline{T}}_E \cdot \underline{\underline{F}}^{-T}$$

$$\boxed{\tilde{\underline{\underline{T}}}_L = J\, \underline{\underline{F}}^{-1} \cdot \underline{\underline{T}}_E \cdot \underline{\underline{F}}^{-T}}$$

Note that $\tilde{\underline{\underline{T}}}_L^T = J\, \underline{\underline{F}}^{-1} \cdot \underline{\underline{T}}_E^T \cdot \underline{\underline{F}}^{-T} = \tilde{\underline{\underline{T}}}_L$

$\boxed{\tilde{\underline{\underline{T}}}_L \text{ is symmetric}}$

$$\boxed{\underline{\underline{T}}_E = J^{-1}\, \underline{\underline{F}} \cdot \tilde{\underline{\underline{T}}}_L \cdot \underline{\underline{F}}^T}$$

also: $\tilde{\underline{\underline{T}}}_L = \underline{\underline{F}}^{-1} \cdot \underline{\underline{T}}_L^T$

Also: $\boxed{\tilde{\underline{\underline{T}}}_L = \underline{\underline{T}}_L \cdot \underline{\underline{F}}^{-T}}$ $\boxed{\underline{\underline{T}}_L = \tilde{\underline{\underline{T}}}_L \cdot \underline{\underline{F}}^T}$

also: $\underline{\underline{T}}_L = \underline{\underline{F}} \cdot \tilde{\underline{\underline{T}}}_L^T$

Using the latter form we can express the momentum equation in terms of $\tilde{\underline{\underline{T}}}_L$

$$\boxed{\rho_0 D_t^2 \underline{x} = \nabla_{\underline{X}} \cdot \left[\tilde{\underline{\underline{T}}}_L \cdot \underline{\underline{F}}^T \right] + \rho_0 \underline{g}_L}$$

~~We also define~~ $\cancel{\tilde{\underline{H}}_L(\underline{X}, t)}$ ~~by~~ $\cancel{\tilde{\underline{H}}_L = \underline{\underline{F}}^{-1} \cdot \underline{H}_L}$

$\cancel{\underline{H}_L = \underline{\underline{F}} \cdot \tilde{\underline{H}}_L}$

Malvern page 232 shows that

$$\int_{V(t)} \underline{\underline{T}}_E : \underline{\underline{\varepsilon}}\, dV = \int_{V(0)} \tilde{\underline{\underline{T}}}_L : D_t \underline{\underline{E}}\, dV \quad \text{where}$$

$$\boxed{\underline{\underline{E}}(\underline{x},t) \equiv \frac{1}{2}\left[\underline{\underline{F}}^{T}\cdot\underline{\underline{F}} - \underline{\underline{I}}\right] \quad \text{is the (finite) strain tensor}}$$

Thus the strictly Lagrangian version of the internal energy equation is:

$$\boxed{\rho_0 D_t u_L + \nabla_X \cdot \underline{H}_L = \rho_0 \gamma_L + \tilde{\underline{\underline{T}}}_L : D_t E}$$

And, finally, the CD ≠ :

$$\boxed{\rho_0 D_t S_L \geq -\nabla_X \cdot \left[\theta_L^{-1} \underline{H}_L\right] + \rho_0 \theta_L^{-1} \gamma_L}$$