Rigid Heat Conducting Bodies:

The simplest kind of macroscopic object is a <u>rigid body</u>.

Rigid bodies are definitely objects which are worth study.

By definition, crudely speaking, a <u>rigid body</u> is a continuum in which each macroparticle never changes its <u>relative</u> position w.r.t. every other macroparticle. In other words, the <u>distance</u> between any two points always remains constant.

> This is the <u>constitutive definition</u> of a <u>rigid body</u>: the distance between <u>any</u> two macroparticles $\underline{x}_1$ and $\underline{x}_2$ is always constant.

> In slightly more mathematical language A rigid body, by definition, is a continuum, the Lagrangian description of whose motion can always be written in the form
>
> $$\underline{r}(\underline{x},t) = \underline{R}(t) + \underline{\underline{Q}}(t) \cdot \underline{x}$$

where $\underline{\underline{Q}}(t)$ is , for any fixed t, an orthogonal tensor and where , for any fixed t, $\underline{R}(t)$ is a vector such that

(i) $\underline{R}(0) = \underline{0}$

(ii) $\underline{\underline{Q}}(0) = \underline{\underline{I}}$

(iii) both $\underline{R}(t)$ and $\underline{\underline{Q}}(t)$ are twice continuously differentiable

Remark: $\underline{\underline{Q}}(t)$ is always proper since $\det \underline{\underline{Q}}(t)$ depends cont. on t and hence must be a constant, as it is always either ± 1. At $t = 0$ it is $+1$, hence it is always $+1$.

Remark: Such a motion clearly satisfies the continuity equation iff $\rho_L(\underline{x}, t)$ is independent of t.

The Lagrangian form of the continuity equation is

$$\rho_L(\underline{x}, t) \left| J\underline{r}(\underline{x}, t) \right| = \rho_L(\underline{x}, 0)$$

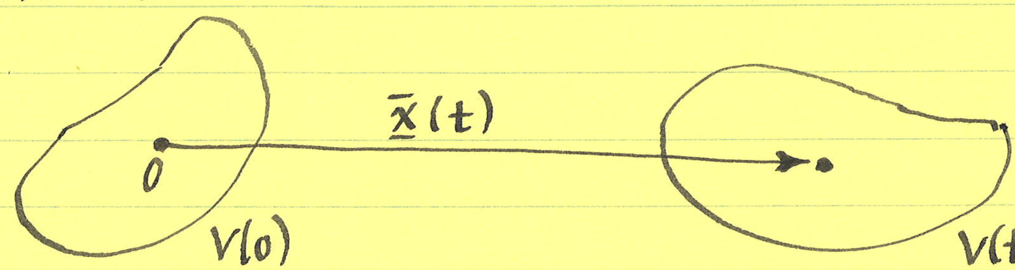
but
$$\begin{aligned} J\underline{r}(\underline{x}, t) &= J\left[\underline{R}(t) + \underline{\underline{Q}}(t) \cdot \underline{x} \right] \\ &= \det \underline{\underline{Q}}(t) \\ &= 1 \end{aligned}$$

q.e.d.

Hence $\rho_L(\underline{x},t) = \rho_L(\underline{x},0)$

Consider now the position of the center of mass of the rigid body.

At $t=0$ At $t=t$



Say the body has total mass M.

The shape of the body remains fixed, but it is free to rotate and translate

The definition of the center of mass $\bar{\underline{x}}(t)$ at any time t is

$$M\,\bar{\underline{x}}(t) \equiv \int_{V(t)} \rho_E(\underline{r},t)\,\underline{r}\,dV$$

for definiteness, we choose 0 the origin of our CAS to coincide with $\bar{\underline{x}}(0)$

Transform the domain of integration to the initial volume $V(t)$

$$\bar{\underline{x}}(t) = \int_{V(0)} \rho_L(\underline{x},0)\left[\underline{R}(t) + \underline{\underline{Q}}(t)\cdot\underline{x}\right]dV$$

$$= \left[\int_{V(0)} \rho_L(\underline{x},0)\,dV\right] R(t) + \underline{\underline{Q}}(t)\cdot\int_{V(0)} \rho_L(\underline{x},0)\,\underline{x}\,dV$$

But $\displaystyle\int_{V(0)} \rho_L(\underline{x},0)\,dV = M$ and

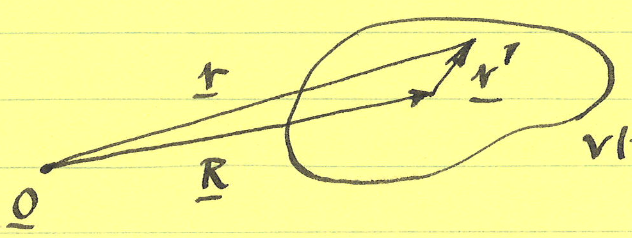
$$\int_{V(0)} \rho_L(\underline{x},0)\,\underline{x}\,dV = \bar{\underline{x}}(0) = \underline{0}$$

Hence $\bar{\underline{x}}(t) = \underline{R}(t)$

> We have shown that in the most general Lagrangian description of the motion of a rigid body $\underline{r}(\underline{x},t) = \underline{R}(t) + \underline{\underline{Q}}(t)\cdot\underline{x}$, the vector $\underline{R}(t)$ is the net translation of the center of mass of the body away from the origin $\underline{R}(0) = \underline{0}$. The remaining term $\underline{\underline{Q}}(t)\cdot\underline{x}$ is the net rotation about the c.o.m.

Let us denote by $\underline{r}'(\underline{x},t)$ the posn at time t w.r.t. the c.o.m. of the particle $\underline{x}$

$$\underline{r}'(\underline{x},t) = \underline{r}(\underline{x},t) - \underline{R}(t) \qquad (*)$$



For a rigid body $\underline{r}'(\underline{x},t) = \underline{\underline{Q}}(t)\cdot\underline{x}$, but for any finite continuum we may always write $(*)$

Let us now consider the application of the conservation laws for an arbitrary continuum.

1. the continuity equation: as we have seen this is trivially satisfied

2. the momentum equation: for a rigid body, the most convenient formulation of the balance law for momentum is the original integral formulation.

$$\frac{d}{dt}\left[\int_{V(t)} \rho_E \underline{u}_E \, dV \right] = \underline{\mathcal{F}}(t)$$

(resultant of all external forces)

We transform the integral on the l.h.s. inside the brackets to a new domain of integration $V(0)$.

note: this step does not depend on the rigid body prescription

$$\int_{V(t)} \rho_E \underline{u}_E \, dV = \int_{V(0)} \rho_L(\underline{x}, 0) \, D_t \underline{r}(\underline{x}, t) \, dV$$

$$= \int_{V(0)} \rho_L(\underline{x}, 0) \left[\dot{\underline{R}}(t) + D_t \underline{r}'(\underline{x}, t) \right] dV$$

$$= M\dot{\underline{R}}(t) + \frac{d}{dt}\int_{V(0)} \rho_L(\underline{x},0)\,\underline{r}'(\underline{x},t)\,dV$$

now transform back to $V(t)$, taking $\underline{R}(t)$ as origin

$$= M\dot{\underline{R}}(t) + \frac{d}{dt}\left[\int_{V(t)} \rho_E(\underline{r}',t)\,\underline{r}'\,dV'\right]$$

but this is the defn of the posn of the c.o.m. in c.o.m. coordinates, i.e. <u>zero</u>

$$= M\dot{\underline{R}}(t)$$

Hence the continuum balance law for momentum necessarily implies, for any continuum (we did <u>not</u> use the rigid body assumption in the above)

$$\boxed{M\ddot{\underline{R}}(t) = \underline{\mathcal{F}}(t)}$$

where M = total mass
$\underline{R}(t)$ = posn at time t of c.o.m.
$\underline{\mathcal{F}}(t)$ = net <u>external</u> force acting at time t. (assumed to be prescribed)

Given $\underline{R}(0) = \underline{0}$ and $\dot{\underline{R}}(0)$, we can always determine $\underline{R}(t)$

Thus we can use the momentum balance law to immediately solve half of the general problem of the mechanics of a rigid body.

$\underline{R}(t)$ can be determined given $\underline{F}(t)$ and given the initial conditions.

The complete specification of the motion of a rigid body requires only the specification of $\underline{R}(t)$ (3 d.f.) and $\underline{\alpha}(t)$ (3 more d.f.).

3. the angular momentum eqn : once again the most convenient formulation is the original integral formulation.

$$\frac{d}{dt}\left[\int_{V(t)} \underline{r} \times \rho_E \underline{u}_E \, dV\right] = \underline{M}(t)$$

(the net external torque)

Once again we transform to $V(0)$

$$\int_{V(t)} \underline{r} \times \rho_E \underline{u}_E \, dV = \int_{V(0)} \rho_L(\underline{x},0)\, \underline{r}(\underline{x},t) \times D_t \underline{r}(\underline{x},t) \, dV$$

$$= \int_{V(0)} \rho_L(\underline{x},0) \left[\underline{R}(t) + \underline{r}'(\underline{x},t) \right] \times \left[\dot{\underline{R}}(t) + D_t \underline{r}'(\underline{x},t) \right] dV$$

$$= \underline{R}(t) \times \underline{M}\dot{R}(t) + \underline{R}(t) \times \frac{d}{dt} \int_{V(0)} \rho_L(\underline{x},0)\, \underline{r}'(\underline{x},t)\, dV$$

$$+ \dot{\underline{R}}(t) \times \int_{V(0)} \rho_L(\underline{x},0)\, \underline{r}'(\underline{x},t)\, dV$$

$$+ \int_{V(0)} \rho_L(\underline{x},0)\, \underline{r}'(\underline{x},t) \times D_t \underline{r}'(\underline{x},t)\, dV$$

now transform back the last three terms, using $\underline{R}(t)$ as origin

$$= \underline{R}(t) \times M\dot{\underline{R}}(t) + \int_{V(t)} \rho_E(\underline{r}',t)\, \underline{r}' \times \underline{u}_E(\underline{r}',t)\, dV'$$

Summarizing $\quad \underline{L}(t) = \underline{R}(t) \times \underline{P}(t) + \underline{L}'(t)$

where

$$\underline{L}(t) = \int_{V(t)} \rho_E(\underline{r},t)\, \underline{r} \times \underline{u}_E(\underline{r},t)\, dV$$

is the angular momentum w.r.t. a fixed (in space) origin $\underline{O}$

$\underline{P}(t) = M\dot{\underline{R}}(t)$ is the <u>net momentum</u>

$$\underline{L}'(t) = \int_{V(t)} \rho_E(\underline{r}',t)\, \underline{r}' \times \underline{u}_E(\underline{r}',t)\, dV'$$

is the "intrinsic" angular momentum w.r.t. an origin always located at the c.o.m. of the body

We have *not* used the rigid body prescription in the above decomposition.
The net angular momentum of any finite continuum always consists of two parts: 1. its "intrinsic" angular momentum in the c.o.m. frame 2. the angular momentum $\underline{R}(t) \times \underline{P}(t)$ due to its motion as a whole.

Let us now consider the *net torque* $\underline{M}(t)$ acting on any finite continuum
There are two possible contributions: 1. body forces $\underline{f}(\underline{r},t)$ 2. surface tractions $\underline{t}(\underline{r},t,\hat{n})$

$$\underline{M}(t) = \int_{V(t)} \underline{r} \times \underline{f}\, dV + \int_{\partial V(t)} \underline{r} \times \underline{t}(\underline{r},t,\hat{n})\, dA$$

If we write $\underline{r}(\underline{x},t) = R(t) + \underline{r}'(\underline{x},t)$
then we may also decompose $\underline{M}(t)$

$$\underline{M}(t) = \underline{R}(t) \times \underline{F}(t) + \underline{M}'(t)$$

where $$\underline{F}(t) = \int_{V(t)} \underline{f}\, dV + \int_{\partial V(t)} \underline{t}(\underline{r}, t, \hat{n})\, dA$$

is the net force

and where $$\underline{M}'(t) = \int_{V(t)} \underline{r}' \times \underline{f}(\underline{r}', t)\, dV' + \int_{\partial V(t)} \underline{r} \times \underline{t}(r', t, \hat{n})\, dA$$

is the net torque exerted about the c.o.m. $\underline{R}(t)$

In words, the net torque $\underline{M}(t)$ exerted on any finite continuum consists of two parts: 1. the torque exerted about the c.o.m. $\underline{R}(t)$ and 2. a term $\underline{R}(t) \times \underline{F}(t)$ where $\underline{F}(t)$ is the net force.

Now we go back to the original formulation of the angular momentum balance

$$\frac{d}{dt} \underline{L}(t) = \underline{M}(t)$$

$$\frac{d}{dt} \left[\underline{L}'(t) + \underline{R}(t) \times M \dot{\underline{R}}(t) \right] = \underline{M}'(t) + \underline{R}(t) \times \underline{F}(t)$$

zero by linear mom. eqn.

$$\frac{d}{dt}\underline{L}'(t) = \underline{M}'(t) + \underline{R}(t) \times \left[\underline{F}(t) - M\ddot{\underline{R}}(t) \right]$$

$$\boxed{\text{Hence} \quad \frac{d}{dt}\underline{L}'(t) = \underline{M}'(t) \qquad (\star\star)}$$

This is the balance law for the "intrinsic" part of the angular momentum. If we measure both angular momentum and the net torque w.r.t. the c.o.m., the usual angular momentum equation is valid.

Note: $(\star\star)$ is valid for any finite continuum, need not be a rigid body.

For a <u>rigid body</u> $\star\star$ has a special significance since it allows a complete solution of the problem of the motion. To see this, we must introduce another way of describing the motion (either we use Euler angles or we use the concept of the instantaneous angular velocity). Lets come back to this later.

For now let us summarize our accomplishments

A *rigid body* is defined *kinematically*. The *mechanical motion* of a rigid body is completely known if we are given $\underline{R}(t)$ the posn of the c.o.m. and $\underline{\underline{Q}}(t)$, the orthogonal operator describing the rotation about the c.o.m.

We have seen that the equations of motion ~~[illegible]~~ allow a similar separation. We can solve separately for $\underline{R}(t)$, the motion of the c.o.m. and $\underline{\underline{Q}}(t)$, the rotation about the c.o.m.

To determine $\underline{R}(t)$, we need only $\underline{F}(t)$ the net external force, M the mass of the body, and initial cond. $\underline{R}(0) = \underline{0}$ and $\dot{\underline{R}}(0)$. We then solve $M\ddot{\underline{R}}(t) = \underline{F}(t)$ for $\underline{R}(t)$.

To determine the motion *about* the center of mass $\underline{\underline{Q}}(t)$, we first compute $\underline{M}'(t)$ the net external torque about the center of mass. We will show that given $\underline{M}'(t)$, given the *inertia tensor* $\underline{\underline{C}}$ of the rigid body, and given initial cond. $\underline{\underline{Q}}(0) = \underline{\underline{I}}$ and $\dot{\underline{\underline{Q}}}(0)$, we can determine $\underline{\underline{Q}}(t)$. by solving $d/dt\ \underline{L}'(t) = \underline{M}'(t)$

This is of course the central problem of rigid body motion. We will discuss it later in some detail. For now let us assume that it can be done.

Note: the kinematical defn of a rigid body allows a complete separation of the mechanics from the thermodynamics. We can completely solve the mechanics problem with no mention of thermo.

Note: What about the stress $\underline{\underline{T}}(\underline{r},t)$ in the rigid body? We can completely solve the mechanical problem without ever having to worry about what are the stresses which maintain the rigidity of the body. The constitutive assumption is purely kinematical. We are at liberty to imagine any stresses we like, but we will exercise Ockham's razor and forget about them.

4. What about the thermodynamics of a rigid body? We have considered the first three balance laws and found that they completely determine the mechanics.

Review

For a <u>complete theory</u> (one which leads to <u>well-posed</u> mathematical problems which allow us to predict the future states of a continuum in terms of given <u>initial conditions</u>, we require <u>constitutive assumptions</u>.

These serve to define various <u>idealized types</u> of continua.

We consider first, as an example of how to go <u>from</u> the balance laws + CD≠ <u>to</u> a <u>complete theory</u>, is the simplest kind of solid material, a rigid heat conducting body

The route, in general, is

<u>Laws of physics</u>
(valid for all continua)
1. balance laws
2. Clausius – Duhem inequality

⊕

Constitutive assumptions
(define idealized types of continua

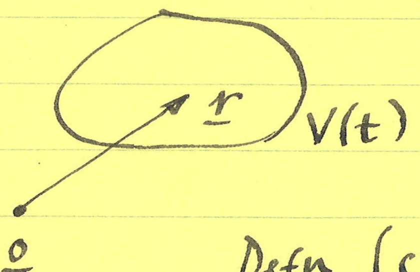
restrictions on constitutive behavior

a complete theory (can be used to solve initial value problems)

First we derived some relations involving the center of mass of any finite continuum

(note)

Given a finite continuum (occupies a finite volume $V(t)$ in space at time t)



Lagrangian descr. of motion $\underline{r}(\underline{x},t)$

Defn (c.o.m.)

$$\bar{\underline{x}}(t) \equiv \int_{V(t)} \rho_E(\underline{r},t)\,\underline{r}\,dV \Big/ \int_{V(t)} \rho_E(\underline{r},t)\,dV$$

$$= \frac{1}{M}\int_{V(t)} \underline{r}\,\rho_E(\underline{r},t)\,dV$$

where M = total mass of body

For convenience, we can choose $\underline{0} \Rightarrow \bar{\underline{x}}(0) = \underline{0}$ — this not necessary in what follows.

We found last time that

Total linear momentum (defined by $\underline{P}(t) \equiv \int_{V(t)} \rho_E(\underline{r},t)\,\underline{v}_E(\underline{r},t)\,dV$)

$$\boxed{\underline{P}(t) = M\dot{\bar{\underline{x}}}(t)}$$

$$M\ddot{\bar{\underline{x}}}(t) = \underline{F}(t)$$

↑ total external force

$$\boxed{\underline{F}(t) = \dot{\underline{P}}(t) \quad \star}$$

Can determine posn of c.o.m. if one knows $\underline{F}(t)$ and i.c.

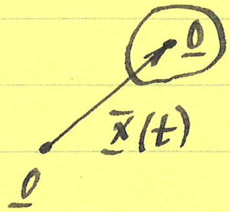
Angular momentum (total) about some arbitrary origin $\underline{0}$

defn: $\underline{L}(t) = \int_{V(t)} \underline{r} \times \rho_E(\underline{r},t)\, \underline{u}_E(\underline{r},t)\, dV$

Can be decomposed into intrinsic + orbital parts

$$\underline{L}(t) = \underline{L}'(t) + \overline{\underline{x}}(t) \times \underline{P}(t)$$

total = intrinsic + orbital



Net torque acting on $V(t)$ similarly decomposed

$$\underline{M}(t) = \underline{M}'(t) + \overline{\underline{x}}(t) \times \underline{F}(t)$$

torque about $\underline{0}$ — torque about c.o.m. — net force

Can determine $\underline{L}'(t)$ given $\underline{M}'(t)$ — Eulers eqn

$$\boxed{\frac{d}{dt}\underline{L}'(t) = \underline{M}'(t)}$$

This true for any finite continuum.

Defn: a <u>rigid body</u> is defined <u>kinematically</u>

$$\underline{r}(\underline{x},t) = \underline{R}(t) + \underline{\underline{Q}}(t) \cdot \underline{x}$$
$$\underline{R}(t) = \bar{\underline{x}}(t)$$

$\underline{\underline{Q}}(t)$ is proper orthogonal, repr. rigid rotation about c.o.m.

For a rigid body, eqns (*) and (**) may used <u>to determine the motion completely</u>

(*) allows one to determine $\underline{R}(t) = \bar{\underline{x}}(t)$ given M and initial cond.

(**) allows one to determine $\underline{\underline{Q}}(t)$ given $\underline{\underline{C}}$ (inertia tensor) and initial cond.

Note: no need to introduce <u>stresses</u> in a rigid body (to do so requires further constitutive assumptions)

Note: mechanics completely separated from thermodynamics

The first three balance laws allow a complete investigation of the <u>mechanics</u> of a rigid body.

We have not examined the <u>internal energy eqn</u>.

$$\rho D_t \mathcal{U} + \nabla \cdot \underline{H} = h + \mathrm{tr}\,(\underline{\underline{T}} : \underline{\underline{\varepsilon}})$$

For a <u>rigid body</u> $\underline{\underline{\varepsilon}}(\underline{r}, t) = 0$, there can be no <u>strain</u>. Hence no matter what $\underline{\underline{T}}(\underline{r}, t)$ is (as long as it is finite), the last term is zero.

$$\rho D_t \mathcal{U} + \nabla \cdot \underline{H} = h \qquad (*)$$

is the form of the internal energy eqn for a rigid body.

To proceed further, we must introduce further constitutive assumptions of a thermodynamical nature.

<u>Heat Conduction in a Rigid Body</u>

We will give the classical assumptions which underly the classical linear theory of heat conduction in a rigid body (Fourier 1822)

First of all, we note that if we utilize a reference frame which moves with the body so that $\underline{r}(\underline{x},t) = \underline{x}$, the eqn $(\star)$ may be thought of as a Lagrangian formulation equation.

This transformation from an Eulerian formulation to a Lagrangian formulation is easy for a rigid body because the motion of a rigid body is so simple.

From now on, we will utilize in our discussion of heat conduction a reference frame which moves with the body. Then $(\star)$ is

$$\rho(\underline{x})\, D_t \mathcal{U}(\underline{x},t) + \nabla \cdot \underline{H}(\underline{x},t) = h(\underline{x},t)$$

Now we introduce the classical constitutive assumptions.

1. Each macroparticle $\underline{x}$ is always in a condition of local thermodynamic equilibrium (l.t.e.) and furthermore at each $\underline{x}$ the equilibrium state can be completely characterized in terms of

only one state variable, the temperature $\theta(\underline{x},t)$.

i.e. we assume that if at any macroparticle $\underline{x}$, we know the temperature $\theta(\underline{x},t)$, then we assume that we can compute $u(\underline{x},t)$ and $S(\underline{x},t)$ from the energetic and caloric eqns of state as if the material were in complete thermodynamic equil. Furthermore we assume that $\theta(\underline{x},t)$ is a complete thermodynamic description, i.e. that the eqns of state are of the form

$$u(\underline{x},t) = \tilde{u}(\theta(\underline{x},t), \underline{x})$$
$$S(\underline{x},t) = \tilde{S}(\theta(\underline{x},t), \underline{x})$$

or $u = \tilde{u}(\theta, \underline{x})$ note: there may be
$S = \tilde{S}(\theta, \underline{x})$ an explicit dependence on $\underline{x}$.

The assumption that $\underline{S}(\underline{x},t)$ is a state variable combined with the first law of thermodynamics tells us that

$$\theta D_t S = D_t u$$

note use of D; we apply first law to an individual macroparticle.

$$\theta(\underline{x},t) = \left.\frac{d\tilde{u}(s(\underline{x},t),\underline{x})}{ds(\underline{x},t)}\right|_{\underline{x}}$$

Thus if one knows the eqn of state in the form $u(\underline{x},t) = \tilde{u}(s(\underline{x},t),\underline{x})$ (this is called the fundamental equation), then one has a complete description of the equilibrium properties. $\underset{\uparrow}{u = \tilde{u}(s,\underline{x})}$

the fund. eqn.

Summarizing, one assumes that $\theta(\underline{x},t)$ and $u(\underline{x},t)$ may be deduced given $s(\underline{x},t)$ as if the material were in thermo. equil. at every $\underline{x},t$, i.e. from the fund. eqn $u = \tilde{u}(s,\underline{x})$

2. We further assume the so-called Fourier's law of heat conduction, namely

$$(**) \quad \underline{H}(\underline{x},t) = -\underline{\underline{K}}(\underline{x},t)\cdot\underline{\nabla}\theta(\underline{x},t)$$

$\underline{\underline{K}}$ is called the thermal conductivity tensor. It is assumed to be a fcn only of $\theta(\underline{x},t)$, although it might be a different such function at each $\underline{x}$

i.e. we assume

$$\underline{\underline{K}}(\underline{x},t) = \tilde{\underline{\underline{K}}}(\theta(\underline{x},t), \underline{x})$$

We further assume that $\underline{\underline{K}}(\underline{x},t)$ is symmetric

$$\underline{\underline{K}}(\underline{x},t) = \underline{\underline{K}}^T(\underline{x},t)$$

This is a typical linearized constitutive assumption. In general it seems to be a very good approx. to the way real materials behave. One can measure $\tilde{\underline{\underline{K}}}(\theta, \underline{x})$ for different materials in the laboratory, (**) is a purely empirical relationship

We now substitute into * our two constitutive assumptions. better to write $D_t \mathcal{U} = \theta D_t S$

$$= \theta \left(\frac{dS}{d\theta}\right)_x D_t \theta$$

$$\rho(\underline{x}) D_t \tilde{\mathcal{U}}(\theta(\underline{x},t), \underline{x}) + \nabla \cdot \left[-\tilde{\underline{\underline{K}}}(\theta(\underline{x},t), \underline{x}) \cdot \underline{\nabla}\theta(\underline{x},t) \right] + h(\underline{x},t)$$

$$\rho(\underline{x}) \left[\left. \frac{d\tilde{\mathcal{U}}(\theta,\underline{x})}{d\theta} \right|_{\underline{x}} \right] D_t \theta(\underline{x},t) + \nabla \cdot \left[-\tilde{\underline{\underline{K}}}(\theta(\underline{x},t), \underline{x}) \cdot \underline{\nabla}\theta(\underline{x},t) + h(\underline{x},t) \right.$$

$$\text{or } \theta \left.\frac{d\tilde{s}}{d\theta}\right|_{\underline{x}}$$

The conventional notation for $\left.\dfrac{d\tilde{u}(\theta, \underline{x})}{d\theta}\right|_{x}$

is $\tilde{c}_v(\theta, \underline{x})$. It is called the

specific heat at constant volume (more specifically it is the specific heat at constant strain)

$c_v(\theta, x)$ is the increment by which the internal energy of particle $\underline{x}$ is increased if the temperature is increased slightly away from $\theta(\underline{x}, t)$

Finally

$$\boxed{\rho(\underline{x})\,\tilde{c}_v(\theta(\underline{x},t),\underline{x})\, D_t\theta(\underline{x},t) - \nabla\cdot\left[\underline{\underline{\tilde{K}}}(\theta(\underline{x},t),\underline{x})\cdot \underline{\underline{\nabla}}\theta(\underline{x},t)\right] = h(\underline{x},t)}$$

This is a single p.d.e. which can, in principle be solved for $\theta(\underline{x}, t)$ given certain initial and bdry information and given $h(\underline{x}, t)$, the rate of external heating

One must use lab data (or possibly microscopic physics, e.g. solid state physics to determine the physical properties of each material; these physical properties enter the theory thru the constitutive assumptions

$\tilde{c}_v(\theta, \underline{x})$ specific heat

$\underline{\underline{K}}(\theta, \underline{x})$ thermal conductivity tensor

see, e.g. Carslaw & Jaeger

We will not discuss how to solve the final eqn, nor what i.c. and b.c. are appropriate. Our goal was to see how, in the simplest case we go from constitutive assumptions + the balance eqns to a complete theory which we can use to predict phenomena.

5. We have not yet made use of the one remaining postulated law of physics for an arbitrary continuum, the Clausius-Duhem inequality. We use it in general to place restrictions on our constitutive assumptions.

$$\rho D_t S + \nabla \cdot (\underline{H}/\theta) - \frac{h}{\theta} \geq 0$$

In the body reference frame this may be interpreted as a Lagrangian eqn.
Now for the body under consideration, we also have an equality involving the specific entropy $S(\underline{x}, t)$

$$\rho D_t u + \nabla \cdot \underline{H} = h$$

but the assumption that S is a state variable tells us that

$$D_t u = \theta D_t S$$

$$\rho \theta D_t S + \nabla \cdot \underline{H} = h, \text{ or}$$

$$\rho D_t S + \frac{1}{\theta} \nabla \cdot \underline{H} - \frac{h}{\theta} = 0.$$

hence, the CDI reduces to $\underline{H} \cdot \nabla\left(\frac{1}{\theta}\right) \geq 0$

or

$$-\frac{1}{\theta^2} \underline{H} \cdot \underline{\nabla} \theta \geq 0$$

or $\underline{H} \cdot \underline{\nabla} \theta \leq 0$

Now we have assumed further that $\underline{H} = -\underline{\underline{K}} \cdot \nabla \theta$
Hence the CDI implies

$$\underline{\nabla\theta} \cdot \underline{\underline{K}} \cdot \underline{\nabla\theta} \geqslant 0$$

We assume that this must hold true for every conceivable process $\theta(\underline{x}, t)$
Hence we conclude that the CD ≠ requires that

$\forall \underline{v} \in R^3$, $\underline{v} \cdot \underline{\underline{K}} \cdot \underline{v} \geqslant 0$, i.e. that $\forall \underline{x}, t$ $\tilde{\underline{\underline{K}}}(\theta(\underline{x}, t), \underline{x})$ is a positive definite symmetric tensor.

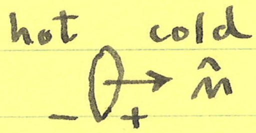
For an isotropic heat conductor (a further assumption valid for most aggregate or amorphous materials), $\underline{\underline{K}}$ is an isotropic tensor

$$\tilde{\underline{\underline{K}}}(\theta(\underline{x}, t), \underline{x}) = \underset{\uparrow}{\kappa}(\theta(\underline{x}, t), \underline{x}) \, \underline{\underline{I}}$$

called the thermal conductivity

The CD ≠ requires that $\kappa \geqslant 0$ for an isotropic material. ($\tilde{\kappa}(\theta(\underline{x}, t), \underline{x} \geqslant 0)$
This is a necessary restriction on our assumed const. relation if our postulated ideal material is to be compatible with the second law of thermo.

In this case, the restriction is that the body must conduct heat *from hot to cold*. If it went the other way spontaneously it would be a violation of the second law.



say $\underline{\nabla}\theta \cdot \hat{n} < 0$

$\underline{H} = -k\underline{\nabla}\theta \Rightarrow \underline{H} \cdot \hat{n} > 0$

if $k \geq 0$.

This is precisely the way in which the CD$\neq$ will be utilized in more general formulations. It always places necessary restrictions on postulated material properties.

We now wish to go back and consider in more detail the purely mechanical problem of rigid body motion. Without loss of generality we will consider rigid body motion with the c.o.m. *fixed*.

Our first task is to introduce some more *kinematical notation*.