# Lecture 23:

## The Motion of a Rigid Body

1. Kinematics of Rigid body motion (about center of mass)

A. definition: (the c.o.m.)

(i) a certain point O does not move with time

(ii) the distance between any two points is constant

pick O as origin, label particles in body by $\vec{x}$

Lagrangian description $\vec{r}(\vec{x}, t)$

For any fixed $t$

$$\vec{r}(\vec{x}, t) = Q_t \vec{x}$$

$Q_t$ a rigid rotation or orthogonal operator

$$\boxed{\underline{r}(\underline{x}, t) = \underline{\underline{Q}}(t) \cdot \underline{x}}$$

definition 2: A rigid body motion is a rule which assign to each time $t$ a $Q(t)$

(i) $Q(0) = I$

(ii) $Q(t)$ is twice cont. differ. (an assumption)

$Q(t)$ is always proper since $\det Q(t)$ depends cont. on $t$ and hence must be a const. as it is always $\pm 1$. At $t=0$ it is $+1$, thus always $+1$.

B. Instantaneous angular velocity

Given the Lagrangian descr., to find the Eulerian.

At time $t$, the position of particle $\vec{x}$ is

$\vec{r}(\vec{x}, t) = \overleftrightarrow{Q}(t) \cdot \vec{x}$

At $t$, its velocity is

$$\underline{v}(\underline{x}, t) = D/Dt\left[\underline{r}(\underline{x}, t)\right] = D_t\left[\underline{\underline{Q}}(t) \cdot \underline{x}\right]$$
$$= \left[D_t \underline{\underline{Q}}(t)\right] \cdot \underline{x}$$

Note $D/Dt \equiv$ partial w.r.t. $t$ holding $\underline{x}$ constant Eulerian derivative — notation might seem peculiar at this time but it is standard.

$$\underline{v}(\underline{x}, t) = D_t \underline{\underline{Q}}(t) \cdot \underline{x}$$

so in terms of its present position $\underline{r}$

$$\underline{v}(\underline{r}, t) = D_t \underline{\underline{Q}}(t) \cdot \left(\underline{\underline{Q}}^{-1}(t) \cdot \underline{r}\right)$$

$$\boxed{\begin{aligned} \underline{v}(\underline{r}, t) &= \underline{\underline{\Omega}}(t) \cdot \underline{r} \quad \text{where} \\ \underline{\underline{\Omega}}(t) &= D_t \underline{\underline{Q}}(t) \cdot \underline{\underline{Q}}^{-1}(t) \\ &= D_t \underline{\underline{Q}}(t) \cdot \underline{\underline{Q}}^{T}(t) \end{aligned}}$$

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Now $D_t \underline{\underline{Q}}^T(t) = \left[D_t \underline{\underline{Q}}(t)\right]^T$ since

$$\left[D_t \underline{\underline{Q}}(t)\right]^T = \lim_{\Delta t \to 0} \left[\frac{\underline{\underline{Q}}(t+\Delta t) - \underline{\underline{Q}}(t)}{\Delta t}\right]^T$$
$$= \lim_{\Delta t \to 0} \left[\frac{\underline{\underline{Q}}^T(t+\Delta t) - \underline{\underline{Q}}^T(t)}{\Delta t}\right]$$
$$= D_t \underline{\underline{Q}}^T(t)$$

All limits are considered to be component-wise.

Now we will show that $\underline{\underline{\Omega}}(t)$ is antisymmetric.

$$\underline{\underline{Q}}(t) \cdot \underline{\underline{Q}}^T(t) = \underline{\underline{I}} \quad \text{for all } t$$

$$D_t [\, \underline{\underline{Q}}(t) \cdot \underline{\underline{Q}}^T(t) \,] = \underline{\underline{0}}$$

$$D_t \underline{\underline{Q}}(t) \cdot \underline{\underline{Q}}^T(t) + \underline{\underline{Q}}(t) \cdot D_t \underline{\underline{Q}}^T(t) = \underline{\underline{0}}$$

$$D_t \underline{\underline{Q}}(t) \cdot \underline{\underline{Q}}^T(t) + \underline{\underline{Q}}(t) \cdot (D_t \underline{\underline{Q}}(t))^T = \underline{\underline{0}}$$

but $\underline{\underline{\Omega}}(t) = D_t \underline{\underline{Q}}(t) \cdot \underline{\underline{Q}}^T(t)$ and

$$\underline{\underline{\Omega}}^T(t) = (\underline{\underline{Q}}^T(t))^T \cdot (D_t \underline{\underline{Q}}(t))^T$$

$$= \underline{\underline{Q}}(t) \cdot [D_t \underline{\underline{Q}}(t)]^T$$

so

$$\boxed{\begin{array}{l} \underline{\underline{\Omega}}(t) + \underline{\underline{\Omega}}^T(t) = 0 \quad \text{or} \\ \underline{\underline{\Omega}}(t) = -\underline{\underline{\Omega}}^T(t) \end{array}}$$

Thus at each instant $t$, $\exists$ a unique <u>vector</u> $\underline{\Omega}(t)$

$$\underline{\Omega}(t) = -\tfrac{1}{2} \wedge \underline{\underline{\Omega}}(t) = -\tfrac{1}{2} \wedge (D_t \underline{\underline{Q}}(t) \cdot \underline{\underline{Q}}^T(t))$$

$$\underline{\underline{\Omega}}(t) = -\underline{\Omega}(t) \cdot \underline{\underline{\underline{A}}}.$$

Thus at each instant $t$ all particles have exactly the velocities they would have if the body were rotating with constant angular velocity $\underline{\Omega}(t)$ about $O$

$$\boxed{\underline{v}(\underline{r}, t) = \underline{\underline{\Omega}}(t) \cdot \underline{r} = \underline{\Omega}(t) \times \underline{r}}$$

$\underline{\Omega}(t)$ is called the <u>instantaneous angular velocity</u> at time $t$.

$$|\underline{v}(\underline{r},t)| = |\underline{\Omega}(t)|\,|\underline{r}| \sin\theta = |\underline{\Omega}(t)|\,\varpi$$

Thus a rigid body motion $\underline{\underline{Q}}(t)$ gives rise to a unique instantaneous angular velocity $\underline{\Omega}(t)$. Is the converse true?

Let $\underline{\Omega}(t)$ be a cont. differ. vector fcn of time. Then there is a unique twice cont. differ. orthogonal tensor valued fcn of time $\underline{\underline{Q}}(t)$ such that

(i) $\underline{\underline{Q}}(0) = \underline{\underline{I}}$

(ii) $D_t \underline{\underline{Q}} \cdot \underline{\underline{Q}}^T = \underline{\underline{\Omega}}(t) = -\underline{\Omega}(t) \cdot \underline{\underline{A}}$.

Proof:
$$D_t \underline{\underline{Q}} \cdot \underline{\underline{Q}}^T = \underline{\underline{\Omega}}$$
$$D_t \underline{\underline{Q}} \cdot \underline{\underline{Q}}^{-1} = \underline{\underline{\Omega}}$$
$$D_t \underline{\underline{Q}} = \underline{\underline{\Omega}} \cdot \underline{\underline{Q}} \quad , \text{ a differ eqn. for } \underline{\underline{Q}}(t)$$

Let's solve it. Introduce a $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$.

$D_t \underline{\underline{Q}} = \underline{\underline{\Omega}} \cdot \underline{\underline{Q}} \qquad D_t Q_{ik} = \Omega_{ij} Q_{jk}$

nine eqns, nine unknowns $Q_{ik}(t)$.

initial conds. $Q_{ik}(0) = \delta_{ik}$

System is linear, hence $\exists$ a unique solution.

Hence $\exists$ a unique second order tensor $\underline{\underline{Q}}(t)$

$$(i)' \quad \underline{\underline{Q}}(0) = \underline{\underline{I}}$$

$$(ii)' \quad D_t \underline{\underline{Q}} = \underline{\underline{\Omega}}(t) \cdot \underline{\underline{Q}}(t)$$

We want to prove that $\underline{\underline{Q}}(t)$ is a rigid body motion and $\underline{\underline{\Omega}}(t)$ is its angular velocity.

$$(ii)' \Rightarrow D_t \underline{\underline{Q}} \cdot \underline{\underline{Q}}^T = \underline{\underline{\Omega}} \cdot \underline{\underline{Q}} \cdot \underline{\underline{Q}}^T$$

but also

$$D_t \underline{\underline{Q}}^T = \underline{\underline{Q}}^T \cdot \underline{\underline{\Omega}}^T \quad \text{and since } \underline{\underline{\Omega}}^T = -\underline{\underline{\Omega}}$$

$$\underline{\underline{Q}} \cdot D_t \underline{\underline{Q}}^T = -\underline{\underline{Q}} \cdot \underline{\underline{Q}}^T \cdot \underline{\underline{\Omega}}$$

adding

$$* \quad D_t (\underline{\underline{Q}} \cdot \underline{\underline{Q}}^T) = \underline{\underline{\Omega}} \cdot \underline{\underline{Q}} \cdot \underline{\underline{Q}}^T - \underline{\underline{Q}} \cdot \underline{\underline{Q}}^T \cdot \underline{\underline{\Omega}}$$

now let $\underline{\underline{M}}(t) = \underline{\underline{Q}}(t) \cdot \underline{\underline{Q}}^T(t)$

$$* \text{ is } \quad D_t \underline{\underline{M}} = \underline{\underline{\Omega}} \cdot \underline{\underline{M}} - \underline{\underline{M}} \cdot \underline{\underline{\Omega}}$$

with i.c. $\underline{\underline{M}}(0) = \underline{\underline{I}}$

as before can show that $\underline{\underline{M}}(t)$ is uniquely det. by $*$ + i.c.

But $\underline{\underline{M}}(t) = \underline{\underline{I}}$ is a solution, hence it is the only solution

$$\underline{\underline{Q}} \cdot \underline{\underline{Q}}^T = \underline{\underline{I}} \quad \text{for all times } t.$$

hence ~~$\not\exists$~~ $\underline{\underline{Q}}(t)$ is orthogonal for all $t$.

Thus $\underline{\underline{Q}}^{-1}(t) = \underline{\underline{Q}}^T(t)$, and from $(ii)'$

$$\underline{\underline{\Omega}}(t) = D_t \underline{\underline{Q}}(t) \cdot \underline{\underline{Q}}^T(t).$$

Hence $\underline{\underline{\Omega}}(t)$ $\underline{\text{is}}$ the associated instantaneous angular velocity

<u>Summary</u>: Thus a rigid body motion may be characterized either by an orthogonal operator $\underline{\underline{Q}}(t)$ (twice cont. differ.) or by an instantaneous angular velocity $\underline{\omega}(t)$ (cont. differ.) The two fcns uniquely determine one another, and their relation is

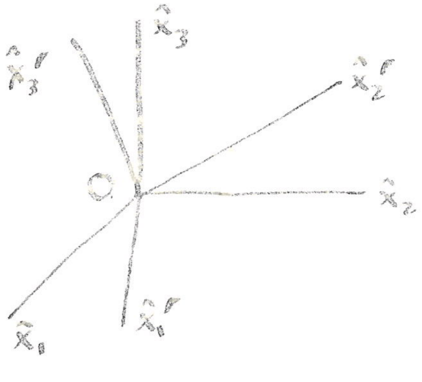
$$\underline{\Omega}(t) = -\tfrac{1}{2}\Lambda\left[D_t\underline{\underline{Q}}(t)\cdot\underline{\underline{Q}}^T(t)\right]$$
$$D_t\underline{\underline{Q}}(t) = -\underline{\Omega}(t)\cdot\underline{\underline{\underline{A}}}\cdot\underline{\underline{Q}}(t)$$
$$\underline{\underline{Q}}(0) = \underline{\underline{I}}$$

Now go thru the usual physicist's proof of this. Euler's Thm → $\underline{\underline{Q}}(t)$ is a <u>rotation</u> at t; consider <u>infinitesimal</u> rotation

2. <u>Newton's laws of motion in a rotating reference frame</u>, <u>Inertial forces due to rotation</u>

Newton's laws of motion hold in an inertial reference. What form do Newton's laws take in a frame of reference which is undergoing rigid body rotations w.r.t. an inertial frame?

$\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ inertial frame fixed $\}$ same
$\{\hat{x}_1', \hat{x}_2', \hat{x}_3'\}$ rotating frame $\}$ origin O




What are relations between <u>fcns of time alone</u>.

$\{\hat{x}_1'(t), \hat{x}_2'(t), \hat{x}_3'(t)\}$

(i) scalars, say the temp of a body or the volume of some body $f(t)$
Both observers will agree

$$D_t f = D_t' f$$

(ii) vectors $\underline{f}(t)$

$$\underline{f}(t) = f_i(t)\,\hat{x}_i = f_i'(t)\,\hat{x}_i'(t)$$

$$D_t f_i = D_t' f_i \quad , \quad D_t f_i' = D_t' f_i'$$

however space observer $D_t \hat{x}_i = 0$, but body observer says $D_t' \hat{x}_i' = 0$
Thus space observer claims

$$D_t \underline{f}(t) = (D_t f_i)\hat{x}_i = (D_t f_i')\hat{x}_i' + f_i' D_t \hat{x}_i'$$

body observer claims

$$D_t' \underline{f}(t) = (D_t' f_i')\,\hat{x}_i'$$

the relation is
$$D_t \underline{f} = D_t' \underline{f} + f_i' D_t x_i'(t)$$

$$D_t \underline{f} = D_t' \underline{f} + (D_t \hat{x}_i')\hat{x}_i' \cdot \underline{f}$$

Now let $\underline{\underline{G}}(t)$ be that unique orthogonal operator

$$\hat{x}_i'(t) = \underline{\underline{G}}(t) \cdot \hat{x}_i$$

$$\hat{x}_j' = \underline{\underline{Q}} \cdot \hat{x}_j$$

$$Q_{ij} = \hat{x}_i \cdot \underline{\underline{Q}} \cdot \hat{x}_j = \hat{x}_i \cdot \hat{x}_j'$$

if $\underline{\underline{Q}} = \hat{x}_k' \hat{x}_k$, then $Q_{ij} = (\hat{x}_i \cdot \hat{x}_k')(\hat{x}_k \cdot \hat{x}_j) = (\hat{x}_i \cdot \hat{x}_k')\delta_{kj} = \hat{x}_i \cdot \hat{x}_j'$

<u>$\underline{\underline{Q}}(t)$ is the rigid rot. which at time $t$ carries unprimed (inertial) frame into primed body frame.</u>

Clearly $\underline{\underline{Q}}(t) = \hat{x}_j'(t)\,\hat{x}_j$

$\underline{\underline{Q}}^T(t) = \hat{x}_j\,\hat{x}_j'(t)$

$Q(t)$ is also clearly the rigid rot. which at $t$ carries the body from its position at $t=0$.

Now in the inertial frame

$$D_t \hat{x}_i' = D_t \underline{\underline{Q}}(t) \cdot \hat{x}_i \quad \text{so}$$

$$(D_t \hat{x}_i')\hat{x}_i' = D_t \underline{\underline{Q}} \cdot \hat{x}_i \hat{x}_i' = D_t \underline{\underline{Q}} \cdot \underline{\underline{Q}}^T$$

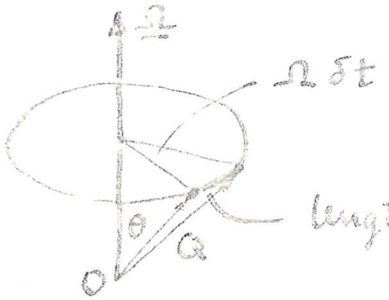
$$\boxed{\begin{aligned} D_t \underline{f} &= D_t' \underline{f} + [D_t \underline{\underline{Q}} \cdot \underline{\underline{Q}}^T] \cdot \underline{f} \\ &= D_t' \underline{f} + \underline{\underline{\Omega}} \cdot \underline{f} \\ D_t \underline{f} &= D_t' \underline{f} + \underline{\Omega} \times \underline{f} \end{aligned}}$$

<u>Here $\underline{\Omega}(t)$ is the instantaneous angular velocity of the body (primed) frame w.r.t. the inertial (unprimed) frame.</u>

$$\left(\frac{D}{Dt}\right)_{space} = \left(\frac{D}{Dt}\right)_{body} + \underline{\Omega}\times \quad (\text{for vectors})$$

Simple physical explanation — ~~[illegible]~~ rate of change in inertial frame due to two independent causes

1. rate of change in body
2. rate at which it would be carried around by instantaneous angular velocity of body if fixed in body frame.




$\vec{Q}$ vector, length $Q$, fixed in body

length $Q \sin\theta\,(\Omega\,\delta t)$

$\vec{\Omega} \times \vec{Q}\,\delta t$ = change in space in time $\delta t$.

(iii) second time derivatives of vectors

$$D_t^2 \underline{f} = D_t (D_t' \underline{f}) + D_t \underline{\Omega} \times \underline{f} + \underline{\Omega} \times D_t \underline{f}$$

$$= D_t' D_t' \underline{f} + \underline{\Omega} \times D_t' \underline{f} + D_t \underline{\Omega} \times \underline{f}$$

$$+ \underline{\Omega} \times (D_t' \underline{f} + \underline{\Omega} \times \underline{f})$$

now $\underline{\Omega}' = \underline{\Omega}$ (the observers will agree on the angular velocity of the ' relative to the un' system. Also

$$D_t \underline{\Omega} = D_t' \Omega + \underline{\Omega} \times \underline{\Omega} = D_t' \underline{\Omega}$$

$$\boxed{\begin{aligned} D_t^2 \underline{f} = (D_t')^2 \underline{f} + 2\underline{\Omega}' \times D_t' \underline{f} + D_t' \underline{\Omega}' \times \underline{f} \\ + \underline{\Omega}' \times \underline{\Omega}' \times \underline{f}. \end{aligned}}$$

#

now if $\underline{f} = \underline{r}(t)$ the position vector of a moving particle $\underline{f} = \underline{r}(x,t)$

$\underline{v}$ and $\underline{v}'$ its velocity
$\underline{a}$ and $\underline{a}'$ its acceleration

$$\boxed{\begin{aligned} \underline{r} &= \underline{r}' \\ \underline{v} &= \underline{v}' + \underline{\Omega} \times \underline{r} \\ \underline{a} &= \underline{a}' + 2\underline{\Omega} \times \underline{v}' + D_t \underline{\Omega} \times \underline{r} + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}) \end{aligned}}$$

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Now Newton's law of motion for a ~~test~~ particle of mass $m$

$\underline{f} = m\underline{a}$, in an inertial frame

$\underline{f} = \underline{f}'$, $m = m'$ In the body reference frame

$$\boxed{\begin{aligned} & m\,[\underline{a}' + 2\underline{\Omega} \times \underline{v} + D_t \underline{\Omega} \times \underline{r} + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}')] = \underline{f} \\ & m\underline{a}' = \underline{f} \underbrace{- 2m\underline{\Omega} \times \underline{v}'}_{\text{Coriolis force}} \underbrace{- m\underline{\Omega} \times (\underline{\Omega} \times \underline{r}')}_{\text{centrifugal force}} \underbrace{- mD_t \underline{\Omega} \times \underline{r}}_{\text{spinning force}} \end{aligned}}$$

Euler force

If left on the left, they are called <u>accelerations</u>

$2\underline{\Omega}\times\underline{v}'$ Coriolis acc.

$\underline{\Omega}\times(\underline{\Omega}\times\underline{r})$ centripetal acc. (away from the center)

$D_t\underline{\Omega}\times\underline{r}$ spin-up acc.

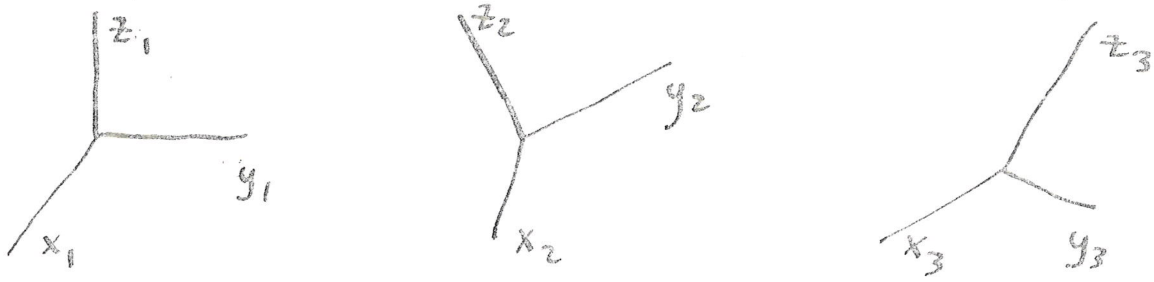
The three imaginary forces are called <u>inertial forces</u>.

Remarks:

1. in most geophysical problems $D_t\underline{\Omega}\times\underline{r}$ is readily neglected, except in discussions of Chandler wobble, etc.
2. measured grav. acceler. on the Earth is $g-\underline{\Omega}\times(\underline{\Omega}\times\underline{r})$
   We have already discussed this.
3. Coriolis acceleration must be taken into account in measuring gravity at sea from a moving ship.
   Ship steaming around equator at 10 knots
   $2\underline{\Omega}\times\underline{v}' = 2\Omega v' \simeq 60$ mgal
   Must have good navigation for grav. at sea. This called the Eötvös effect.
4. cyclones: ccw around a low in N hemis.
5. Baer's law of river displacements (right bank erodes faster in N)

Addition of angular velocities

Suppose we have three different Cartesian reference frames, all same origin.




let $\vec{\Omega}_{ji}$ = angular velocity of frame $j$ as seen from frame $i$. Let $\underline{f}(t)$ be an arbitrary vector fcn of time.

$$D_t^1 \underline{f} = D_t^2 \underline{f} + \underline{\Omega}_{21} \times \underline{f} \qquad ①$$

$$= D_t^3 \underline{f} + \underline{\Omega}_{31} \times \underline{f} \qquad ②$$

$$D_t^2 \underline{f} = D_t^3 \underline{f} + \underline{\Omega}_{32} \times f \qquad ③$$

from ① and ③ $\quad D_t^1 \underline{f} = D_t^3 \underline{f} + \underline{\Omega}_{32} \times f + \underline{\Omega}_{21} \times \underline{f} \qquad ④$

from ② and ④ $\quad \underline{\Omega}_{31} \times \underline{f} = (\underline{\Omega}_{32} + \underline{\Omega}_{21}) \times \underline{f}$, but $\underline{f}$ is arbitrary

addition law for angular velocities

$$\boxed{\underline{\Omega}_{31} = \underline{\Omega}_{32} + \underline{\Omega}_{21}}$$

makes sense

Problem: (could be (iii) tensors of order 2)
Given a second order tensor $\underline{\underline{T}}(t)$ function of time. What is the relation between $D_t \underline{\underline{T}}$ and $D_t' \underline{\underline{T}}$?

$$\underline{\underline{T}}(t) = T_{ij}(t)\,\hat{x}_i \hat{x}_j = T_{ij}'(t)\,\hat{x}_i' \hat{x}_j'(t)$$

$$D_t T_{ij} = D_t' T_{ij}, \quad D_t T_{ij}' = D_t' T_{ij}'$$

space observer claims

$$D_t \underline{\underline{T}}(t) = (D_t T_{ij})\,\hat{x}_i \hat{x}_j$$

$$= (D_t T_{ij}')\,\hat{x}_i' \hat{x}_j' + T_{ij}'(D_t \hat{x}_i')\,\hat{x}_j' + T_{ij}'\,\hat{x}_i'\, D_t \hat{x}_j'$$

body observer claims

$$D_t' \underline{\underline{T}}(t) = (D_t' T_{ij}')\,\hat{x}_i' \hat{x}_j'$$

The relation is

$$D_t \underline{\underline{T}} = D_t' \underline{\underline{T}} + T_{ij}'(D_t \hat{x}_i')\,\hat{x}_j' + T_{ij}'\,\hat{x}_i'(D_t \hat{x}_j')$$

$$= D_t' \underline{\underline{T}} + D_t \hat{x}_i'(\hat{x}_i' \cdot \underline{\underline{T}}) + D_t x_j'(\underline{\underline{T}} \cdot \hat{x}_j')$$

now once again let $\underline{\underline{Q}}(t)$ be that unique orthog. operator $\ni$

$$\hat{x}_i'(t) = \underline{\underline{Q}}(t) \cdot \hat{x}_i.$$

then
$$\underline{\underline{Q}}(t) = \hat{x}_j'(t)\,\hat{x}_j$$
$$\underline{\underline{Q}}^T(t) = \hat{x}_j\,\hat{x}_j'(t)$$

now in the inertial frame $D_t \hat{x}_i' = D_t \underline{\underline{Q}}(t) \cdot \hat{x}_i$

so

$$(D_t \hat{x}_i')\hat{x}_i' = D_t \underline{\underline{Q}} \cdot \hat{x}_i \hat{x}_i' = D_t \underline{\underline{Q}} \cdot \underline{\underline{Q}}^T$$

and

$$\hat{x}_j'(D_t \hat{x}_j') = x_j' D_t \underline{\underline{Q}} \cdot \hat{x}_j = x_j' \hat{x}_j \cdot D_t \underline{\underline{Q}}^T$$
$$= \underline{\underline{Q}} \cdot D_t \underline{\underline{Q}}^T$$

now letting $\underline{\underline{\Omega}} = D_t \underline{\underline{Q}} \cdot \underline{\underline{Q}}^T$, $\underline{\underline{\Omega}}$ is antisymmetric

$$\underline{\underline{\Omega}}^T = \underline{\underline{Q}} \cdot D_t \underline{\underline{Q}}^T = -\underline{\underline{\Omega}} = -D_t \underline{\underline{Q}} \cdot \underline{\underline{Q}}^T$$

$$\boxed{\begin{array}{l} D_t \underline{\underline{T}} = D_t' \underline{\underline{T}} + \underline{\underline{\Omega}} \cdot \underline{\underline{T}} - \underline{\underline{T}} \cdot \underline{\underline{\Omega}} \\ \\ \text{or} \quad D_t \underline{\underline{T}} = D_t' \underline{\underline{T}} + \underline{\underline{\Omega}} \times \underline{\underline{T}} - \underline{\underline{T}} \times \underline{\underline{\Omega}} \end{array}}$$

## 3. Angular momentum and kinetic energy of rigid body motion – the inertia tensor

of body V

Total angular momentum about O, is


$$\boxed{\underline{L}(t) = \int_{V(t)} dV\,[\,\underline{r}(t) \times \rho\, \underline{v}(t)\,]}$$

But since the motion is rigid body motion about O

$\underline{v} = \underline{\omega} \times \underline{r}$ , $\underline{\omega}(t)$ the instantaneous angular velocity.

$$\underline{L}(t) = \int_{V(t)} \rho\, dV [\, \underline{r} \times (\underline{\omega} \times \underline{r})\,] = \int_{V(t)} \rho\, [\, r^2 \underline{\omega} - \underline{r}\,\underline{r}\cdot\underline{\omega}\,]\, dV$$

Thus where

$$\boxed{\begin{aligned} \underline{L}(t) &= \underline{\underline{C}}(t) \cdot \underline{\omega}(t) \\ \underline{\underline{C}}(t) &= \int_{V(t)} \rho\, [\, r^2 \underline{\underline{I}} - \underline{r}\,\underline{r}\,]\, dV \end{aligned}}$$

origin $\underline{O}$ must be c.o.m. to define $\underline{C}$.

If O is the c.o.m., then $\underline{\underline{C}}(t)$ is called the <u>inertia tensor</u> of the body. The inertia tensor is clearly symmetric.

Maybe mention the parallel axis theorem.

Define: $I(\underline{0},\hat{n}) \equiv \int_{V(t)} \rho_E \left[ r^2 - (\underline{r}\cdot\hat{n})^2 \right] dV$

Define moment of inertia about some arbitrary axis (thru the c.o.m.) $\hat{n}$

$I_n = \hat{n}\cdot\underline{\underline{C}}\cdot\hat{n}$ — call this $I(\underline{0},\hat{n})$

$\underline{\underline{C}}(t)$ is that symmetric tensor such that the angular momentum is a linear operator acting on the instantaneous angular velocity $\underline{\omega}(t)$

$\underline{\underline{C}}(t)$ is symmetric so there is a Cartesian axis system ~~made of~~ of eigenvectors $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ such that

$$\underline{\underline{C}} = A\,\hat{x}_1\hat{x}_1 + B\,\hat{x}_2\hat{x}_2 + C\,\hat{x}_3\hat{x}_3$$

these axes are called the principal axes of inertia of the body, and $A, B, C$ (the eigenvalues of $\underline{\underline{C}}$) are called the principal moments of inertia

We shall arrange them so that $A \le B \le C$

In any other $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ $\exists$ in general off-diagonal products of inertia

of the form $\int_V x_i x_j \rho\, dV$

With respect to the principal axes of inertia $L_1 = A\omega_1$, $L_2 = B\omega_2$, $L_3 = C\omega_3$

$$\boxed{\text{kinetic energy } T(t) \rightarrow T = \frac{1}{2}\int_{V(t)} \left[\rho\, \underline{v}(t)\cdot\underline{v}(t)\right] dV}$$

$$T = \frac{1}{2}\int_{V(t)} \rho\, \underline{v}\cdot(\underline{\omega}\times\underline{r})\, dV = \frac{1}{2}\underline{\omega}\cdot\int_{V(t)} \rho\, \underline{r}\times\underline{v}\, dV$$

$$= \frac{1}{2}\underline{\omega}\cdot\underline{L}$$

$$T = \tfrac{1}{2}\underline{\omega}\cdot\underline{L} = \tfrac{1}{2}\underline{\omega}\cdot\underline{\underline{C}}\cdot\underline{\omega}$$

$$T(t) = \tfrac{1}{2}\underline{\omega}(t)\cdot\underline{L}(t) = \tfrac{1}{2}\underline{\omega}(t)\cdot\underline{\underline{C}}(t)\cdot\underline{\omega}(t)$$

Let $\underline{\omega}(t) = \omega(t)\,\hat{n}(t)$

Then $T = \tfrac{1}{2}\omega^2\,\hat{n}\cdot\underline{\underline{C}}\cdot\hat{n}$

$T = \tfrac{1}{2} I_n(t)\,\omega^2(t)$ where

$$I_n = \hat{n}\cdot\underline{\underline{C}}\cdot\hat{n} = \int_{V(t)} \rho\, dV\,[\,r^2 - (\hat{n}\cdot\underline{r})^2\,]$$

$\equiv$ moment of inertia about the axis of rotation

Remarks:

1. Recall that we expressed the $l=2$ terms in the expansion of the grav. pot in terms of the components of the inertia tensor. Our expressions looked particularly nice if the $\hat{z}$-axis was taken to be the axis of greatest inertia — not necessary that $\hat{x}, \hat{y}$ also be principal axes.

$$C_{10} = C_{11} = S_{11} = 0 \quad (\text{origin at c.o.m.})$$
$$C_{21} = S_{21} = 0$$

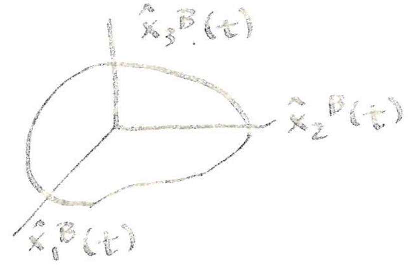
$$\frac{C - \frac{1}{2}(A+B)}{Ma^2} = -\sqrt{5}\,C_{20} = J_2 \qquad \frac{B-A}{Ma^2} = \sqrt{\frac{20}{3}}\left(C_{22}^2 + S_{22}^2\right)^{1/2}$$

historical

2. the associated ellipsoid called the inertia ellipsoid

4. Eulerian nutation of a rigid body

We have a rigid body principal moments $A, B, C$, principal axes of inertia $\hat{x}_1^B, \hat{x}_2^B, \hat{x}_3^B$ fixed in the body & always an orthonormal set.




$\hat{x}_1^B(t)$, $\hat{x}_2^B(t)$, $\hat{x}_3^B(t)$ as the body moves

At any instant the inertia tensor of the body is

$$\underline{\underline{M}}(t) = A\,\hat{x}_1^B(t)\hat{x}_1^B(t) + B\,\hat{x}_2^B(t)\hat{x}_2^B(t) + C\,\hat{x}_3^B(t)\hat{x}_3^B(t)$$
$$= M_{ij}^B\,\hat{x}_i^B\hat{x}_j^B$$

We wish to investigate the motion in the absence of external forces and torques. The dynamical law governing such a motion is conservation of angular momentum

$$\boxed{\underline{L} = \text{constant}}$$

Now such a body can spin at constant angular velocity about any principal axis $L_1 = A\omega_1$, $L_2 = B\omega_2$, $L_3 = C\omega_3$, but about no other axis. These are 3 possible motions. We ask, what motions can it execute which are nearly a spin at constant angular velocity about a principal axis.

Two reasons for asking:

1. stability analysis
2. if stable $\rightarrow$ small oscillations about stable steady rotation $\rightarrow$ what is the nature of these small oscillations.

$\underline{L}(t)$ = angular momentum of body

$\underline{\Omega}(t)$ = inst. angular velocity relative to a spatial or inertial frame $\{\hat{x}_1^s, \hat{x}_2^s, \hat{x}_3^s\}$

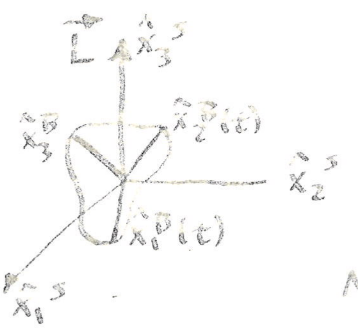
$$\underline{L} = \underline{\underline{M}}(t) \cdot \underline{\Omega}(t) = \text{constant}$$

Now *define* $\hat{x}_3^s$ and $\Omega_0$ by the demand

$$\boxed{\underline{L} = \hat{x}_3^s\, \Omega_0 C}$$

$\hat{x}_3^s$ is direction of fixed angular momentum

$\Omega_0$ is an average angular velocity (of which we examine stability)




Origin of both $\{x_i^B\}$ and $\{x_i^s\}$ is at c.o.m. of body

Now define a new frame for purposes of computation, the O frame

$\{\hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t)\}$ by $\hat{x}_3 = \hat{x}_3^s$, frame rotating at constant angular velocity $\Omega_0$ relative to the inertial frame

$$\boxed{\vec{\Omega}_0 = \Omega_0\, \hat{x}_3^s = \Omega_0\, \hat{x}_3}$$

Now *define* $\underline{\underline{M}}_0 = A\hat{x}_1\hat{x}_1 + B\hat{x}_2\hat{x}_2 + C\hat{x}_3\hat{x}_3$

Then $$\boxed{\underline{L} = \underline{\underline{M}}_0 \cdot \underline{\Omega}_0}$$ $= M_{ij}^B \hat{x}_i \hat{x}_j$

Now we wish to consider motions which are *nearly* a rotation of constant ang. velocity $\Omega_0$ about $\hat{x}_3$. Thus $\hat{x}_i^B(t)$ will be very close to $\hat{x}_i(t)$, so the orthogonal transformation

$$\underline{\underline{Q}} \cdot \hat{x}_i(t) = \hat{x}_i^B(t) \qquad \text{or}$$
$$\underline{\underline{Q}} = \hat{x}_i^B(t)\,\hat{x}_i(t)$$

will be *close to the identity*. $\underline{\underline{Q}} = \underline{\underline{I}} + \underline{\underline{K}}$

$\underline{\underline{Q}}$ carries the unif. rotating frame into the body frame

$$\begin{aligned}\hat{x}_i^B(t) &= (\underline{\underline{I}} + \underline{\underline{K}}) \cdot \hat{x}_i(t)\\ &= \hat{x}_i(t) + \underline{\underline{K}} \cdot \hat{x}_i(t)\\ &= \hat{x}_i(t) + \underline{K} \times \hat{x}_i(t)\end{aligned}$$

$$\boxed{\hat{x}_i^B(t) = \hat{x}_i(t) + \underline{K} \times \hat{x}_i(t) + O(K^2)}$$

$K_1, K_2, K_3$ are the small rotations of the body about $\hat{x}_1, \hat{x}_2, \hat{x}_3$ ~~[illegible]~~ at time $t$. $K_1(t), K_2(t), K_3(t)$.

Now let the <u>angular velocity of frame B relative to frame O be $\underline{\omega}$</u>

Then

$$\underline{\Omega} = \underline{\Omega}_0 + \underline{\omega} \quad , \text{ or } \underline{\Omega} = \underline{\omega} + \underline{\Omega}_0$$

$$\left( \underset{\substack{\uparrow \\ \text{B relative to S}}}{\underline{\Omega}_{BS}} = \underline{\Omega}_{BO} + \underline{\Omega}_{OS} \right)$$

Thus $\quad \underline{L} = \underline{\underline{M}} \cdot \underline{\Omega} = \underline{\underline{M}} \cdot (\underline{\Omega}_0 + \underline{\omega})$

$$\underline{\underline{M}} = M_{ij}{}^B \hat{x}_i{}^B \hat{x}_j{}^B = (\hat{x}_k{}^B \hat{x}_k) \cdot \hat{x}_i M_{ij}{}^B \hat{x}_j \cdot (\hat{x}_\ell \hat{x}_\ell{}^B)$$

$$= \underline{\underline{Q}} \cdot \underline{\underline{M}}_0 \cdot \underline{\underline{Q}}^T$$

$$= (\underline{\underline{I}} + \underline{\underline{K}}) \cdot \underline{\underline{M}}_0 \cdot (\underline{\underline{I}} - \underline{\underline{K}}) + O(K^2)$$

$$\boxed{\underline{\underline{M}} = \underline{\underline{M}}_0 + \underline{\underline{K}} \cdot \underline{\underline{M}}_0 - \underline{\underline{M}}_0 \cdot \underline{\underline{K}} + O(K^2)}$$

Also clearly $\underline{\omega} = O(K)$, so

$$\boxed{\underline{L} = \underline{\underline{M}}_0 \cdot \underline{\Omega}_0 + (\underline{\underline{K}} \cdot \underline{\underline{M}}_0 - \underline{\underline{M}}_0 \cdot \underline{\underline{K}}) \cdot \underline{\Omega}_0 + \underline{\underline{M}}_0 \cdot \underline{\omega} + O(K^2)}$$

But also $\underline{L} = \underline{\underline{M}}_0 \cdot \underline{\Omega}_0$, thus

$$\underline{\underline{M}}_0 \cdot \underline{\omega} = (\underline{\underline{M}}_0 \cdot \underline{\underline{K}} - \underline{\underline{K}} \cdot \underline{\underline{M}}_0) \cdot \underline{\Omega}_0$$

Now $D_t^0 \equiv$ time derivative as seen by frame 0.
By definition

$$\underline{\underline{\omega}} = \underline{\underline{\Omega}}_{BO} = (D_t^0 \underline{\underline{Q}}) \cdot \underline{\underline{Q}}^T = (D_t \underline{\underline{K}}) \cdot (\underline{\underline{I}} + \underline{\underline{K}}^T)$$

(the anti-symmetric tensor assoc. with $\underline{\omega}$)

$$= D_t^0 \underline{\underline{K}} + O(K^2)$$

$$\underline{\underline{\omega}} = D_t^0 \underline{\underline{K}} \quad , \quad \text{so} \quad \underline{\omega} = D_t^0 \underline{K}$$

vector eqn.

hence

$$\boxed{\underline{\underline{M}}_0 \cdot (D_t \underline{\underline{K}}) = (\underline{\underline{M}}_0 \cdot \underline{\underline{K}} - \underline{\underline{K}} \cdot \underline{\underline{M}}_0) \cdot \underline{\underline{\Omega}}_0}$$

We want r.h.s. in terms of the vector $\underline{K}$ to obtain a diff. eqn. for $\underline{K}$.

$$\underline{\underline{K}} \cdot \underline{\underline{M}}_0 \cdot \underline{\underline{\Omega}}_0 = \underline{\underline{K}} \cdot (\underline{\underline{M}}_0 \cdot \hat{x}_3)\, \Omega_0 = \underline{\underline{K}} \cdot \hat{x}_3\, (C\Omega_0)$$

switch order for clarity

$$= (\underline{K} \times \hat{x}_3)\, C\Omega_0 = (\underline{K} \times \hat{x}_3) \cdot C\underline{\underline{I}}\, \Omega_0$$

$$= \Omega_0 C\, \underline{\underline{I}} \cdot (\underline{K} \times \hat{x}_3)$$

$$\underline{\underline{M}}_0 \cdot \underline{\underline{K}} \cdot \underline{\underline{\Omega}}_0 = \Omega_0 \underline{\underline{M}}_0 \cdot \underline{\underline{K}} \cdot \hat{x}_3 = \Omega_0 \underline{\underline{M}}_0 \cdot (\underline{K} \times \hat{x}_3)$$

$$\boxed{\underline{\underline{M}}_0 \cdot (D_t \underline{K}) = \Omega_0 (\underline{\underline{M}}_0 - C\underline{\underline{I}}) \cdot (\underline{K} \times \hat{x}_3)}$$

now $\underline{K} \times \hat{x}_3 = K_2 \hat{x}_1 - K_1 \hat{x}_2$ and

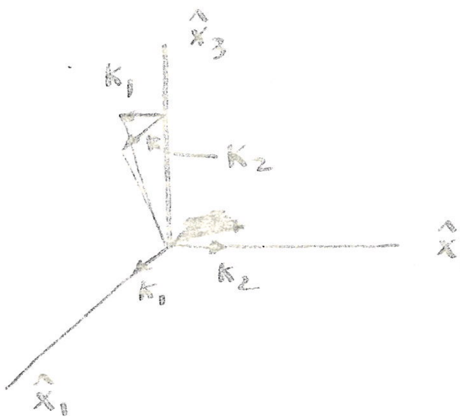
$\underline{\underline{M}}_0 - C \underline{\underline{I}} = (A-C)\hat{x}_1\hat{x}_1 + (B-C)\hat{x}_2\hat{x}_2$

$$\boxed{\begin{aligned} A D_t^0 K_1 &= \Omega_0 (A-C) K_2 \\ B D_t^0 K_2 &= -\Omega_0 (B-C) K_1 \\ C D_t^0 K_3 &= 0 \end{aligned}}$$

written out in components.

Begin lecture 25 here.

$\boxed{K_3 = \text{constant}}$ now what are $K_1, K_2, K_3$?




in the 0 frame

$+K_2$ and $-K_1$ are the $\hat{x}_1$ and $\hat{x}_2$ components of the $\hat{x}_3^B$ axis of figure w.r.t. $\hat{x}_3 \equiv$ angular momentum axis.

$K_1, K_2, K_3$ are rotations of $B$ frame relative to $0$ frame.

$+K_2$ and $-K_1$ are direction cosines of $\hat{x}_3^B$ relative to $\hat{x}_1$ and $\hat{x}_2$

Our first result is $K_3 =$ const., motion of B relative to 0 can be described by $K_1, K_2$ as above.

Now for $K_1, K_2$ $\qquad AB(D_t^0)^2 K_1 = -\Omega_0^2 (C-A)(C-B) K_1$

$$\boxed{(D_t^0)^2 K_1 = -\left[\Omega_0^2 \frac{(C-A)(C-B)}{AB}\right] K_1}$$

now if $C$ = intermediate moment of inertia.
then $(C-A)(C-B) < 0$ and
rotation about $\hat{x}_3$ is unstable dynamically
If $C$ = max. or min. moment of inertia, then
$(C-A)(C-B) > 0$ ; and system is dynamically
stable

Steady rotation about intermediate principal axis is dynamically unstable. Steady rotation about axes of max. and min. inertia dynamically stable.

Must also investigate secular stability, or stability in the presence of dissipation.
Example from Whittaker, p. 203 due to Lamb.


rotating bowl
smooth bowl → stable at bottom, if a small perturbation knocks it up it oscillates around bottom
if ∃ the slightest friction and if $\Omega$ large enough,


ball works its way out in a spiral path.

lunar tidal friction another example of secular instability.

Recall kinetic energy $T = \frac{1}{2} \underline{L} \cdot \underline{\Omega}$ , $\underline{L} = \underline{\underline{M}} \cdot \underline{\Omega}$
For a given $\underline{L}$, $\underline{\Omega}$ is a ~~maxi~~ minimum when rotating about axis of greatest inertia $\Rightarrow$
$T$ is min.

For a given angular momentum $\vec{L}$,

$T =$ min. when rotating about axis of max inertia
$T =$ max. " " " " " min. inertia

$\therefore$ If there is any dissipation present, steady rotation about axis of least inertia is secularly unstable.
Any perturbation will decrease $T$, any motion away from this perturbation will decrease $T$ due to dissipation $\rightarrow$ instability

secular instability : e-folding time $\propto$ amt. of dissipation

From now on assume $C > B > A$ ; $\hat{x}_3$ is axis of greatest inertia.
Now define

$$\xi = \frac{K_1}{\sqrt{B(C-A)}} \quad , \quad \eta = \frac{K_2}{\sqrt{A(C-B)}}$$

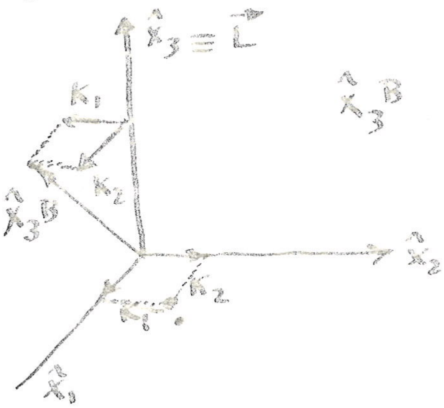
$$D_t^\circ \xi = -\Omega_0 \alpha \eta$$
$$D_t^\circ \eta = \Omega_0 \alpha \xi \qquad \alpha = \sqrt{\frac{(C-A)(C-B)}{AB}}$$

$$D_t^\circ(\xi + i\eta) = i\Omega_0\alpha(\xi + i\eta)$$

$$\boxed{\xi + i\eta = (\xi_0 + i\eta_0)\, e^{i\alpha\Omega_0 t}}$$

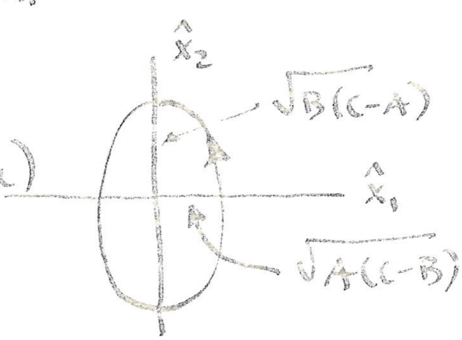
Thus the vector $K_1\hat{x}_1 + K_2\hat{x}_2$ describes an *ellipse* in the counter clockwise (positive) sense relative to $\hat{x}_3 \equiv$ axis of $\vec{L}$.

The axis of figure $\hat{x}_3{}^B$ has direction cosines $K_2\hat{x}_1 - K_1\hat{x}_2$




$\hat{x}_3{}^B$ also goes ccw and is $\frac{\pi}{2}$ out of phase ($\frac{\pi}{2}$ behind)

Motion of $\hat{x}_3{}^B$:
(in the $\hat{x}_1\hat{x}_2$ plane)




frequency of this motion of $K_1\hat{x}_1 + K_2\hat{x}_2$

$$\omega_N = \alpha\Omega_0 = \sqrt{(C-A)(C-B)/AB}$$

nutation, Eulerian free nutation

Now what about the motion of the rotation axis

$$\omega_1 = D_t^0 K_1 = \sqrt{B(C-A)}\, D_t^0 \xi = -\Omega_0 \alpha \eta \sqrt{B(C-A)}$$

$$= -\Omega_0 \left(\frac{C-A}{A}\right) K_2$$

$$\omega_2 = D_t^0 K_2 = \sqrt{A(C-B)}\, D_t^0 \eta = \Omega_0 \alpha \xi \sqrt{A(C-B)}$$

$$= \Omega_0 \left(\frac{C-B}{B}\right) K_1$$

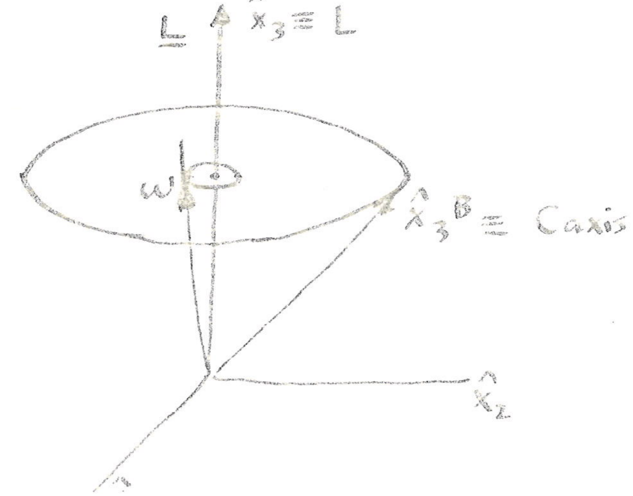
$$\boxed{\begin{aligned} \omega_1 &= -\Omega_0 \left(\frac{C-A}{A}\right) K_2 \\ \omega_2 &= \Omega_0 \left(\frac{C-B}{B}\right) K_1 \\ \omega_3 &= 0 \end{aligned} \quad *}$$

now $\frac{\omega_1}{\Omega_0}$, $\frac{\omega_2}{\Omega_0}$ are direction cosines of $\underline{\omega}$ axis in 0 frame. Recall direction cosines of $\hat{x}_3^B$ figure axis $f_1$, $f_2$

$\omega_1/\Omega_0 = -\left(\frac{C-A}{A}\right) K_2$ ; $f_1 = K_2$

$\omega_2/\Omega_0 = \left(\frac{C-B}{B}\right) K_1$ ; $f_2 = -K_1$

So in the 0-frame the motion looks like a precession (bad word) of the $\underline{\omega}$ and figure $\hat{x}_3^B$ axes




for the Earth

$A \approx B$ and $\frac{C-A}{A} \sim \frac{1}{300}$

$\therefore$ ellipse is $\sim$ a circle (300 days)

$\therefore$ period $\sim$ 10 months

$\therefore$ inner circle $\underline{\omega} \ll$ outer circle ($\sim \frac{1}{300}$ less)

What are the observable effects of the free nutation in the body frame $\hat{x}_i^B(t)$, where we are sitting.
To find out, we must determine $\Omega_i^B(t)$, the <u>components</u> of $\underline{\Omega}(t)$ <u>in the body frame</u>.
To first order in K, we have

$$\Omega_1^B = \underline{\Omega}\cdot\hat{x}_1^B = (\underline{\Omega}_0 + \underline{\omega})\cdot\hat{x}_1^B$$
$$= (\underline{\Omega}_0+\underline{\omega})\cdot(\underline{\underline{I}} + \underline{\underline{K}})\cdot\hat{x}_1$$
$$= (\Omega_0\hat{x}_3 + \underline{\omega})\cdot(\hat{x}_1 + \underline{K}\times\hat{x}_1)$$
$$= \Omega_0\hat{x}_3\cdot(\underline{K}\times\hat{x}_1) + \underline{\omega}\cdot\hat{x}_1$$
$$= -\Omega_0\hat{x}_2\cdot\underline{K} + \omega\cdot\hat{x}_1$$
$$\therefore\ \Omega_1^B = -\Omega_0 K_2 + \omega_1 = -\Omega_0 K_2 - \Omega_0 K_2\left(\frac{C-A}{A}\right)$$

$$\Omega_2^B = \underline{\Omega}\cdot x_2^B = (\Omega_0\hat{x}_3 + \underline{\omega})\cdot(\underline{\underline{I}}+\underline{\underline{K}})\cdot\hat{x}_2$$
$$= (\Omega_0\hat{x}_3 + \underline{\omega})\cdot(\hat{x}_2 + \underline{K}\times\hat{x}_2)$$
$$= \Omega_0\hat{x}_3\cdot(K\times\hat{x}_2) + \underline{\omega}\cdot\hat{x}_2$$
$$\therefore\ \Omega_2^B = \Omega_0 K_1 + \omega_2 = \Omega_0 K_1 + \Omega_0 K_1\left(\frac{C-B}{B}\right)$$

$$\Omega_3^B = \underline{\Omega}\cdot\hat{x}_3^B = (\Omega_0\hat{x}_3 + \underline{\omega})\cdot(\hat{x}_3 + K\times\hat{x}_3)$$
$$= \Omega_0 + \omega_3\ ,\quad \text{but}\ \omega_3 = 0$$

$$\boxed{\begin{aligned}\Omega_1^B &= -\Omega_0\left(\frac{C}{A}\right)K_2\\ \Omega_2^B &= \Omega_0(C/B)K_1\\ \Omega_3^B &= \Omega_0 \quad \star\end{aligned}}$$

Also of course $\Omega_1^B/\Omega_0$ and $\Omega_2^B/\Omega_0$ are the direction cosines of $\underline{\Omega}$ in $\hat{x}_1^B, \hat{x}_2^B$

$$\boxed{\begin{aligned}\Omega_1^B/\Omega_0 &= -\frac{C}{A}K_2 \quad &\text{also}\quad L_1^B/L &= -K_2\\ \Omega_2^B/\Omega_0 &= \frac{C}{B}K_1 & L_2^B/L &= K_1\end{aligned}}$$

Comparing ∗ with ∗ we see that motion of $\underline{\Omega}$ in space ≪ motion of $\underline{\Omega}$ in body axes. The ellipse traced out by $\underline{\Omega}$ in space is smaller by $\frac{C-A}{C}$ and $\frac{C-B}{C}$ than the ellipse traced out by $\underline{\Omega}$ in the body axes.

In real inertial space motions of $\underline{\Omega}$ and $\hat{x}_3^B$ are complicated further by the diurnal rotation $\Omega_0$ of the zero frame we have been using.

This small force-free motion of a rigid body called the Eulerian free nutation (Euler, 1758).

<u>Comment</u> on terms precession and nutation.

1. geometry: precession  P the vector P is said to precess about O
2. physics: precession – a motion produced by an external torque
   nutation – Eulerian <u>free</u> nutation
3. astronomy – precession – precession of the equinoxes
   nutation – <u>astronomical nutation</u> – an additional 18.6 year precession of the $\vec{L}_{\oplus}$ relative to the ecliptic caused by the effect on the ⊕ of the 18.6 year regression of the lunar nodes.
   astronomers call the Eulerian free nutation the variation of latitude