Homework problem set # 4 : For the massive and/or tense.

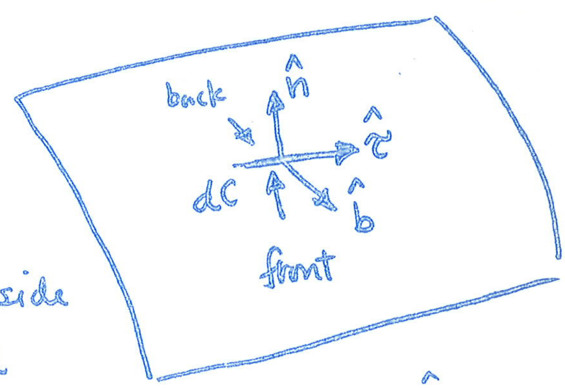
1. You are about deduce the nature of the dynamical boundary condition for a somewhat more complicated kind of boundary, a so-called massive, tense boundary. A massive, tense boundary has a net mass density and is also under isotropic surface tension. A good example of such a boundary is a soap bubble which may be considered to be the boundary between the two continua inside and outside (both air).

not true; should say: the boundary between soapy water and air.

Given an interface $\tilde{A}$ between two continua 1 and 2, let $\tilde{A}$ have a mass density μ gm/cm^2, and let its particles move with velocitys $\underline{Q}$ and accelerations $\underline{a} = \frac{D\underline{Q}}{Dt}$.

Suppose also that there is a force $\underline{f}_S$ dyne/cm^2 acting on $\tilde{A}$ (e.g. since it has mass there will be gravitational force in the Earth's field). Suppose finally that $\tilde{A}$ is under isotropic surface tension. By this we mean:

Let dC be any very short, nearly straight curve in $\tilde{A}$. Let $\hat{\tau}$ be a unit vector tangent to dC (and hence to $\tilde{A}$), let $\hat{n}$ be the unit normal to $\tilde{A}$ and let $\hat{b} = \hat{\tau} \times \hat{n}$. Then we assume that the surface material in $\tilde{A}$ just in front (side $\hat{b}$) of DC exerts on the surface material just behind DC a force $\gamma \hat{b}\, dC$, where γ is independent of $\hat{b}$.

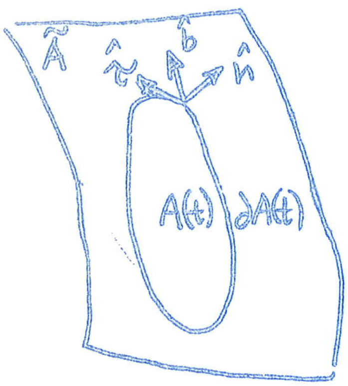


The theory can be worked out for anisotropic surface tension (one assumes only that the force is tangent to $\tilde{A}$ and proportional

to dC, permitting its magnitude and direction to depend on $\hat{\tau}$)
We shall not consider this case.

The number $\gamma(\underline{r},t)$ is the surface tension in the surface at position $\underline{r}$ at time t.

Now let's apply Newton's second law of motion to a finite patch $A(t)$ on the surface $\tilde{A}$



$$\frac{d}{dt}\int_{A(t)} \mu \underline{Q}\, dA = \int_{A(t)} \hat{n}\cdot(\underline{\underline{T}}_1 - \underline{\underline{T}}_2)\, dA + \int_{A(t)} \underline{f}_S\, dA + \int_{\partial A(t)} \gamma \hat{b}\, dC$$

this with $\hat{n}$ pointing into (1)

Now we need to express all terms in this equation as surface integrals over $A(t)$, so we can then apply the obvious extension of the vanishing integral theorem for surfaces.

(a) show that $\displaystyle \frac{d}{dt}\int_{A(t)} \mu \underline{Q}\, dA = \int_{A(t)} \mu \frac{D\underline{Q}}{Dt}$

(hint: give μ and $\underline{Q}$ their Lagrangian descriptions, and transform variables to $A(0)$, the initial position of the patch)

Now consider the integral $\displaystyle\int_{\partial A(t)} \gamma \hat{b}\, dC$

(b) show that $\displaystyle\int_{\partial A(t)} \gamma b_i\, dC = \epsilon_{ijk}\int_{\partial A(t)} (\gamma n_k \hat{x}_j)\cdot\hat{\tau}\, dC$ (easy)

For any fixed j,k this integral is in the form to which Stoke's theorem is applicable, except that the vector field $\gamma n_k \hat{x}_j$ is defined only on $\tilde{A}$, not in space. This prevents us from calculating $\nabla\times(\gamma n_k \hat{x}_j)$.

To circumvent this we use the following artifice. We simply extend the domain of definition of γ', defining γ' above and below $\tilde{A}$ <u>in any</u> <u>manner whatever</u> (so long as $\nabla\gamma'$ is continuous). What can we do about $\hat{n}$. Well, on $\tilde{A}$ if $\psi(\underline{r},t)=0$ is the equation of $\tilde{A}$, we know $\hat{n}=\nabla\psi/|\nabla\psi|$. We simply <u>define</u> $\hat{n}$ at points not on $\tilde{A}$ in the same way $\hat{n}=\nabla\psi/|\nabla\psi|$. Now we can calculate $\nabla\times(\gamma' n_k \hat{x}_j)$. Pretty sly, huh?.

(c) now you may use Stoke's theorem to show that

$$\int_{\partial A(t)} \gamma' \hat{b}\, dC = \int_{A(t)} \left[-\gamma\hat{n}(\nabla\cdot\hat{n}) + \left(\nabla\gamma - \hat{n}(\hat{n}\cdot\nabla\gamma)\right)\right] dA$$

Now we have everything tidily tucked away under a $\int_{dA(t)}$ sign, we apply the vanishing integral theorem

The differential form of Newton's law for a massive Tense interface is $[\hat{n}\cdot T]^{\pm}$

$$\boxed{\mu \frac{DQ}{Dt} = \hat{n}\cdot(\underline{\underline{T}}_{(1)} - \underline{\underline{T}}_{(2)}) + \underline{f}_S - \gamma\hat{n}(\nabla\cdot\hat{n}) + [\nabla\gamma - \hat{n}(\hat{n}\cdot\nabla\gamma)] \quad (*)}$$

This is the required dynamical boundary condition. Let's look at it. Look first at $[\nabla\gamma' - \hat{n}(\hat{n}\cdot\nabla\gamma')]$. This quantity can be calculated by differentiating γ' only in directions tangent to $\tilde{A}$ ($\nabla\gamma - \hat{n}(\hat{n}\cdot\nabla\gamma)$ is the $\perp$ projection of $\nabla\gamma'$ onto $\tilde{A}$. Thus the arbitrary way in which we extended the definition of γ above and below $\tilde{A}$ produces no arbitrariness in (*). If it did, we'd be in trouble

If (*) is to be of any use, $\nabla\cdot\hat{n}$ must be calculated

(d). Let $\psi(\underline{r},t)=0$ be the equation of $\tilde{A}$. Show that

$$\nabla\cdot\hat{n} = \partial n_i/\partial r_i = \frac{|\nabla\psi|^2(\nabla^2\psi) - \sum_{i,j} \frac{\partial\psi}{\partial r_i}\frac{\partial\psi}{\partial r_j}\left(\frac{\partial^2\psi}{\partial r_i \partial r_j}\right)}{|\nabla\psi|^3}$$

(e) let $\tilde{A}$ be the surface of a sphere of radius a, center at origin

$$\psi(\underline{x}) = (x_1)^2 + (x_2)^2 + (x_3)^2 - a^2$$

Show that $\nabla \cdot \hat{n} = \frac{2}{a}$

It is a well-known fact in differential geometry that in general $\nabla \cdot \hat{n}$ is the sum of the two principal curvatures of $\tilde{A}$. The above result is a special case.

Specialize now to a surface on which $\mu = 0$ (massless) and $\underline{f}_S = 0$ (no "body" forces acting)

Furthermore let γ be constant on $\tilde{A}$ (no surface tension gradient)

Furthermore let the continua (1) and (2) have isotropic stress tensors $\underline{\underline{T}}_{(1)} = -P_{(1)} \underline{\underline{I}}$, $\underline{\underline{T}}_{(2)} = -P_{(2)} \underline{\underline{I}}$.

(f) Show that the b.c. reduces to

$$\boxed{P_{(2)} - P_{(1)} = +\gamma \nabla \cdot \hat{n}}$$

this with $\hat{n}$ pointing into (1)

$$[p]^{+}_{-} = -\gamma \nabla \cdot \hat{n}$$

$\hat{n}$ points to + side

(g) assume air and water have isotropic stress tensors

Find the pressure (in atmospheres) inside a raindrop 10^{-4} cm in diameter. Use $\gamma = 72$ dyne/cm for distilled H_2O

(h) If $\gamma = 24$ dyne/cm for soapy air water interface, how does the pressure inside a soap bubble depend on its radius?

(i) Find $\nabla \cdot \hat{n}$ on the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Massive tense answers #4:

(a) $\frac{d}{dt}\int_{A(t)} \mu Q \, dA$. Let u, v be a parameterization of $A(0)$ at time zero. As (u,v) runs over a certain region B in the u,v plane, the vector $\underline{r}_0(u,v)$ runs over the surface $A(0)$. At any later time t, the particle at $\underline{r}_0(u,v)$ has moved to a point $\underline{r}_t(u,v)$ on $A(t)$. Now $dA = \left|\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right| du\,dv$ so

$$\int_{A(t)} \mu \, dA = \int_B \mu_t \left|\frac{\partial \underline{r}_t}{\partial u} \times \frac{\partial \underline{r}_t}{\partial v}\right| du\,dv.$$ But then

$$\int_{A(0)} \mu_0 \, dA_0 = \int_B \mu_0 \left|\frac{\partial \underline{r}_0}{\partial u} \times \frac{\partial \underline{r}_0}{\partial v}\right| du\,dv$$ so by the vanishing integral theorem

$$\mu_0 \left|\frac{\partial \underline{r}_0}{\partial u} \times \frac{\partial \underline{r}_0}{\partial v}\right| = \mu_t \left|\frac{\partial \underline{r}_t}{\partial u} \times \frac{\partial \underline{r}_t}{\partial v}\right| \quad \text{(conservation of surface mass)}$$

Then
$$\int_{A(t)} \mu Q \, dA = \int_B \mu_t \left|\frac{\partial \underline{r}_t}{\partial u} \times \frac{\partial \underline{r}_t}{\partial v}\right| Q \, du\,dv = \int_B \mu_0 \left|\frac{\partial \underline{r}_0}{\partial u} \times \frac{\partial \underline{r}_0}{\partial v}\right| Q \, du\,dv$$

Hence
$$\frac{d}{dt}\int_{A(t)} \mu Q \, dA = \frac{d}{dt}\int_B \mu_0 \left|\frac{\partial \underline{r}_0}{\partial u} \times \frac{\partial \underline{r}_0}{\partial v}\right| Q \, du\,dv = \int_B \mu_0 \left|\frac{\partial \underline{r}_0}{\partial u} \times \frac{\partial \underline{r}_0}{\partial v}\right| \frac{\partial Q}{\partial t} \, du\,dv$$

Here $\frac{\partial Q}{\partial t}$ means the derivative of Q holding u, v fixed, i.e. holding the particle fixed. Thus $\frac{\partial Q}{\partial t} \equiv \frac{DQ}{Dt}$, and

$$\frac{d}{dt}\int_{A(t)} \mu Q \, dA = \int_B \mu_0 \left|\frac{\partial \underline{r}_0}{\partial u} \times \frac{\partial \underline{r}_0}{\partial v}\right| \frac{DQ}{Dt} \, du\,dv = \int_B \mu_t \left|\frac{\partial \underline{r}_t}{\partial u} \times \frac{\partial \underline{r}_t}{\partial v}\right| D_t Q \, du\,dv$$

$$= \int_{A(t)} \mu D_t Q \, dA$$

(b) $\hat{b} = \hat{\tau} \times \hat{n}$ so $b_i = \epsilon_{ijk} \tau_j n_k = (\epsilon_{ijk} n_k \hat{x}_j) \cdot \hat{\tau}$. Thus

$$\int_{\partial A} \gamma b_i \, dC = \epsilon_{ijk} \int_{\partial A} (\gamma n_k \hat{x}_j) \cdot \hat{\tau} \, dC$$

(c)

$$\nabla\times(\gamma n_k \hat{x}_j) = -\hat{x}_j \times \nabla(\gamma n_k) = -\hat{x}_j \times \hat{x}_l\, \partial_l(\gamma n_k)$$

$$= -\epsilon_{jlm}\hat{x}_m\, \partial_l(\gamma n_k)$$

Thus $$\int_{\partial A}\gamma b_i\, dC = \epsilon_{ijk}\int_A \hat{n}\cdot[\nabla\times(\gamma n_k \hat{x}_j)]\, dA \quad \text{(Stokes)}$$

$$= -\epsilon_{ijk}\epsilon_{jlm}\int_A \hat{n}\cdot\hat{x}_m\, \partial_l(\gamma n_k)\, dA = -\int_A(\delta_{im}\delta_{kl}-\delta_{il}\delta_{km})\, n^m \partial_l(\gamma n_k)\, dA$$

The integrand is $n_i\partial_k(\gamma n_k) - n_k\partial_i(\gamma n_k)$

$$= \gamma n_i \partial_k n_k + n_i(n_k\partial_k\gamma) - n_k n_k \partial_i\gamma - \gamma n_k \partial_i n_k$$

But $n_k n_k = 1$ so $n_k\partial_i n_k = 0$

$$\int_{\partial A}\gamma b_i\, dC = -\int_A[\gamma n_i(\nabla\cdot\hat{n}) + n_i(\hat{n}\cdot\nabla\gamma) - \partial_i\gamma]\, dA$$

$$\int_{\partial A}\gamma\hat{b}\, dC = \int_A\Big[-\gamma\hat{n}(\nabla\cdot\hat{n}) + \underbrace{\big(\nabla\gamma - \hat{n}(\hat{n}\cdot\nabla\gamma)\big)}_{\text{the so-called \underline{surface} gradient}}\Big]\, dA$$

(d) $$\hat{n} = \frac{\nabla\psi}{|\nabla\psi|}, \qquad n_i = \frac{\partial\psi}{\partial r_i}\left(\frac{\partial\psi}{\partial r_j}\frac{\partial\psi}{\partial r_j}\right)^{-1/2}$$

$$\nabla\cdot\hat{n} = \partial_i n_i = \frac{|\nabla\psi|^2\nabla^2\psi - \underline{\nabla\psi}\cdot\underline{\underline{\nabla\nabla\psi}}\cdot\underline{\nabla\psi}}{|\nabla\psi|^3}$$

(e) $|\nabla\psi| = 2a$

$\nabla^2\psi = 6$ $\qquad\qquad$ $\nabla\cdot\hat{n} = \frac{2}{a}$

$\underline{\nabla\psi}\cdot\underline{\underline{\nabla\nabla\psi}}\cdot\underline{\nabla\psi} = 8a^2$

$\hat{n}$ pointing into (1).

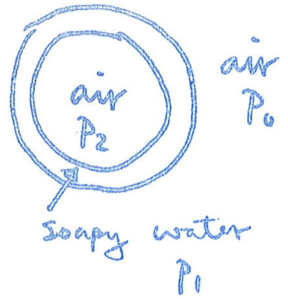
(f) easy $\quad P(2) - P(1) = +\gamma(\nabla\cdot\hat{n})$

(g) radius $= \frac{1}{2}\cdot 10^{-4}$ cm, $\quad \frac{2}{a} = 4\cdot 10^{4}$ cm^{-1}

$P_{in} - P_{out} = (4\cdot10^4)(72) = 2.9\cdot 10^6$ dyne/cm^2 = 2.9 atm.

if P_{out} = 1 atm $\qquad P_{in}$ = 3.9 atm.

(b)



$$P_1 - P_0 = \frac{2}{a}\gamma$$

$$P_2 - P_1 = \frac{2}{a}\gamma$$

$$\boxed{P_2 - P_0 = \frac{4}{a}\gamma}$$

(c) $\psi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$$\nabla\psi = \frac{2x}{a^2}\hat{x} + \frac{2y}{b^2}\hat{y} + \frac{2z}{c^2}\hat{z}$$

$$\nabla^2\psi = \frac{2}{a^2} + \frac{2}{b^2} + \frac{2}{c^2}$$

$$\underline{\nabla}\psi \cdot \underline{\underline{\nabla\nabla}}\psi \cdot \underline{\nabla}\psi = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)\begin{pmatrix} \frac{2}{a^2} & 0 & 0 \\ 0 & \frac{2}{b^2} & 0 \\ 0 & 0 & \frac{2}{c^2} \end{pmatrix}\begin{pmatrix} \frac{2x}{a^2} \\ 2y/b^2 \\ 2z/c^2 \end{pmatrix}$$

$$= 8\left(\frac{x^2}{a^6} + \frac{y^2}{b^6} + \frac{z^2}{c^6}\right)$$

$$\nabla\cdot\hat{n} = \frac{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - \left(\frac{x^2}{a^6} + \frac{y^2}{b^6} + \frac{z^2}{c^6}\right)}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{3/2}}$$

Not all problems have pretty answers.