II. The mechanics of fluids (non-viscous)

References:

general: Landau & Lifshitz, Fluid Mechanics
Batchelor, Intro. to Fluid Dynamics

classical: Lamb, Hydrodynamics – mostly homogeneous perfect fluids

special: Eckart, Hydrodynamics of Oceans and Atmospheres – linearized theory of non-homogeneous perfect fluids

Turner: Buoyancy Effects in Fluids

Tritton: Physical Fluid Dynamics

1. Hydrostatics

Definition: a fluid is a continuum whose stress tensor is always and everywhere isotropic when the continuum is at rest in some reference frame. If we write $\underline{\underline{T}}(\underline{r}) = -p(\underline{r})\underline{\underline{I}}$ then $p(\underline{r})$ is called the fluid pressure at posn $\underline{r}$.

No mention is made of fluids in motion. The above defn applies to viscous fluids as well.

Suppose a fluid is at rest in a gravitational field $\underline{g}(\underline{r}) = -\nabla\phi(\underline{r})$. The body force is thus $\rho(\underline{r})\underline{g}(\underline{r}) = \underline{f}(\underline{r})$. The momentum eqn becomes

$$\rho \frac{D\underline{u}}{Dt} = \nabla \cdot \underline{\underline{T}} + \underline{f} \qquad \text{or}$$

since $\underline{u} = \underline{0}$ $\qquad 0 = -\nabla p - \rho \nabla \phi$

$$\boxed{\nabla p + \rho \nabla \phi = 0}$$

This is the hydrostatic equation or the eqn of hydrostatic equilibrium.

This eqn may be used to prove the so-called

<u>Fundamental Theorem of Hydrostatics</u>
If a fluid is at rest in an inertial frame under the influence of a grav. pot. $\phi(\underline{r})$ and no other body forces, then the level surfaces of constant density, pressure, and grav. pot all coincide. That is $p(\underline{r})$ and $\rho(\underline{r})$ are fcns of $\phi(\underline{r})$ alone.

proof: first we prove $\forall\, f(\underline{r}), h(\underline{r})$

<u>lemma</u>: The functions $f(\underline{r})$ and $h(\underline{r})$ have the same level surfaces iff $\nabla f \times \nabla h = 0$

~~[illegible]~~ proof: If f and h have the same level surfaces then their normals $\nabla f(\underline{r})$ and $\nabla h(\underline{r})$ have everywhere the same direction so $\nabla f \times \nabla h = 0$. For the converse consider a particle starting at $\underline{r}_0$ and moving so that $f(\underline{r}) = f(\underline{r}_0)$ i.e. on the level surface of $f(\underline{r})$ thru $\underline{r}_0$. Then $Df/Dt = 0$ or $\partial f/\partial t + d\underline{r}/dt \cdot \nabla f = 0$ but $\partial f/\partial t = 0$ so $d\underline{r}/dt \cdot \nabla f = 0$. But if $\nabla f \times \nabla h = 0$ then $d\underline{r}/dt \cdot \nabla h = 0$ also. Since $\partial h/\partial t = 0$, also $Dh/Dt = 0$ hence $h(\underline{r}) = h(\underline{r}_0)$. Any curve lying in a level surface of f lies in a level surface of h. Q.E.D.

now we proceed

clearly $\nabla p + \rho \nabla \phi = 0 \Rightarrow \nabla p = -\rho \nabla \phi$

so $\nabla p \times \nabla \phi = -\rho \nabla \phi \times \nabla \phi = 0$

hence

$$\nabla p \times \nabla \phi = 0$$

now take the curl of $\nabla p + \rho \nabla \phi = 0$

$$\underbrace{\nabla \times \nabla p}_{=0} + \rho \underbrace{\nabla \times \nabla \phi}_{=0} + \nabla \rho \times \nabla \phi = 0$$

$$\nabla \rho \times \nabla \phi$$

then also

$$\nabla \rho \times \nabla p = 0$$

and the fund. thm. is proven.

Best order of presentation

Thermostatics of fluids (single-phase)

$\nabla p + \rho \nabla \phi$ guarantees mechanical equilibrium

Consider the thermostatics of a fluid at rest.

Concept of local thermostatic equilibrium.

A parcel of fluid at $\underline{x}$ has a caloric eqn of state

$$u = \tilde{u}(\tau, s) \qquad \text{fund. eqn}$$

Defn: of fluid, equil. state completely described by two state variables.

From first law of thermo $\quad \theta = \left.\frac{\partial \tilde{u}}{\partial s}\right|_{\tau}, \; -p = \left.\frac{\partial \tilde{u}}{\partial \tau}\right|_{s}$

Convenient to use θ and ρ as state variables

$$\left.\begin{aligned} p &= \tilde{p}(\theta, \rho) \\ s &= \tilde{s}(\theta, \rho) \\ u &= \tilde{u}(\theta, \rho) \end{aligned}\right\| \quad \text{thermal eqns of state}$$

Thermal equil. condition $\quad \theta(\underline{r}, t) = \theta_0 = \text{constant}$

A fluid can be in mech. equil. without being in thermal equil.

Note: eqn of state is same for all particles defn of a homog. fluid. In a grav field, properties, e.g. pressure need not be uniform

Now define a non-viscous fluid (in l.t.e. even when moving)

Review:

defn of a fluid:

1. mechanical: at rest, $\underline{\underline{T}}(\underline{r}) = -p(\underline{r})\underline{\underline{I}}$
2. thermal: at rest, local thermostatic equil, only 2 state variables required for a complete description
   fund. eqn. of state $U = \tilde{U}(S, \tau)$
   applicable at every pt. $\underline{r}$
   $U(\underline{r}) = \tilde{U}(S(\underline{r}), \tau(\underline{r}))$
3. geometric, even when moving, isotropic
   any little moving parcel looks the same from all directions (we define this more carefully later)

this obvious in hydrostatic situation, since all relevant variables are scalars

fund. thm. of hydrostatics in a grav. pot. $\phi(\underline{r})$
all of $\phi(\underline{r})$, $\rho(\underline{r}) = \tau^{-1}(\underline{r})$, $S(\underline{r})$, $\theta(\underline{r})$, $U(\underline{r})$, $p(\underline{r})$ have the same level surfaces

defn of a non-viscous fluid:

1. it is a fluid (1.2.3. above)
   ~~[illegible] even when in motion~~
2. $\underline{\underline{T}}(\underline{r}, t) = -p(\underline{r}, t)\,\underline{\underline{I}}$ even when in motion
3. in l.t.e. even when in motion.

entropy conservation law for a non-viscous fluid

HYDRODYNAMICS OF OCEANS AND ATMOSPHERES

by

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2. Thermodynamics

The state of a pure fluid can be specified by any two of the four variables v, p, θ, η, which are respectively its specific volume, pressure, temperature and entropy.[3,6] Its internal energy, ϵ, (erg/g) is best expressed as a function of v and η. Then its equation of state can be obtained by eliminating η between the two fundamental equations

$$p = -\partial\epsilon/\partial v, \qquad \theta = \partial\epsilon/\partial\eta. \tag{1}$$

These can also be used to transform from one pair of independent variables to another.

On differentiating Eq. (2–1) the result may be written

$$\begin{aligned} \delta p &= -X\,\delta v + Y\,\delta\eta, \\ \delta\theta &= -Y\,\delta v + Z\,\delta\eta, \end{aligned} \tag{2}$$

where $+X$, $-Y$ and Z are the three second derivatives of ϵ. They can be expressed in terms of more commonplace quantities as follows:

$$\begin{aligned} X &= \rho^2 c^2, \\ Y &= \rho(\gamma-1)/a, \\ Z &= \theta/C_v, \end{aligned} \tag{3}$$

where $\rho = 1/v$ = density,
c = velocity of sound,
a = coefficient of thermal expansion,
C_v = specific heat, constant volume,
γ = ratio of specific heats.

The derivation of these equations is as follows: The definition of the velocity of sound, as given in text books on acoustics,[9] may be reduced to

$$c^2 = \left(\frac{\delta p}{\delta\rho}\right)_{\delta\eta=0}.$$

Since $\delta v = -\delta\rho/\rho^2$, the first of Eqs. (2–1) immediately yields $X = \rho^2 c^2$. It will be noted that ρc is the acoustic impedance of the fluid.[9] In the same way, it is a matter of definition that

$$C_v = \left(\frac{\delta\epsilon}{\delta\theta}\right)_{\delta v=0} = \theta\left(\frac{\delta\eta}{\delta\theta}\right)_{\delta v=0},$$

whereupon it follows at once that $Z = \theta/C_v$. To establish the formula for Y, it is first necessary to calculate the specific heat at constant pressure, defined by

$$C_p = \left(\frac{\delta\epsilon + p\,\delta v}{\delta\theta}\right)_{\delta p=0} = \theta\left(\frac{\delta\eta}{\delta\theta}\right)_{\delta p=0}.$$

When $\qquad \delta p = 0, \qquad \delta v = (Y/X)\,\delta\eta,$

and so $\qquad \delta\theta = \left(1 - \frac{Y^2}{XZ}\right) Z\,\delta\eta.$

From this, it follows that

$$C_p = \left(1 - \frac{Y^2}{XZ}\right)^{-1} C_v,$$

or

$$C_v/C_p = 1/\gamma = 1 - \frac{Y^2}{XZ}.$$

Consequently

$$Y^2 = \frac{\gamma-1}{\gamma} XZ = \frac{(\gamma-1)\rho^2 c^2\theta}{\gamma C_v}. \tag{4}$$

This equation would enable one to determine Y (except for sign) without reference to the coefficient of thermal expansion. This last is defined as

$$a = \frac{1}{v}\left(\frac{\delta v}{\delta\theta}\right)_{\delta p=0}.$$

Using the foregoing expression for γ, this leads to $Y = \rho(\gamma-1)/a$, as stated above. Equation (2–4) may then be transformed to

$$\gamma(\gamma-1)C_v = a^2c^2\theta, \tag{5}$$

which is a thermodynamic identity of some importance.

3. The thermodynamic functions of an ideal gas

As a more concrete example of the general thermodynamic relations, and one that will be useful later, the case of an ideal gas may be considered. An ideal gas is one whose molecules are all identical and interact only by collisions, moving in straight lines between collisions. Its specific heats are constants. The general principles of statistical mechanics(7) can then be used to show that its internal energy

$$\epsilon = Av^{-\gamma+1} \exp [(\gamma - 1)\eta/R], \tag{1}$$

A, γ and R being constants.

In particular, γ is the ratio of specific heats, as defined above. For air of molecular weight 29, $R = 2{\cdot}87 \times 10^6$ erg/g deg and $\gamma = 1{\cdot}40$. The value of A will not be needed.

Then it follows from Eq. (2–1) that

$$p = -\frac{\partial \epsilon}{\partial v} = (\gamma - 1)Av^{-\gamma} \exp [(\gamma - 1)\eta/R], \tag{2}$$

$$\theta = \frac{\partial \epsilon}{\partial \eta} = [(\gamma - 1)A/R]v^{-\gamma+1} \exp [(\gamma - 1)\eta/R].$$

These equations may be transformed into

$$\begin{aligned} pv &= R\theta, \\ pv^{\gamma} &= (\gamma - 1)A \exp [(\gamma - 1)\eta/R], \end{aligned} \tag{3}$$

which are the *equation of state* and the *equation of the isentropes*, respectively.

The second derivative of ϵ yields

$$\begin{aligned} X &= \frac{\partial^2 \epsilon}{\partial v^2} = \gamma(\gamma - 1)Av^{-\gamma-1} \exp [(\gamma - 1)\eta/R] \\ &= \gamma \rho^2 R\theta. \end{aligned} \tag{4}$$

Combined with the previous result $X = \rho^2 c^2$, it follows that the velocity of sound, c, is given by Laplace's formula:

$$c^2 = \gamma R\theta. \tag{5}$$

Similarly

$$\begin{aligned} +Y &= -\frac{\partial^2 \epsilon}{\partial v\, \partial \eta} = [(\gamma - 1)^2/R]Av^{-\gamma} \exp [(\gamma - 1)\eta/R] \\ &= (\gamma - 1)\rho\theta. \end{aligned} \tag{6}$$

Combined with $Y = \rho(\gamma - 1)/a$, this shows that the coefficient of thermal expansion is

$$a = 1/\theta. \tag{7}$$

Finally

$$\begin{aligned} Z &= \frac{\partial^2 \epsilon}{\partial \eta^2} = [(\gamma - 1)/R]^2 Av^{-\gamma+1} \exp [(\gamma - 1)\eta/R] \\ &= (\gamma - 1)\theta/R. \end{aligned} \tag{8}$$

From $Z = \theta/C_v$, one finds that the specific heat is

$$C_v = R/(\gamma - 1). \tag{9}$$

Substitution of Eqs. (3–5), (3–7) and (3–9) into Eq. (2–5) verifies that the latter is satisfied.

Last term we utilized the fundamental theorem of hydrostatics to deduce the equil. shape of a rotating Earth in hydrostatic equilibrium (Clairaut's eqn).

This concludes our discussion of hydrostatics

2. Non-viscous fluids

Introduce concept of eqn of state first in hydrostatic situation.

Defn: A homogeneous non viscous fluid is a fluid which, even when moving, has two properties:

1. its stress tensor is always isotropic $\underline{\underline{T}}(\underline{r},t) = -p(\underline{r},t)\,\underline{\underline{I}}$, and furthermore the fluid pressure $p(\underline{r},t)$ is always the local thermodynamic equilibrium pressure determined by the equation of state from the local temperature and density.

always in a state of local thermodynamic equil. at every pt. $\underline{r}, t$

thus at every pt. $\underline{r},t$ $p(\underline{r},t)$ is a fcn $\tilde{p}(\rho(\underline{r},t), \theta(\underline{r},t))$

$$p(\underline{r},t) = \tilde{p}(\rho(\underline{r},t), \theta(\underline{r},t))$$

$p = \tilde{p}(\rho, \theta)$ is called the eqn of state of the fluid

2. Since the non-viscous fluid is always in a state of <u>local</u> thermodynamic equilibrium, then at every pt. $\underline{r}, t$ the <u>entropy</u> S per gram $S(\underline{r},t)$ is a well-defined fcn of state

$$S(\underline{r},t) = \tilde{S}\,(\rho(\underline{r},t)\,,\,\theta(\underline{r},t))$$

The entropy $S(\underline{r},t)$ is related to the other state variables also by the first law of thermodynamics

$S(\underline{r},t) \equiv$ entropy per gram
$\theta(\underline{r},t) \equiv$ temp.
$\tau(\underline{r},t) \equiv$ specific volume or vol. per gm.

$$\theta\, DS = DU + p\, D\tau \qquad \text{or}$$

$$\theta\, D_t S = D_t E + p\, D_t \tau$$

note use of D
one applies first law to an inf. small <u>parcel of fluid</u> about a material particle $\underline{x}$.

But the <u>energy</u> <u>equation</u> is

$$\rho\, D_t U + \nabla \cdot \underline{H} = \operatorname{tr}\,(\underline{\underline{T}}^T \cdot \nabla \underline{u}) + h$$

in a non-viscous fluid

$$\mathrm{tr}(\underline{\underline{T}}^T \cdot \underline{\underline{\nabla u}}) = -p\nabla\cdot\underline{u} = (\text{by cont. eqn.}) = -p D_t \tau$$

$$\rho(D_t U + p D_t \tau) + \nabla\cdot\underline{H} = h$$

$$\rho\theta D_t S + \nabla\cdot\underline{H} = h$$

or dividing by θ

$$\boxed{\rho D_t S + \nabla\cdot(\underline{H}/\theta) = h/\theta + \underline{H}\cdot\nabla\frac{1}{\theta}}$$

Thus in a non-viscous fluid, the first law of thermo. can be written as a conservation law for the entropy.
This is true in general whenever the entropy $S(\underline{r},t)$ can be written as a fcn of state of the relevant thermodynamic variables.

~~For non-viscous fluids, we have enough a complete system of eqns.~~
~~Unknowns $\rho, \theta, S, p, \underline{u}, \underline{H}$~~
~~if we assume a heat flow eqn (and if we assume $\underline{f}$, h given)~~
~~Then unknowns $\rho, \theta, S, p, \underline{u}, \underline{H}$~~

lets examine this cons. law. It can be put in the usual form by adding the continuity eqn.

$$\rho(\partial_t S + \underline{u}\cdot\nabla S) + \nabla\cdot(\underline{H}/\theta) = \frac{h}{\theta} + H\cdot\nabla\frac{1}{\theta}$$

$$\partial_t \rho + \nabla\cdot(\rho\underline{u}) = 0$$

$$\boxed{\partial_t(\rho S) + \nabla\cdot(\rho S\underline{u} + \underline{H}/\theta) = \frac{h}{\theta} + H\cdot\nabla\frac{1}{\theta}}$$

ϕ = entropy

$\phi = \rho S$ = entropy density (per unit volume)

$\underline{K} = \rho S\underline{u} + \underline{H}/\theta$ current density of entropy in space

$\rho S\underline{u}$: advected entropy

$\underline{H}/\theta$: current of entropy due to heat flow

$k = h/\theta + \underline{H}\cdot\nabla\frac{1}{\theta}$ = rate of production of entropy per cm^3 per sec

h/θ: entropy prod. due to internal heating

$\underline{H}\cdot\nabla\frac{1}{\theta}$: entropy production due to heat flowing down temp. gradients (vanishes in complete thermo. equilibrium)

See remarks on p. 121

For non-viscous fluids we have a complete system of eqns only if we assume a heat flow eqn

unknowns $\rho, \theta, S, p, \underline{u}, H$ ($\underline{f}, h$ given) equations:

continuity: $\partial_t \rho + \nabla\cdot(\rho \underline{u}) = 0$

momentum: $\rho D_t \underline{u} + \nabla p = \underline{f}$

entropy: $\rho D_t S + \nabla\cdot(\underline{H}/\theta) = h/\theta + H\cdot\nabla\frac{1}{\theta}$

two eqns of state: $p = \tilde{p}(\rho, \theta)$, $S = \tilde{S}(\rho, \theta)$

necessary to also assume → heat flow eqn: often one takes the simple linear relation known as Fourier's law $\underline{H} = -\kappa \underline{\nabla\theta}$ (first term in a Taylor series)

where $\kappa \equiv$ thermal conductivity

remark on isotropic fluid ⇒ $\underline{\underline{\kappa}}$ is isotropic tensor

10 eqns, 10 unknowns

Also in general $\kappa(r,t) = \tilde{\kappa}(\rho(\underline{r},t), \theta(\underline{r},t))$

must in general be used in any convection problem.

What form does the Clausius – Duhem inequality take for a non-viscous fluid?

$$\rho D_t S + \nabla\cdot(\underline{H}/\theta) - h/\theta \geq 0$$

this is second law of thermo. for any continuum. For a non-viscous fluid

$$\rho D_t S + \nabla\cdot(\underline{H}/\theta) - h/\theta = H\cdot\nabla\frac{1}{\theta}$$

hence

$$\boxed{H\cdot\nabla\frac{1}{\theta} \geq 0}$$

or

$$-\frac{1}{\theta^2} H\cdot\nabla\theta \geq 0$$

or just

$$\boxed{-\underline{H}\cdot\nabla\theta \geq 0}$$

$$\boxed{\underline{H} \cdot \underline{\nabla}\theta \le 0}$$

This is just a statement that heat must flow from hot temperatures to cold temperatures and indeed this is the main import of the second law. Note that if we assume ~~[illegible]~~ Fourier's law as an approx. heat flow eqn

remark here: $\underline{\underline{K}} = K \underline{\underline{I}}$ a necessary consequence of part 3 of defn of fluid

$$-[-K(\nabla\theta \cdot \nabla\theta)] = K\,|\nabla\theta|^2 \geq 0$$

so the thermal conductivity

$$K \geq 0$$

The second law (Clausius – Duhem inequal.) has been used to impose a necessary restriction on a proposed constitutive eqn. This is usually its main usefulness. We'll see this again for viscous fluids and for elastic solids.

Non-viscous fluids have 10 eqns, 10 unknowns, and we must assume separately a law of heat flow. Still a very complicated system. In many circumstances it is a very good approx. to make the even simpler assumption

stop here 28 Feb, 1972

Variable counting

- $D_t \rho + \rho \nabla \cdot \underline{u} = 0$
- $\rho D_t \underline{u} + \nabla p = \rho \underline{g}$ given
- $\rho D_t \mathcal{U} + \nabla \cdot \underline{H} = h - p(\nabla \cdot \underline{u})$ reversible — no dissipation

$$\rho D_t S + \nabla \cdot (\underline{H}/\theta) = \frac{h}{\theta} + H \cdot \nabla \left(\frac{1}{\theta}\right)$$

no entropy generation due to motion

$\rho, \underline{u}, p, \theta, \mathcal{U}, \underline{H}$ — 10 unknowns (let's forget about S)

5 cons. laws •

2 equs of state — say use p, θ as ind. variables

$$\rho = \rho(p,\theta) \qquad \mathcal{U} = \mathcal{U}(p,\theta)$$

ideal gas

$$p\tau = \tilde{R}\theta \text{ or } \rho = \frac{p}{\tilde{R}\theta}$$

$$\mathcal{U} = c_v \theta$$

3 equs — Fourier's law $\underline{H} = -k \nabla \theta$

If S included as unknown need $S = S(p,\theta)$

e.g. ideal gas $S = \frac{\tilde{R}}{\gamma - 1} \ln \left[\frac{c_v \theta}{A} \left(\frac{\tilde{R}\theta}{p} \right)^{1-\gamma} \right]$

Lecture # 12 Review

Began discussion of fluid mechanics

hydrostatics: a fluid is a continuum in which $\underline{\underline{T}}(\underline{r}) = -p(\underline{r})\underline{\underline{I}}$ (isotropic) when the cont. is at rest.

in a gravitational field $\underline{g}(\underline{r}) = -\nabla\phi(\underline{r})$

$$\nabla p + \rho\nabla\phi = 0$$

prove fundamental theorem: level surfaces of $p(\underline{r})$, $\rho(\underline{r})$, $\phi(\underline{r})$ coincide. Thus $p(\underline{r})$ and $\rho(\underline{r})$ are fcns of grav. altitude alone
If fluid is in a state of local thermodynamic equilibrium.

$$\left.\begin{aligned} \theta(\underline{r}) &= \tilde{\theta}(p(\underline{r}), \rho(\underline{r})) \\ S(\underline{r}) &= \tilde{S}(p(\underline{r}), \rho(\underline{r})) \end{aligned}\right\} \text{eqns of state}$$

Thus $\theta(\underline{r})$ and $S(\underline{r})$ are also fcns only of grav. altitude.

What is meant by local thermodynamic equilibrium?

$\nabla p + \rho\nabla\phi = 0$ hydrostatic eqn only guarantees mechanical equilibrium.

To be in complete thermal equilibrium as well, the fluid must be all at the same temp. (isothermal)

Thus a fluid can be in mechanical equilibrium (at rest) and not in thermal equilibrium. In any such case there will in general be heat flowing $\underline{H}$

A fluid will be said to be in local thermodynamic equilibrium whenever at any pt. $\underline{r} \in$ fluid, the equations of state of the fluid can be used to deduce any thermodynamical variable $\rho(\underline{r})$, $p(\underline{r})$, $\theta(\underline{r})$, $S(\underline{r})$, $U(\underline{r})$ in terms of just two variables.

non viscous fluids: fluids in local thermodynamic equil even when moving

1. $\underline{\underline{T}}(\underline{r}, t) = -p(\underline{r}, t)\,\underline{\underline{I}}$ isotropic
   $p(\underline{r}, t) = \tilde{p}\,(\theta(\underline{r}, t), \rho(\underline{r}, t))$
2. $S(\underline{r}, t) = \tilde{S}\,(\theta(\underline{r}, t), \rho(\underline{r}, t))$

Two eqns of state locally satisfied.
For a non-viscous fluid, energy eqn can be written as a conservation law for entropy

$$\boxed{\partial_t(\rho S) + \nabla\cdot\left(\rho S \underline{u} + \underline{H}/\theta\right) = h + H\cdot\nabla\frac{1}{\theta}}$$

<u>entropy generation</u> by

1. internal heating h/θ
2. <u>heat flow</u> down <u>temp gradients</u> $\underline{H}\cdot\nabla\frac{1}{\theta}$

NOTE: <u>no entropy generation due to motion of fluid</u>. This is key aspect of non-viscous fluid. Entropy only generated by heat flowing (and heat input).

The heat flow gives rise to no <u>mechanical effects</u>; it only contributes to the generation of entropy

Thus in a fluid <u>at rest</u> (in mechanical equil.) but <u>not</u> in complete thermal equil (not isothermal, so $\nabla\theta \neq 0$)

Entropy conservation law takes form

$$\partial_t(\rho S) + \nabla\cdot(\underline{H}/\theta) = h + \underline{H}\cdot\nabla\frac{1}{\theta}$$

exactly the same except no <u>entropy advection</u>

In a fluid in mechanical equil. but not in thermal equilibrium, $\exists$ entropy generation due to 1. internal heating h and 2. heat <u>flow</u> trying to attain thermal equil. $\underline{H}$.
(since <u>irreversible</u>, generates entropy)

In general a non-viscous fluid has 10 unknowns $\rho, \theta, S, p, \underline{u}, \underline{H}$ and 10 eqns (if one assumes a heat flow eqn)

In many circumstances it is a very good approximation to make the even simpler assumption.

3. ~~Inhomogeneous~~ Baroclinic Perfect Fluids

Defn: an inhomogeneous perfect fluid is one in which at each particle $\underline{x}$ at each instant t, the density ρ is a known fcn of $\underline{x}, t$ and the pressure p

Note

Note: better to leave out t as an argument in this discussion

$$\boxed{\rho = \tilde{\rho}(p, \underline{x}, t)}$$

maybe better to write
$p = \tilde{P}(\rho, \underline{x}, \cancel{t})$

there is no explicit dependence on the the temp θ or the entropy S.
The function $\tilde{\rho}$ may however be different for different particles $\underline{x}$ of the fluid.

In an inhomog. perfect fluid, one has a complete system of eqns in terms of only mechanical variables (no thermodynamic quantities need enter)

unknowns	equations
$\rho, \underline{u}, p, \underline{x}$ all fcns of $\underline{r}, t$	$D_t\rho + \rho\nabla\cdot\underline{u} = 0$ $\rho D_t \underline{u} = -\nabla p + \underline{f}$ $\rho = \tilde{\rho}(p, \underline{x}, t)$ $\underline{u} = D_t \underline{r}(\underline{x}, t)$

System of 8 eqns, 8 unknowns
There are 3 simple examples where this approx. might be valid.

Examples where non-viscous fluids may be approximated as inhomog. perfect fluids. What are the necessary circumstances?

$\boxed{D_t \rho \equiv 0}$

Example 1: a fluid has such a small compressibility and coeff. of thermal expansion that $D_t \rho = 0$ is a very good approx. (an incompr. fluid). Then $\rho = \tilde{\rho}(\underline{x})$ where $\tilde{\rho}(\underline{x})$ = density at $t=0$ of particle $\underline{x}$.

What happens to the energy eqn here? Since we can't change τ, $D_t \tau = 0$, we can only change E by changing the temp θ

$$D_t U = D_t \tau \left(\frac{\partial U}{\partial \tau}\right)_\theta + D_t \theta \left(\frac{\partial U}{\partial \theta}\right)_\tau$$

$$= 0 + c_v D_t \theta$$

spec. heat at const. volume

$$D_t E = c_v D_t \theta$$

Now assuming ~~[illegible]~~ Fourier's law $\underline{H} = -\kappa \nabla \theta$
The internal energy eqn becomes

$$\boxed{\rho c_v D_t \underline{\theta} = \nabla \cdot (\kappa \nabla \theta) + h}$$

~~[illegible]~~ $\kappa[\rho, \theta]$ in general

The familiar heat conduction eqn. — as in a rigid body.

we can find $\underline{u}, \rho$ without it, once $\underline{u}, \rho$ known we can solve (*) for θ if we know h

($\underline{u}$ enters as $D_t\theta = \partial_t\theta + \underline{u}\cdot\nabla\theta$, heat is advected by the velocity)

Tritton calls this forced convection. Temp. fluctuations have no effect on flow.

Example 2: The temp. of each fluid particle is constant in time $\theta(\underline{x},t) = \theta(\underline{x})$. Then if we write eqn. of state as $\rho = \rho(\theta, p)$ we have $\tilde{\rho}(\underline{x}, p) = \rho(\theta(\underline{x}), p)$ and the fluid is an inhomog. perfect fluid

$\boxed{D_t\theta \equiv 0}$

Such a fluid might be called an isothermal perfect fluid.

There seems to be no known example of a non-viscous fluid which can be treated as an inhomog. perfect fluid in exactly this way.

Newton, not having invented thermodynamics (what?), thought this was a good model for propagation of sound in atmosphere. Now we know he should have used example 3.

Example 3 (most important): Suppose that in a non-viscous fluid $\underline{H} = 0$ and $h = 0$

no internal heat sources $h = 0$

no heat allowed to flow $\underline{H} = 0$

in that case, the entropy / (really energy) eqn becomes

$$\rho D_t S + \nabla \cdot (H/\theta) = h/\theta + H \cdot \nabla \frac{1}{\theta}$$

$$\rho D_t S = 0 \quad \text{or} \quad \boxed{D_t S = 0}$$

the entropy per gram at each fluid particle does not vary with time. The <u>energy eqn reduces to</u> $D_t S = 0$ or

$$S(\underline{x}, t) = S(\underline{x}) = \underline{\text{initial entropy/gm}} \text{ at particle } \underline{x}.$$

we write eqn of state in form $\rho = \rho(S, p)$ then

$$\tilde{\rho}(\underline{x}, p) = \rho(S(\underline{x}), p)$$

write as $p = \tilde{p}(\rho, \underline{x}) = p(\rho, S(\underline{x}))$

and such a fluid is an <u>inhomog. perfect fluid</u>.

The conditions under which examples 2. and 3. might be adequate approximations may be seen by an examination of the ~~ent~~ energy eqn in general

$$\rho(D_t U + p D_t \tau) = -\nabla \cdot \underline{H} + h$$

say this is zero

assuming Fourier's law $\underline{H} = -\kappa \nabla\theta$

also

$$D_t U = \left(\frac{\partial U}{\partial \tau}\right)_\theta D_t \tau + c_v D_t \theta$$

~~Better to use the fact that~~ $\cancel{c_p = \left(\frac{\partial U}{\partial \theta}\right)_p + p\left(\frac{\partial \tau}{\partial \theta}\right)_p}$ ~~so~~ $\cancel{\rho(D_t U + p D_t \tau) = \rho c_p D_t \theta}$ ~~if the pressure can be assumed const.~~

$$\rho\left[\, p + \left(\frac{\partial U}{\partial \tau}\right)_\theta \right] D_t \tau + \rho c_v D_t \theta = + \nabla\cdot(\kappa \nabla\theta) + h$$

Say this zero

If this the small term, then case 1. $D_t p \cong 0$

$$\rho\left[\left(\frac{\partial U}{\partial \tau}\right)_\theta - \left(\frac{\partial U}{\partial \tau}\right)_s\right] D_t \tau + \rho c_v D_t \theta \,\cancel{=}\, - \nabla\cdot(\kappa\nabla\theta) \,\cancel{+h}$$

$= 0$

Then the three terms must be zero, together.

the important thing is the ratio of the two terms (these are the two extremes $D_t\theta \approx 0$ + $D_t s \approx 0$)

$$\rho c_v D_t \theta \quad \text{and} \quad \nabla\cdot(\kappa\nabla\theta)$$

let $\kappa/\rho c_v = \tilde{\kappa} =$ <u>diffusivity</u> [cm^2/sec]

let T = time scale of motion
L = length scale of motion

we wish to compare $\frac{\theta}{T}$ and $\tilde{\kappa}\frac{\theta}{L^2}$

or

$$\frac{L^2}{T} \quad \text{and} \quad \tilde{\kappa}$$

2. if $\tilde{\kappa} \gg L^2/T$, then the fluid is an excellent thermal conductor. For such a fluid $D_t\theta$ can be neglected w.r.t. the heat flow term $\nabla\cdot\underline{H}$ and this

gives us example 2.

$\tilde{K}$ is so large (compared to L^2/T) that the temp. of every particle just remains the same

3. if $L^2/T \gg \tilde{K}$ if the length scale is sufficiently great and the periods sufficiently short (compared to the thermal diffusivity) then ~~$\nabla \cdot \underline{H}$ can~~ $\nabla \cdot \underline{H}$ and hence $\underline{H}$ can be neglected in the energy eqn. It then reduces to $\partial_t S = 0$ (if $h = 0$)

For such a medium heat transport by advection is much more efficient than heat transport by conduction.

For air at $20°$ C. $\kappa = \dfrac{k}{\rho c_v} \sim 0.2$ cm^2/sec

For sea water $\kappa \sim 1.4 \cdot 10^{-3}$ cm^2/sec

For igneous rocks $\kappa \sim 0.1$ cm^2/sec

Thus for sea water say wavelengths

$c \sim 1000'/\text{sec}$ $\sim$ cm , time scales $\sim$ seconds . Then

$\sim 3 \cdot 10^4$ cm/sec $L^2/T \sim 1$ cm^2/sec $\gg 10^{-3}$ cm^2/sec

$T \sim 10^{-3} - 10^{-4}$ sec give 2 examples

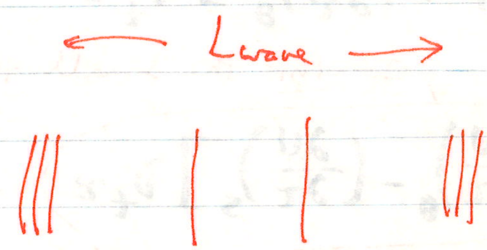
1. sound waves $L/T \sim c$, $L \sim Tc$, $L^2/T \sim c^2 T$

$10^{+9} \cdot 10^{-4}$ For larger scale motions , usually slower,

$\sim 10^5 \gg$ adiabatic approx can still be a good one

10^{-3} In geophysics , one usually interested in L larger at least ~ 1 m.

Sound wave



$$L_{heat} \approx \sqrt{\kappa T}, \qquad L_{wave} \approx cT$$

$$L_{heat} \ll L_{wave} \quad \text{if} \quad T \gg \kappa c^{-2}$$

In air $\quad T \gg \dfrac{2\cdot 10^{-5}}{300^2} \sim 2\cdot 10^{-10}$ secs

ex: ocean mesoscale

$L \gtrsim$ 100 km

$= 10^7$ cm

$T \sim 1$ mo $\sim 10^6$ sec

$L^2/T \sim 10^{14}/10^6 \sim 10^8$

<u>note</u>: for sound waves, the shorter the period T, the <u>worse</u> the approx: this goes against intuition.

i.e. the higher the freq

Many of these are attacked by linearization

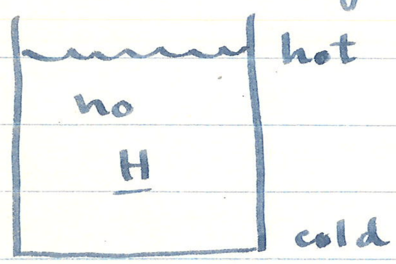
Inhomogeneous perfect fluids are the subject of a great deal of current experimental and theor. research. Many phenomena of interest, e.g. internal waves, stabilization of shear flow by a density gradient, flow of stratified fluids over obstacles.

since maybe superseded by Turner, Buoyancy Effects in Fluids

Best single reference is Carl Eckart's book: a treatise on the linearized theory of inhomogeneous perfect fluids.

This is the theory of stratified fluids

e.g.



in the initial stratified configuration $S(\underline{x})$ is depth-dependent.

He considers only example 3

We will focus attention on an even simpler approximation.

4. Homogeneous perfect fluids, usually just called perfect fluids. Some people call the above pseudo-perfect fluids or stratified fluids

Defn: a perfect fluid is an inhomog. perfect fluid which is in fact homog; $\tilde{\rho}(p, \underline{x}, t)$ has no explicit dependence on $\underline{x}$.

other words in use: <u>baroclinic</u> and <u>barotropic</u>

i.e. $\boxed{\rho = \rho(p, t)}$ The density at a pt. $\underline{r}, t$ is a fcn only of the pressure $\underline{r}, t$

let's see what this implies for our above three examples.

<u>Example 1:</u> now we assume $\tilde{\rho}(\underline{x})$ is ind. of $\underline{x}$ as well as $D_t \rho = 0$. Thus $\rho =$ const at all points $\underline{r}$ and times t.

homog. ρ

This is called an <u>incompressible perfect fluid</u>. The theory for these is particularly well-developed. The best reference is Lamb

<u>Example 2:</u> now we assume $\theta = \theta_0 =$ const. at all points and times. This example is of little physical interest. Laplace thought the <u>isothermal</u> atmosphere was a good model (1832)

homog. θ

<u>Example 3:</u> now we assume $S(\underline{x})$ is ind. of $\underline{x}$ as well as $D_t S = 0$. <u>The entropy per gram is the same everywhere at all times</u>. Such a fluid is said to be an <u>isentropic perfect fluid</u>.

homog. S

An example is a neutrally stratified fluid, one which satisfies the so-called Adams-Williamson eqn (so-called by seismologists).

Perfect fluids were the subject of 19th century hydrodynamics. They are often a

Lecture # 13 Review

Inhomogeneous perfect fluids or pseudo-perfect fluids

A good way to model phenomena in stratified fluids when effects of viscosity can be neglected
See Eckart, *Hydrodynamics of Oceans and Atmospheres*

Perfect fluids (HOMOGENEOUS)

A non-viscous fluid for which the density is a fcn only of the pressure

$\rho(\underline{r},t) = \tilde{\rho}(p(\underline{r},t))$ which we write as $\rho = \tilde{\rho}(p,t)$	density at $\underline{r},t$ depends only on $p(r,t)$

barotropic eqn of state.

Complete system of equations involving only mechanical variables

$\rho, \underline{u}, p$	$D_t\rho + \rho\nabla\cdot\underline{u} = 0$ $\rho D_t\underline{u} + \nabla p = \underline{f}$ $\rho = \tilde{\rho}(p,t)$

Three examples where this might be a good approximation

1. homogeneous incompressible fluid
$\rho(\underline{x}, 0) = \rho_0 = \text{const}$ <u>and</u>
$D_t \rho = 0$, so $\rho = \rho_0 = \text{const}$ for all $\underline{r}, t$
clearly a special case of $\rho = \tilde{\rho}(p, t)$

2. <u>isothermal perfect fluid</u>
initial configuration $\theta(\underline{x}, 0) = \theta_0 = \text{const}$
and diffusivity $\tilde{\kappa} \gg L^2/T$ so that
$D_t \theta \stackrel{\sim}{=} 0$
very good heat conductor
eqn of state $\rho = \tilde{\rho}(\theta, p) = \tilde{\rho}(\theta_0, p)$
if no dependence on θ, clearly only a fcn of p

3. <u>isentropic perfect fluid</u>
initial configuration $S(\underline{x}, 0) = S_0 = \text{const}$
and $L^2/T \gg \tilde{\kappa}$ so that $D_t S = 0$
motion has sufficiently <u>large length scale</u> and <u>short period</u> that $\underline{H} = 0$
(much more efficient to transport heat by convection)
eqn of state $\rho = \tilde{\rho}(S, p) = \tilde{\rho}(S_0, p)$,
only a fcn of p.

very good approximation to the behavior of real fluids except in thin <u>boundary layers</u> near solid surfaces.

The equations of motion for <u>perfect fluids</u> are:

unknowns	equations
$\rho, p, \underline{u}$	$D_t\rho + \rho\nabla\cdot\underline{u} = 0$ cont. $\rho D_t \underline{u} = -\nabla p + \underline{f}$ momentum $\rho = \rho(p,t)$ eqn. of state

We will deal with these full <u>non linear</u> eqns.

System of 5 eqns, 5 unknowns.
These can be considerably simplified by the introduction of a

end here 1 Mar

5. Work function for a perfect fluid

These eqns do not at first sight appear to be a whole lot simpler. However the fact that <u>the density</u> $\rho(\underline{r},t)$ <u>is a function only of the pressure</u> $p(\underline{r},t)$ makes possible the following simplification.

(*) this is the key assumption about the <u>perfect fluid</u>.

At any instant t, lets define the so-called <u>work function</u>

$$\boxed{q(\underline{r},t) = \int_{p_1}^{p(\underline{r},t)} \frac{dp}{\tilde{\rho}(p)}}$$

<u>an auxiliary variable</u>

where p_1 is some conveniently fixed, arbitrary pressure

Why is this function useful? Because in terms of it, the momentum eqn takes a very simple form. Note that

$$\nabla q = \frac{\partial q}{\partial p} \nabla p = \frac{1}{\rho} \nabla p \qquad \text{so}$$

$$D_t \underline{u} = -\nabla q + \underline{f}/\rho$$

<u>We will consider only perfect fluids in gravitational fields</u> $\quad \underline{f} = \rho \underline{g} = -\rho \nabla \phi$

The mom. eqn. is thus

$$\boxed{D_t \underline{u} = -\nabla(\phi + q)}$$

The grav. pot $\phi(\underline{r},t)$ has units of energy / gram. Hence the work function $\underline{q}(\underline{r},t) = \underline{q}(p(\underline{r},t),t)$ has the same units.

Let's see what $\underline{q}$ is in the three examples

Example 1: constant density incompr. fluid

$$\underline{q} = \int_0^p \frac{dp}{\rho} = \frac{P}{\rho} = p\tau \qquad \boxed{\underline{q} = P/\rho}$$

Example 2: non-viscous isothermal fluid
first law of thermodynamics

$$\theta\, dS = dU + p\, d\tau$$

$$d(\theta S) - S\, d\theta = dU + d(p\tau) - \tau\, dp$$

~~$\tau dp - S d\theta = d$~~

$$d(U + p\tau - S\theta) = \tau\, dp - S\, d\theta$$

now if $\theta = \theta_0$, a constant throughout the fluid for all times

$$\tau\, dp = \frac{dp}{\rho} = d(U + p\tau - S\theta)$$

but $U + p\tau - S\theta = G =$ Gibbs free energy per gram

hence $d\underline{q} = dG$ and $\boxed{\underline{q} = G}$

Example 3: non-viscous isentropic fluid

$$\theta\, dS = dU + p\, d\tau$$
$$= dU + d(p\tau) - \tau\, dp$$

$$d(U + p\tau) = \tau\, dp + \theta\, dS$$

now if $S = S_0 =$ const. throughout

$$\tau\, dp = \frac{dp}{\rho} = d(U + p\tau)$$

but $U + p\tau = H =$ enthalpy per gram
and for this case

$$\boxed{q = H = \text{enthalpy}/\text{gm}}$$

Returning to the momentum equation

$$D\underline{u}/Dt = -\nabla(\phi + q)$$

If the fluid is at rest (hydrostatic case)
$D\underline{u}/Dt = 0$ so
$\nabla(\phi + q) = 0$ or (over)

$$\cancel{\boxed{\phi + q = \text{constant (a fcn of } t \text{ only)}}}$$

$$\boxed{\phi(\underline{r},t) + \underline{q}(\underline{r},t) = \underset{\substack{\uparrow \\ \text{a fcn of time only}}}{F(t)}}$$

if the fluid is <u>incompressible</u> this is the best we can do.

If however the fluid is the least bit compressible (ρ varies as p varies) then a fcn of p may be considered a fcn of ρ.

Then $q(p,t)$ may be considered a fcn $q(\rho,t)$. Then

$$\partial q/\partial t = (\partial q/\partial \rho)(\partial \rho/\partial t)$$

but in a fluid at rest $\partial_t \rho = -\overbrace{\nabla\cdot(\rho \underline{u})}^{\text{zero}}$, so $\partial q/\partial t = 0$

If $\partial_t \phi = 0$ also then $\partial_t F = 0$

(example 2 or 3)

in a <u>compressible</u> fluid at rest in a <u>time-dependent</u> grav. field

$$\phi(\underline{r},t) + q(\underline{r},t) = \underset{\substack{\uparrow \\ \text{ind of } \underline{r},t}}{F}$$

why is there this difference for incompr. fluids?

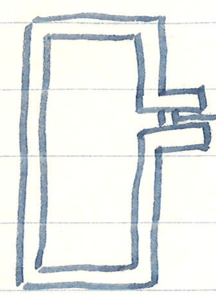
incompr. fluid $\underline{q} = P/\rho$, so

$$\frac{p(\underline{r},t)}{\rho} + \underset{\substack{\uparrow \\ \text{time } \underline{\text{independent}}}}{\phi(\underline{r})} = F(t)$$

say $\underline{g} = -g\hat{z}$ $\underline{\text{const}}$ $\underline{\text{grav. field}}$

$$\boxed{P/\rho + gz = F(t)}$$

Why is $F(t)$ a fcn of t



$\mathcal{F}(t)$ force

now at top of tank $z = 0$ $p = \mathcal{F}(t)/A$
A = area of piston

$$\mathcal{F}(t)/A\rho + g(0) = F(t) \rightarrow F(t) = \frac{\mathcal{F}(t)}{A\rho}$$

$$p/\rho + gz = \frac{\mathcal{F}(t)}{A\rho}$$

if fluid compr. we don't have this situation e.g. if a ~~[illegible]~~ compressible, perfect fluid in a grav. field is in thermal as well as hydrostatic equilibrium, so that

$\theta = \theta_0 =$ constant (isothermal)

$q = G$ and

Generalization of $G =$ const in a fluid with p, θ both constant

$$\boxed{\begin{array}{l} G + \phi = \text{constant} \\ \text{cond. for thermal equil. of a perfect fluid (actually of any fluid)} \end{array}}$$

Now $D_t \underline{u} = -\nabla(\phi + q)$ can be written in another form

we have

$$\underline{u} \cdot \underline{\underline{\nabla u}} = (\nabla \times \underline{u}) \times \underline{u} + \nabla\left(\tfrac{1}{2} u^2\right)$$

proof:

$$[(\nabla \times \underline{u}) \times \underline{u}]_i = \epsilon_{ijk} (\nabla \times \underline{u})_j u_k$$

$$= \epsilon_{ijk} \epsilon_{jlm} (\partial_l u_m) u_k = (\delta_{im}\delta_{kl} - \delta_{il}\delta_{km}) u_k \partial_l u_m$$

$$= u_k \partial_k u_i - u_k \partial_i u_k = \underline{u} \cdot \underline{\underline{\nabla u}} - \partial_i\left(\tfrac{1}{2} v_k v_k\right)$$

$$= \underline{u} \cdot \underline{\underline{\nabla u}} - \nabla\left(\tfrac{1}{2} u^2\right)$$

so

$$\boxed{\partial_t \underline{u} + (\nabla \times \underline{u}) \times \underline{u} = -\nabla\left[\phi + q + \tfrac{1}{2} u^2\right]}$$

6. The vorticity and the Helmholtz vorticity equation

Let's take the curl of the above eqn.

$$\nabla \times \partial_t \underline{u} + \nabla \times [(\nabla \times \underline{u}) \times \underline{u}] = -\nabla \times \nabla [q + \phi + \tfrac{1}{2} u^2]$$

but $\nabla \times \nabla [\;] = 0$ curl (grad) = 0

and

$$\nabla \times \partial_t \underline{u} = \partial_t (\nabla \times \underline{u})$$

note: $\nabla \times D_t \underline{u} \neq D_t \nabla \times \underline{u}$ since D_t holds $\underline{x}$ fixed not $\underline{r}$

Now define the vorticity $\underline{\omega}(\underline{r}, t)$ of a fluid motion to be

$$\boxed{\underline{\omega}(\underline{r}, t) = \nabla \times \underline{u}(\underline{r}, t)}$$

and our eqn is

generation of vorticity by baroclinicity $\nabla \rho \times \nabla p$ a ~~very~~ very important process in GFD (see Pedlosky)

$$\boxed{\begin{gathered} \partial \underline{\omega} / \partial t = \nabla \times (\underline{u} \times \underline{\omega}) \\ \text{or} \quad \partial_t \underline{\omega} + \nabla \times (\underline{\omega} \times \underline{u}) = \underline{0} \end{gathered}}$$

baroclinic vorticity equation in GFD is

$$\boxed{\partial_t \underline{\omega} + \nabla \times (\underline{\omega} \times \underline{u}) = \rho^{-2} (\nabla \rho \times \nabla p)}$$

(barotropic)

This is the Helmholtz / vorticity equation

note that we also have $\nabla \cdot \underline{\omega} = 0$
since $\underline{\omega} = \nabla \times \underline{u}$ and div (curl) = 0

equations for vorticity	valid for any perfect fluid
$$\partial \underline{\omega} / \partial t = \nabla \times (\underline{u} \times \underline{\omega})$$ $$\nabla \cdot \underline{\omega} = 0$$	

Before looking into the consequences of these equations, lets examine more closely the physical significance of the vorticity.

7. Kinematic interpretation of vorticity
Local deformation of a continuum.

Given a continuum at time t has Eulerian velocity $\underline{u}(\underline{x}, t)$

Consider a particular material particle $\underline{x}_0$
Lets examine how the material near $\underline{x}_0$ is distorted between times t and δt

What happens to an inf. small sphere of material surrounding $\underline{x}_0$ at time t?

At t, $\underline{x}_0$ is at posn $\underline{r}_0 = \underline{r}(\underline{x}, t)$

$\uparrow$ Lag. descr.

At $t + \delta t$ it is at $\underline{r}_0 + \underline{u}(\underline{r}_0, t)\,\delta t + O(\delta t^2)$
$= \underline{r}_0'$

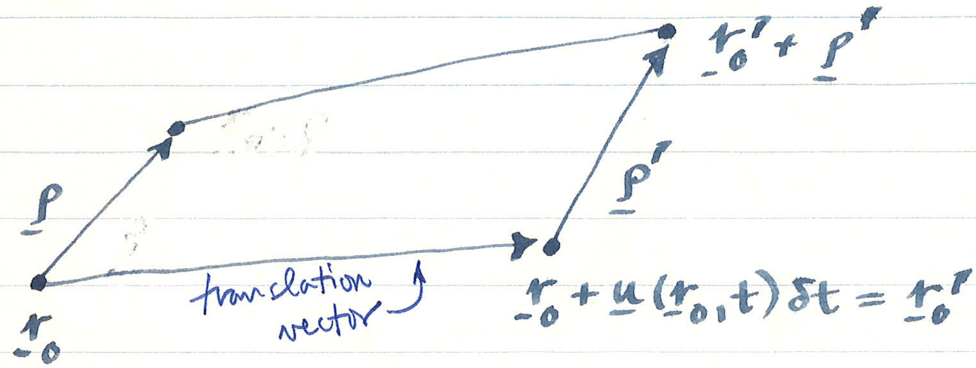
$\boxed{\text{draw picture now}}$

Now to examine the distortion of the surrounding material, we must examine the relative motion of two nearby particles

Thus to $O(\delta t^2)$ the center of a small sphere of material surrounding $\underline{x}$ has been translated thru a distance $\underline{u}(\underline{r}_0, t)\,\delta t$

What about the rest?

A particle which was at $\underline{r}_0 + \underline{\rho}$ at time t has moved to $\underline{r}_0' + \underline{\rho}' = \underline{r}_0 + \underline{\rho} + \underline{u}(\underline{r}_0 + \underline{\rho}, t)\,\delta t + O(\delta t^2)$



How is the new relative position vector $\underline{\rho}'$ related to the old one $\underline{\rho}$?

$$\underline{\rho}' = \underline{\rho} + [\underline{u}(\underline{r}_0+\rho, t) - \underline{u}(\underline{r}_0, t)]\,\delta t + O(\delta t)^2$$

or

$$\boxed{\underline{\rho}' = \underline{\rho} + \underline{\rho}\cdot\underline{\underline{\nabla u}}(\underline{r}_0, t)\,\delta t + O(\rho^2) + O(\delta t^2)}$$

So if we consider an infin. small sphere of radius ρ about $\underline{r}_0$, and if we consider the motion during an infin. small time increment δt

$$\underline{\rho}' = \underline{\rho} + \underline{\rho}\cdot\underline{\underline{\nabla u}}(\underline{r}_0, t)\,\delta t \qquad \text{or}$$

$$\boxed{\underline{\rho}' = \underline{\rho}\cdot[\,\underline{\underline{I}} + \underline{\underline{\nabla u}}(\underline{r}_0, t)\,\delta t\,]}$$

the relative position of a certain material particle relative to $\underline{x}$ at time $t+\delta t$ is given in terms of its relative position at time t by a linear operator

Lets examine the nature of this linear operator

$$\underline{\underline{I}} + \underline{\underline{\nabla u}}(\underline{r}_0, t)\,\delta t$$

posn of $\underline{x}$ at $t=0$

Since δt is infinitesimally small, this is a linear operator near the identity

You should remember all about these.

Define the symmetric and antisymmetric parts of the operator $\underline{\underline{\nabla u}}(\underline{r}, t)$

$$\boxed{\begin{aligned} \underline{\underline{\epsilon}}(\underline{r},t) &= \tfrac{1}{2}\left[\underline{\underline{\nabla u}} + (\underline{\underline{\nabla u}})^T\right] \\ \underline{\underline{\Omega}}(\underline{r},t) &= -\tfrac{1}{2}\left[\underline{\underline{\nabla u}} - (\underline{\underline{\nabla u}})^T\right] \end{aligned}}$$

now clearly $\underline{\underline{\nabla u}} = \underline{\underline{\epsilon}} - \underline{\underline{\Omega}}$ so

$$\underline{\rho}' = \underline{\rho} \cdot \left[\underline{\underline{I}} + \underline{\underline{\epsilon}}\,\delta t - \underline{\underline{\Omega}}\,\delta t\right] + O(\delta t^2)$$

$$= \underline{\rho} \cdot \left[(\underline{\underline{I}} + \underline{\underline{\epsilon}}\,\delta t) \cdot (\underline{\underline{I}} - \underline{\underline{\Omega}}\,\delta t)\right] + O(\delta t^2)$$

Gibbs dot product

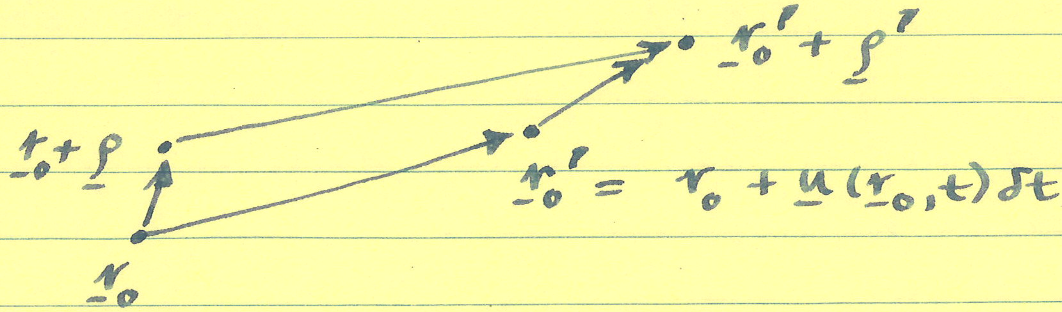
now $\underline{\underline{\epsilon}}^T = \underline{\underline{\epsilon}}$ and $\underline{\underline{\Omega}}^T = -\underline{\underline{\Omega}}$

Lecture # 14 Review

We were discussing the local deformation of a continuum

Given a continuum (not necessarily a perfect fluid) with Eulerian velocity kinematic description $\underline{u}(\underline{r}, t)$

Consider a particle $\underline{x}_0$. At time t
$\underline{r}(\underline{x}_0, t) = \underline{r}_0$ say.
What happens to an inf. small sphere of material surrounding $\underline{x}_0$ at time t (between t and $t + \delta t$)?



During a time increment δt particle $\underline{x}_0$ (center of sphere) gets translated an amount

$$\underline{u}(\underline{r}_0, t)\,\delta t \;+\; O(\delta t^2)$$

so center of sphere gets translated
What about rest of sphere

How is the new relative position vector $\underline{\rho}'$ related to the old relative position vector $\underline{\rho}$? How does the relative position of a nearby particle change

Answer:

$$\boxed{\underline{\rho}' = \underline{\rho} \cdot \left[\underline{\underline{I}} + \underline{\underline{\nabla u}}(\underline{r}_0, t)\, \delta t \right] + O(\delta t^2) + O(\rho^2)}$$

neglect $O(\delta t^2)$ inf. small time increment
$O(\rho^2)$ inf. small sphere

$\underline{\rho}'$ related to $\underline{\rho}$ by means of a linear operator which for small δt is near the identity.

$\boxed{\underline{\underline{\nabla u}}(\underline{r}, t) \text{ deformation rate tensor}}$

define $\underline{\underline{\epsilon}}(\underline{r}, t) = \frac{1}{2}\left[\underline{\underline{\nabla u}} + (\underline{\underline{\nabla u}})^T \right]$ strain rate tensor

$\underline{\underline{\Omega}}(\underline{r}, t) = -\frac{1}{2}\left[\underline{\underline{\nabla u}} - (\underline{\underline{\nabla u}})^T \right]$ rotation rate tensor

$\underline{\underline{\epsilon}}(\underline{r},t)$ is a symmetric linear operator
$\underline{\underline{\Omega}}(\underline{r},t)$ is an antisymmetric linear operator, and

$$\nabla \underline{u}(\underline{r},t) = \underline{\underline{\epsilon}}(\underline{r},t) - \underline{\underline{\Omega}}(\underline{r},t)$$

so

$$\underline{\underline{I}} + \nabla \underline{u}(\underline{r},t)\,\delta t = \underline{\underline{I}} + \underline{\underline{\epsilon}}\,\delta t - \underline{\underline{\Omega}}\,\delta t$$

$$= \left(\underline{\underline{I}} + \underline{\underline{\epsilon}}\,\delta t\right) \cdot \left(\underline{\underline{I}} - \underline{\underline{\Omega}}\,\delta t\right) + O(\delta t^2)$$

Gibbs • product

note: can also write in form
$\underline{\rho}' = [\underline{\underline{I}} + \underline{\underline{\Omega}}\,\delta t] \cdot [\underline{\underline{I}} + \underline{\underline{\epsilon}}\,\delta t] \cdot \underline{\rho}$, i.e. order irrelevant.

so

$$\boxed{\underline{\rho}' = [\underline{\underline{I}} + \underline{\underline{\epsilon}}\,\delta t] \cdot [\underline{\underline{I}} + \underline{\underline{\Omega}}\,\delta t] \cdot \underline{\rho} + O(\delta t^2) + O(\rho^2)}$$

$\underline{\underline{I}} + \underline{\underline{\epsilon}}\,\delta t$ is a symmetric lin. operator, to first order $\underline{\underline{I}} + \underline{\underline{\Omega}}\,\delta t$ is an orthogonal linear operator

$$\underline{\underline{Q}} = \underline{\underline{I}} + \underline{\underline{\Omega}}\,\delta t \Rightarrow \underline{\underline{Q}} \cdot \underline{\underline{Q}}^T = (\underline{\underline{I}} + \underline{\underline{\Omega}}\,\delta t) \cdot$$

$$(\underline{\underline{I}} - \underline{\underline{\Omega}}\,\delta t) = \underline{\underline{I}} + \underline{\underline{\Omega}}\,\delta t - \underline{\underline{\Omega}}\,\delta t + O(\delta t^2)$$
$$= \underline{\underline{I}} + O(\delta t^2)$$

$(\underline{\underline{I}} + \underline{\underline{\Omega}}\, \delta t)$ represents an inf. small <u>rigid rotation</u>

$\boxed{\text{define a unique vector associated with the antisymmetric operator } \underline{\underline{\Omega}} \qquad \underline{\Omega} = -\tfrac{1}{2} \wedge \underline{\underline{\Omega}}}$

in a Cart. axis system $\hat{x}_1, \hat{x}_2, \hat{x}_3$ (<u>rt. handed</u>)

$$\left.\begin{aligned} \Omega_i &= -\tfrac{1}{2} \epsilon_{ijk} \Omega_{jk} \\ \Omega_{ij} &= \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \end{aligned}\right\} \text{relation between } \underline{\Omega} \text{ and } \underline{\underline{\Omega}}$$

$\boxed{(\underline{\underline{I}} + \underline{\underline{\Omega}}\, \delta t) \text{ as a linear operator represents an inf. small rigid rotation about a direction } \underline{\Omega} = -\tfrac{1}{2} \wedge \underline{\underline{\Omega}} \text{ thru the angle } |\underline{\Omega}\, \delta t|}$

don't say this until <u>after</u> drawing picture on page 143.5

$\Omega_i = -\frac{1}{2} \varepsilon_{ijk} \Omega_{jk}$

$= -\frac{1}{4} \varepsilon_{ijk} (\partial_k u_j - \partial_j u_k) = \frac{1}{2} \varepsilon_{ijk} \partial_j u_k = \frac{1}{2} (\nabla \times \underline{u})_i$

$\Rightarrow \quad \underline{\omega} = 2\times$ inst. angular velocity

$$\boxed{\underline{\rho}' = [\,\underline{\underline{I}} + \underline{\underline{\epsilon}}\,\delta t\,][\,\underline{\underline{I}} + \underline{\underline{\Omega}}\,\delta t\,]\cdot\underline{\rho} + O(\delta t^2) + O(\rho^2)}$$

Now recall that since $\underline{\underline{\Omega}}$ is <u>antisymmetric</u> and small, that it represents a <u>proper</u> <u>orthogonal operator</u>. end here 3 feb.

> An infin. small <u>rigid rotation</u> about a direction $\underline{\Omega} = -\frac{1}{2}\Lambda\underline{\underline{\Omega}}$ thru the angle $|\underline{\Omega}\,\delta t|$

in a part. Cartesian axis system

$$(\underline{\underline{I}} + \underline{\underline{\Omega}}\,\delta t)_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}\delta t$$

$$[(\underline{\underline{I}} + \underline{\underline{\Omega}}\,\delta t)\cdot\underline{\rho}]_1 = \rho_1 + (\Omega_2\rho_3 - \Omega_3\rho_2)\,\delta t$$

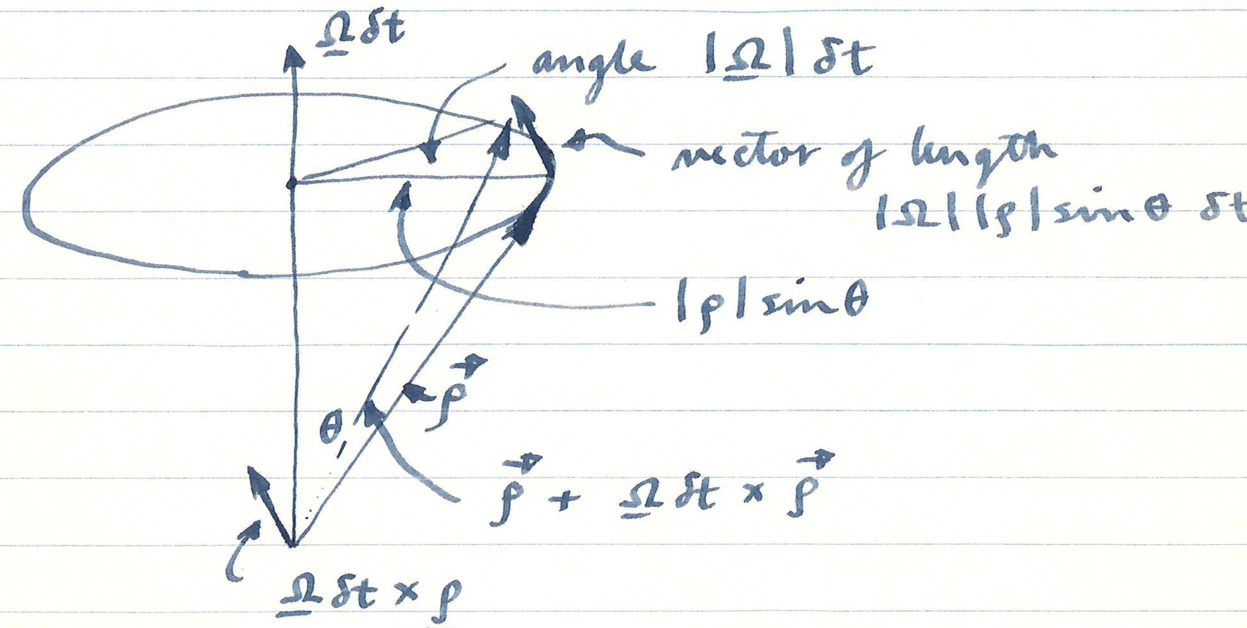
$$[(\underline{\underline{I}} + \underline{\underline{\Omega}}\,\delta t)\cdot\underline{\rho}]_2 = \rho_2 + (\Omega_3\rho_2 - \Omega_1\rho_3)\,\delta t$$

$$[(\underline{\underline{I}} + \underline{\underline{\Omega}}\,\delta t)\cdot\underline{\rho}]_3 = \rho_3 + (\Omega_1\rho_2 - \Omega_2\rho_1)\,\delta t$$

$$\boxed{(\underline{\underline{I}} + \underline{\underline{\Omega}}\,\delta t)\cdot\underline{\rho} = \underline{\rho} + \underline{\Omega}\,\delta t \times \underline{\rho}}$$

rigid rotation

$$(\underline{\underline{I}} + \underline{\underline{\Omega}}\,\delta t)\cdot\underline{\rho} = \underline{\rho} + \underline{\Omega}\,\delta t \times \underline{\rho}$$



$\boxed{\text{$\underline{\Omega}(\underline{r},t)$ is the local instantaneous angular velocity.}}$

$$\boxed{\underset{\sim}{\rho}' = [\underset{\approx}{I} + \underset{\approx}{\epsilon}\,\delta t]\cdot[\underset{\approx}{I} + \underset{\approx}{\Omega}\,\delta t]\cdot \underset{\sim}{\rho}}$$

Thus we see that the effect of the motion during a small time δt is to produce

1. a <u>translation</u> thru $\underset{\sim}{u}(\underset{\sim}{r}_0, t)\,\delta t$
2. a <u>rigid rotation</u> about direction $\underset{\sim}{\Omega} = -\frac{1}{2}\Lambda\underset{\approx}{\Omega}$ thru the angle $|\underset{\sim}{\Omega}|\,\delta t$
3. a <u>symmetric</u> squeeze $\underset{\approx}{I} + \underset{\approx}{\epsilon}\,\delta t$

Consider now the <u>symmetric squeeze or strain</u> following (or preceding) the local rotation.

$$(\underset{\approx}{I} + \underset{\approx}{\epsilon}\,\delta t)$$

a <u>symmetric</u> linear operator : hence has 3 real <u>eigenvalues</u> and three mutually $\perp$ <u>eigenvectors</u>.

Let $\hat{x}_1, \hat{x}_2, \hat{x}_3$ be the eigenvectors of $\underset{\approx}{\epsilon}$ and $\epsilon_1, \epsilon_2, \epsilon_3$ the eigenvalues

then $\underline{\underline{I}} + \underline{\underline{\epsilon}}\,\delta t$ has same eigenvectors and eigenvalues
$1 + \epsilon_1 \delta t$
$1 + \epsilon_2 \delta t$
$1 + \epsilon_3 \delta t$

$$(\underline{\underline{I}} + \underline{\underline{\epsilon}}\,\delta t)_{ij} = \begin{pmatrix} 1+\epsilon_1\delta t & 0 & 0 \\ 0 & 1+\epsilon_2\delta t & 0 \\ 0 & 0 & 1+\epsilon_3\delta t \end{pmatrix}$$

An inf. small sphere of material about $\underline{r}_0$ gets squeezed or strained into an ellipsoid with axes $\hat{x}_1, \hat{x}_2, \hat{x}_3$ of lengths $1+\epsilon_1\delta t$, $1+\epsilon_2\delta t$, $1+\epsilon_3\delta t$

So now we have an interpretation of the strain rate tensor

$$\underline{\underline{\epsilon}}(\underline{r},t) = \tfrac{1}{2}\left[\underline{\underline{\nabla u}} + (\underline{\underline{\nabla u}})^T\right]$$

and we see the reason for the name.

The fact that it is the strain rate tensor that appears as a source term in the energy equation

$$\rho D_t U + \nabla \cdot \underline{H} = h + tr\,(\underline{\underline{T}} \cdot \underline{\underline{\epsilon}})$$

indicates that energy is only dissipated or stored as a result of local straining (actual local changes of shape) Local rigid rotations do not contribute to the dissipation of energy.

We have an interpretation of the strain rate tensor $\underline{\underline{\epsilon}}\,(\underline{r}, t)$

We also have an interpretation of its trace

$$tr\,\underline{\underline{\epsilon}} = \frac{1}{2}\left[\, tr\,\underline{\underline{\nabla u}} + tr\,(\underline{\underline{\nabla u}})^T \,\right]$$

$$= \nabla \cdot \underline{u} = \operatorname{div} \underline{u}$$

and from cont. eqn $\quad \nabla \cdot \underline{u} = \frac{1}{\tau}\frac{D\tau}{Dt}$

(local relative rate of change of volume)

What is physical interpretation of curl $\nabla \times \underline{u}$

The local rigid rotation is about $\underline{\Omega}$ axis thru an angle $|\underline{\Omega}|\,\delta t$ in time δt.

Clearly $\underline{\Omega}(\underline{r},t)$ is the <u>local instantaneous angular velocity</u> $\underline{\Omega} = -\frac{1}{2}\wedge\underline{\underline{\Omega}}$

$$\Omega_i = -\tfrac{1}{2}\epsilon_{ijk}\,\Omega_{jk} = -\tfrac{1}{4}\epsilon_{ijk}\left(\partial_k v_j - \partial_j v_k\right)$$

$$= \tfrac{1}{2}\epsilon_{ijk}\,\partial_j v_k = \tfrac{1}{2}(\nabla\times\underline{u})_i$$

$\frac{1}{2}\left[\nabla\times\underline{u}(\underline{r},t)\right] =$ <u>instantaneous angular velocity</u>

recall defn. $\underline{\omega}(\underline{r},t) \equiv \nabla\times\underline{u}(\underline{r},t)$
<u>vorticity</u>

$\frac{1}{2}\underline{\omega}(\underline{r},t) =$ inst. ang. velocity

$\underline{\omega} = 2\underline{\Omega} =$ twice inst. ang. velocity

Summary:

$\underline{\underline{\epsilon}}(\underline{r},t)$ strain rate tensor

$\nabla\cdot\underline{u}(\underline{r},t) = \mathrm{tr}\,\underline{\underline{\epsilon}}(\underline{r},t)$ local relative rate of change of volume

$\frac{1}{2}\underline{\omega}(\underline{r},t) = \frac{1}{2}\nabla\times\underline{u}(\underline{r},t)$ instantaneous angular velocity

$\underline{\underline{\nabla u}}(\underline{r},t) = \underline{\underline{\epsilon}}(\underline{r},t) - \underline{\underline{\Omega}}(\underline{r},t)$ deformation rate tensor

8. Solution to the Vorticity Eqn.

Brief review: we are studying perfect fluids.

defn: density $\rho(\underline{r},t)$ either constant (special case: homog. incompr.) or a function only of the pressure $p(\underline{r},t)$

$$\rho(\underline{r},t) = \tilde{\rho}(p(\underline{r},t))$$

can thus define a work function $q(\underline{r},t)$

$$q(\underline{r},t) = \int_{P_0}^{P} \frac{dp}{\tilde{\rho}(p(\underline{r},t))}$$

in terms of $q(\underline{r},t)$ the momentum eqn governing the motion of a perfect fluid in a gravitational field $\phi(\underline{r},t)$

$$\boxed{D_t \underline{u} = -\nabla(q+\phi)}$$

which can be rewritten

$$\partial_t \underline{u} + (\nabla\times\underline{u})\times\underline{u} = -\nabla\left(q+\phi+\tfrac{1}{2}u^2\right)$$

upon taking the curl (define $\underline{\omega} = \nabla \times \underline{u}$)

$$\boxed{\begin{array}{ll} \partial_t \underline{\omega} = \nabla \times (\underline{u} \times \underline{\omega}) & \text{Helmholtz vorticity eqn.} \\ \partial_t \underline{\omega} + \nabla \times (\underline{\omega} \times \underline{u}) = 0 & \\ \nabla \cdot \underline{\omega} = 0 & (\text{since } \underline{\omega} = \nabla \times \underline{u}) \end{array}}$$

Now we pose a purely mathematical problem

Given a fluid in motion with Eulerian description $\underline{u}(\underline{r}, t)$

Given a vector field quantity $\underline{B}(\underline{r}, t)$ whose value is known at time $t = 0$ $\underline{B}(\underline{r}, 0)$, and which satisfies the equations

$$\partial_t \underline{B} = \nabla \times (\underline{u} \times \underline{B})$$

$$\nabla \cdot \underline{B} = 0$$

problem: determine $\underline{B}(\underline{r}, t)$ at any later time t.

The problem is of course trivial for $\underline{B}(\underline{r}, t) = \underline{\omega}(\underline{r}, t)$

Full dynamo eqn. 15

$$\partial_t \underline{B} = \underbrace{\frac{1}{\mu_0 \sigma} \nabla^2 \underline{B}}_{\text{mag. diffusion term (dissipative)}} + \nabla \times (\underline{u} \times \underline{B})$$

σ ← elec. cond.

the above problem has a solution in general

Let me just quote and not derive the result.

Most easily expressed in terms of the <u>Lagrangian description</u>

$\underline{u}_E(\underline{r},t)$ implies knowledge of Lagrangian description of motion $\underline{r}(\underline{x},t)$

Can also utilize L. description of $\underline{B}(\underline{r},t)$

$$\underline{B}_L(\underline{x},t) = \underline{B}_E(\underline{r}(\underline{x},t),t)$$

$\boxed{\text{We are given } \underline{B}_L(\underline{x},0) = \underline{B}_E(\underline{x},0) \text{ and } \underline{r}(\underline{x},t). \text{ What is } \underline{B}_L(\underline{x},t) \text{ at time } t.}$

Solution:

$$\boxed{\tau_L(\underline{x},t)\,\underline{B}_L(\underline{x},t) = \tau_L(\underline{x},0)\,\underline{B}_L(\underline{x},0)\cdot\underline{\underline{\nabla \underline{r}}}(\underline{x},t) \qquad (*)}$$

$$\tau(\underline{x},t) \equiv \tau_L(\underline{x},t) = \tau_E(\underline{r}(\underline{x},t),t)$$

local specific volume. Known if $\underline{u}(\underline{r},t)$ is known from cont. eqn.

Note: $\tau \underline{B}$ behaves just like $\underline{\underline{\nabla r}}$.
We say that $\tau \underline{B}$ "moves with the fluid".

Eqn ($\star$) is the exact solution to the mathematical problem we have posed.

We will utilize this solution in the following way
If $\underline{B}(\underline{r},t)$ is $\underline{\omega}(\underline{r},t)$ the vorticity

$$\partial_t \underline{\omega} = \nabla \times (\underline{u} \times \underline{\omega})$$
$$\nabla \cdot \underline{\omega} = 0$$

this system of eqns is integrable

$$\tau(\underline{x},t)\,\underline{\omega}_L(\underline{x},t) = \tau(\underline{x},0)\,\underline{\omega}_L(\underline{x},0) \cdot \underline{\underline{\nabla r}}(\underline{x},t)$$

From this eqn it follows that

> If $\underline{\omega}_L(\underline{x},0) = \underline{\omega}_E(\underline{r},0) = 0$ at time $t=0$, then $\omega_L(\underline{x},t) = \omega_E(\underline{r}(\underline{x},t),t) \equiv 0$ for all time

i.e. if at any time $t=0$, the motion of a perfect fluid in a grav. field is such that $\underline{\omega}(\underline{r},t) = \nabla \times \underline{u}(\underline{r},t) = 0$, then the motion will for all later times be such that $\underline{\omega}(\underline{r},t) = 0$

Motion for which $\underline{\omega}(\underline{r},t) = \nabla \times \underline{u}(\underline{r},t) = 0$ is said to be irrotational flow

If at $t = 0$, the flow of a perfect fluid in a grav. field is irrotational, then the flow will be irrotational for all later times

A perfect fluid moving in a gravitational field cannot generate vorticity. If at time $t = 0$, there is some initial vorticity, then this initial vorticity will be carried with the fluid flow, but if $\nabla \times u(r,t) = 0$ at $t = 0$, then $\nabla \times \underline{u}(r,t) = 0$ for all times thereafter

$$\partial_t \underline{\omega} + \nabla \times (\underline{\omega} \times \underline{u}) = \frac{1}{\rho^2}(\nabla \rho \times \nabla p)$$

baroclinic vorticity eqn.

Discuss production or generation of vorticity by baroclinicity here (very important in GFD)

This is an extremely important result. It is the basis of 19th century hydrodynamics.

e.g. Lamb proves this result on p. 17 and virtually the entire rest of the book is based on it.

Before showing why its so important, let me give another proof, since the last one was pulled out of the hat

end here 6 Feb.

Lecture # 15 Review

Discussing the motion of a perfect fluid in a gravitational potential $\phi(\underline{r},t)$

The grav. pot $\phi(\underline{r},t)$ can in general be time-varying
For a perfect fluid ρ a fcn of p only one can define a work function $q(\underline{r},t)$ and the mom. eqn. may be written in the form

$$D_t \underline{u} = -\nabla(\phi + q)$$

If one defines the vorticity $\omega(\underline{r},t) = \nabla \times \underline{u}(\underline{r},t)$, a measure of the local rotation ($\frac{1}{2}\omega(\underline{r},t) \equiv$ inst. angular velocity of small parcel of fluid at posn $\underline{r}$ at time t)
the above eqn equivalent to:

$$\partial_t \underline{\omega} = \nabla \times (\underline{u} \times \underline{\omega})$$ Helmholtz vorticity eqn

this eqn + the cond. $\nabla \cdot \underline{\omega}$ is an integrable system allowed us to deduce the following extremely important theorem

> If at $t=0$ the flow of a perfect fluid in a grav. field is <u>irrotational</u> $\nabla \times \underline{u}(\underline{r},0) = 0$ in a certain region $V(0)$ then in the set $V(t)$ moving with the material the flow will be irrotational for all later times

Potential forces alone cannot generate vorticity in a perfect fluid.

In particular if at $t=0$ the fluid is at rest, then $\nabla \times \underline{u}(\underline{r},t) = 0$ for all later times.

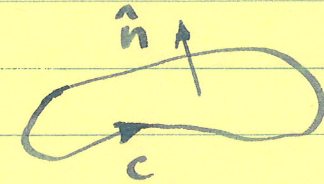
Since this result is so important, let's deduce it in another way.

9. The Helmholtz Circulation Theorem

following Lamb p. 36

definition: the circulation of a vector field $\underline{A}(\underline{r})$ around a directed closed curve C is

$$\oint_C \underline{A} \cdot d\underline{r}$$



note that if S is any surface with bdry C and $\hat{n}$ is unit normal (in positive sense determined by direction of C) then by Stokes theorem the circulation of $\underline{A}$ around C is

$$\oint_C \underline{A} \cdot d\underline{r} = \int_S \hat{n} \cdot (\nabla \times \underline{A})\, dA$$

the flux of $\nabla \times \underline{A}$ thru the surface S

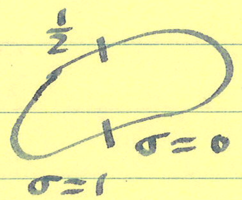
(any surface S will do)

Theorem (Helmholtz circulation theorem)
If the closed directed curve C is in and moves with a perfect fluid in a grav. field then the circulation of the fluid velocity $\underline{u}(\underline{r}, t)$ around C is a constant, independent of time

proof: given a curve C moving with the perfect fluid in a grav. field

C always consists of the same material particles

Let σ be a parameter for the curve C $\quad 0 \le \sigma \le 1$



Let $\underline{r}(\underline{x}, t)$ be a Lagrangian description of the motion

As σ goes from 0 to 1, the points $\underline{x}(\sigma)$ represent all the particles on the curve C.

Hence as σ goes from zero to one $\forall$ fixed t, $\underline{r}(\underline{x}(\sigma), t)$ traces out C at time t

The circulation around C is

$$C(t) = \oint \underline{u}_E(\underline{r}, t) \cdot d\underline{r} = \int_0^1 \underline{u}_L(\underline{x}(\sigma), t) \cdot \frac{\partial \underline{r}(\underline{x}(\sigma), t)}{\partial \sigma} \, d\sigma$$

$$C(t) = \int_0^1 \underline{u} \cdot \frac{\partial \underline{r}}{\partial \sigma} \, d\sigma$$

then $$dC/dt = \int_0^1 D_t \underline{u} \cdot \partial \underline{r}/\partial\sigma \, d\sigma + \int_0^1 \underline{u} \cdot D_t \frac{\partial \underline{r}}{\partial \sigma} \, d\sigma$$

but $$D_t \frac{\partial \underline{r}}{\partial \sigma} = \frac{\partial}{\partial \sigma} D_t \underline{r} = \frac{\partial}{\partial \sigma} D_t \underline{r}(\underline{x}(\sigma), t) = \frac{\partial}{\partial \sigma} \underline{u}_L(\underline{x}(\sigma), t) = \frac{\partial}{\partial \sigma} \underline{u}$$

thus
$$\frac{d}{dt} C(t) = \int_0^1 D_t \underline{u} \cdot \frac{\partial \underline{r}}{\partial \sigma} \, d\sigma + \int_0^1 \frac{\partial}{\partial \sigma} \left(\tfrac{1}{2} u^2\right) d\sigma$$

$$= \oint D_t \underline{u} \cdot d\underline{r} + \int_0^1 \frac{\partial}{\partial \sigma} \left[\tfrac{1}{2} u_L^2(\underline{x}(\sigma), t)\right] d\sigma$$

$$\downarrow$$

$$\left[\tfrac{1}{2} u_L^2(\underline{x}(\sigma), t)\right]_{\sigma=0}^{\sigma=1}$$

the second integral vanishes since $\underline{x}(0) = \underline{x}(1)$
consider the first integral

we haven't used yet

for a <u>perfect fluid</u> in a <u>grav. field</u>

$$D_t \underline{u} = -\nabla(\phi + q)$$

$$\oint D_t \underline{u} \cdot \underline{dr} = - \oint \nabla(\phi + g) \cdot \underline{dr}$$

$$= 0$$

the circulation of any irrotational field (derivable from a scalar) being zero

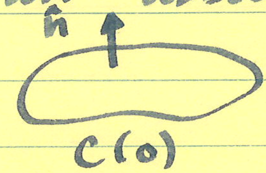
hence $\dfrac{dC(t)}{dt} = 0$ q.e.d.

The Helmholtz circulation theorem is of course essentially a statement that a perfect fluid moving in a grav. field does not generate vorticity.

If at time $t = 0$, the circulation around every closed curve is zero, then the circulation around every closed curve is zero for all times t

Corollary: If at time $t=0$, the motion of a perfect fluid is ~~[illegible]~~ irrotational $\underline{\omega}(\underline{r}, t) = 0$, then it remains irrotational for all times.

proof: Consider the circulation around an arbitrary closed curve at time $t=0$



$$\oint_{C(0)} \underline{u} \cdot d\underline{r} = \int_S (\nabla \times \underline{u}) \cdot \hat{n} \, dA = 0$$

$S \uparrow$ any surface

Now let $C(t)$ be the curve obtained by letting $C(0)$ move with the fluid then Then by Helmholtz

$$\oint_{C(t)} \underline{u} \cdot d\underline{r} = 0$$

hence $\int_S \hat{n} \cdot (\nabla \times \underline{u}) \, dA = 0$ for any arbitrary surface. Hence $\nabla \times \underline{u} = 0$ for all times.

From now on we will study only <u>irrotational flow</u>. We will not consider the case where the <u>initial vorticity</u> $\omega(\underline{r}, 0) \neq 0$
(it could have been induced by non-potential forces) e.g. stirring with a stick.

10. Irrotational or Potential Flow

The reason why the study of irrotational flow is so easy is that we can introduce a velocity potential function.

If $\underline{u}(\underline{r},t)$ is irrotational so that
$\underline{\omega}(\underline{r},t) = 0$ (or $\nabla \times \underline{u}(\underline{r},t) = 0$)
(or so that the circulation around every closed curve C is zero)
then $\exists$ a scalar potential $\psi(\underline{r},t)$

$$\boxed{\underline{u}(\underline{r},t) = \nabla\psi(\underline{r},t)}$$

do not confuse ψ (velocity pot.) with ϕ.

For this reason irrotational flow is called potential flow.
Potential flow is interesting because

Theorem: Consider a perfect fluid moving in a grav. field. If at $t=0$ the motion is a potential flow in a region $V(0)$, then it is a potential flow at any time t in $V(t)$, moving with the fluid.

e.g. if at $t=0$ a perfect fluid in a gravitational field is at rest, then at all times later its motion is a potential flow.

remark: note that if $\underline{u}(\underline{r},t) = \nabla\psi(\underline{r},t)$ that $\psi(\underline{r},t)$ is <u>not</u> <u>unique</u>.
It is determined only to within an additive fcn of time

if $f(t)$ is a fcn of time only, then

$$\nabla(\psi + f) = \nabla\psi$$

either ψ or $\psi' = \psi + f$ can serve as a velocity potential for $\underline{u}(\underline{r},t)$

conversely if $\underline{u}(\underline{r},t) = \nabla\psi(\underline{r},t) = \nabla\psi'(\underline{r},t)$ then $\nabla(\psi - \psi') = 0$ so $\psi - \psi'$ is a fcn only of t.

11. Bernoulli theorems

Consider a non-viscous in a grav. field with potential ϕ the acceleration of gravity is $\underline{g} = -\nabla\phi$

The equation of motion is

$$\rho D_t \underline{u} = -\nabla p - \rho \nabla \phi$$

$$D_t \underline{u} = -\frac{\nabla p}{\rho} - \nabla \phi$$

note: the fluid is not yet necessarily perfect

now $D_t \underline{u} = \partial_t \underline{u} + \underline{u} \cdot \underline{\underline{\nabla u}}$

$$\underline{u} \cdot \underline{\underline{\nabla u}} = (\nabla \times \underline{u}) \times \underline{u} + \nabla\left(\tfrac{1}{2}u^2\right)$$
$$= \underline{\omega} \times \underline{u} + \nabla\left(\tfrac{1}{2}u^2\right)$$

the momentum eqn thus becomes (for any non-viscous fluid)

$$\boxed{\partial_t \underline{u} + \underline{\omega} \times \underline{u} = -\nabla\left(\phi + \tfrac{1}{2}u^2\right) - \frac{1}{\rho}\nabla p}$$

We now proceed to derive three very similar results all of which are given the general rubric of <u>Bernoulli theorems</u>

I. <u>Potential flow of a perfect fluid</u>

Assume the fluid is <u>perfect</u> and moves in a <u>potential flow</u>. Then we have

$$\underline{u}(\underline{r},t) = \nabla\psi(\underline{r},t)$$
$$\underline{\omega}(\underline{r},t) = 0$$
$$1/\rho\, \nabla p = \nabla q$$

our eqn becomes

$$\boxed{\nabla\left(\partial_t \psi + \phi + \tfrac{1}{2}u^2 + q\right) = 0}$$

Bernoulli's theorem for potential flow of a perfect fluid: if $\psi(\underline{r},t)$ is any velocity potential for pot. flow of a perfect fluid in a grav. field ϕ, $\exists$ a fcn $f(t)$ of t alone $\ni$

$$\boxed{\partial_t \psi + \phi + \tfrac{1}{2}u^2 + q = \underset{\substack{\uparrow \\ \text{fcn of } t \text{ only}}}{f(t)}}$$

II. Steady Potential Flow in a steady gravit. potential

defn: a steady flow is a flow whose Eulerian description is independent of time

$$\partial_t \underline{u}_E = \partial_t p = \partial_t \rho = \partial_t U = \partial_t q = 0$$

in the above eqn.

$\exists$ a steady flow potential $\partial_t \psi = 0$

$$\phi + \tfrac{1}{2}u^2 + q = f(t)$$

but $\partial_t \phi = 0$ (steady grav. pot)

$\partial_t u = \partial_t q = 0$ (steady flow)

so $\partial_t f = 0$

Bernoulli theorem: $\forall$ steady potential flow in a steady grav. field ϕ

$$\tfrac{1}{2}u^2 + \phi + q = K \quad , \text{ a constant}$$

Recall that a perfect fluid at rest in a steady grav. potential ϕ satisfies

$$\underline{q} + \phi = \text{constant}$$ (actually only a non-incompr. fluid)

now

if the flow of that perfect fluid is a steady potential flow

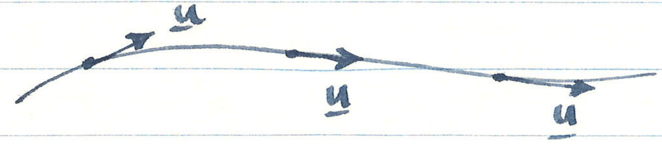
$$\frac{1}{2}u^2 + \underline{q} + \phi = \text{constant}$$

III. Steady non-potential flow

Given a non-viscous fluid flowing steadily in a steady grav. pot.
There is then still a Bernoulli theorem
We define a

streamline: any curve everywhere tangent to $\underline{u}(\underline{r}, t)$ at the instant t. This is only a useful concept in a steady flow in steady flow a streamline always consists of the same fluid particles.

i.e. in a steady flow, (but not in general otherwise) the paths of the fluid particles coincide with the streamlines



in steady flow the streamlines have the same form for all times

thus in steady flow we can define a "work function" for each streamline

$$g(\underline{r}_1) - g(\underline{r}_0) = \int_{\underline{r}_0}^{\underline{r}_1} dp/\rho = \int_{\underline{r}_0}^{\underline{r}_1} \frac{\nabla p}{\rho} \cdot d\underline{r}$$

the integral being taken along the streamline connecting $\underline{r}_0$ and $\underline{r}_1$

Note: since the flow is steady $g(\underline{r})$ is not a fcn of t.

in a steady flow the eqn. of motion is

$$\partial_t \underline{u} + \underline{\omega} \times \underline{u} = -\nabla(\phi + \tfrac{1}{2}u^2) - \tfrac{1}{\rho}\nabla p$$

$\downarrow$

zero in steady flow

we integrate this eqn. on a streamline from $\underline{r}_0$ to $\underline{r}_1$

$\boxed{\text{go to 165}}$

$$\left[\phi + \tfrac{1}{2}u^2 + \underline{g} \right]_{\underline{r}_0}^{\underline{r}_1} = - \int_{\underline{r}_0}^{\underline{r}_1} (\underline{\omega} \times \underline{u}) \cdot \underline{dr}$$

but on a streamline $\underline{u}$ is $\parallel$ to $\underline{dr}$
so $\underline{\omega} \times \underline{u} \cdot \underline{dr} = 0$ and

$$\boxed{\phi(\underline{r}_1) + \tfrac{1}{2}u^2(\underline{r}_1) + \underline{g}(\underline{r}_1) = \phi(\underline{r}_0) + \tfrac{1}{2}u^2(\underline{r}_0) + \underline{g}(\underline{r}_0)}$$

Thus $\phi + \underline{g} + \frac{1}{2}u^2$ is constant along any given streamline
The const. depends on the streamline
(varies from streamline to streamline)

For pot. flow the constant is the same for all streamlines.

end here 8 feb.

<u>Example</u> (of the use of Bernoulli theorems)

A body (e.g. airplane) moving with velocity v in a non-viscous fluid

In <u>body</u> reference frame flow is steady

Lecture # 16 Review

Discussing the flow of non-viscous fluids in grav. fields.

<u>Important result</u> : if at $t = 0$ motion in a region $V(0)$ is <u>irrotational</u>, then at any later time t, motion in $V(t)$ is still irrotational.

<u>Bernoulli theorems:</u>

I. <u>Potential flow</u>, <u>perfect fluid</u>

$$\partial_t \psi + \phi + \tfrac{1}{2} u^2 + \underline{q} = f(t)$$

or

$$\partial_t \psi + \phi + \underline{q} + \tfrac{1}{2} |\nabla \psi|^2 = f(t)$$

↑ fcn of time only.

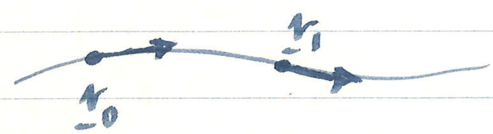
This is all we can say for <u>non-steady</u> potential flow. We will make use of and interpret this result later.

II. <u>Steady (non time-varying all $\partial_t \equiv 0$) Potential Flow of a Perfect Fluid</u>

$$\phi + q + \tfrac{1}{2}u^2 = \underset{\substack{\uparrow \\ \text{a } \underline{\text{constant}}}}{K}$$

III. <u>Steady</u> flow of <u>any</u> non-viscous fluid (<u>non-potential</u> flow)

define <u>streamlines</u> : everywhere tangent to local Eulerian velocity, in steady flow



these are <u>particle paths</u>.

can define a <u>work fcn</u> along an individual streamline

$$q(\underline{r}_1) - q(\underline{r}_0) = \int_{\underline{r}_0}^{\underline{r}_1} \frac{1}{\rho} \nabla p \cdot \underline{r}$$

then

$$\phi(\underline{r}_1) + q(\underline{r}_1) + \tfrac{1}{2}u^2(\underline{r}_1) = \phi(\underline{r}_0) + q(\underline{r}_0) + \tfrac{1}{2}u^2(\underline{r}_0)$$

$\phi + q + \frac{1}{2}u^2$ is a constant along any given streamline. Constant varies from streamline to streamline. For irrot. or potential flow, const. same for all.

Note the obvious analogy
The Bernoulli theorem is merely an energy integral of the system

Consider the motion of an isolated particle mass m (point mass) moving in a grav. pot. $\phi(\underline{r},t)$

$$m\ddot{\underline{r}} = \underline{f} = -m\nabla\phi(\underline{r},t)$$

now take $\phi(\underline{r})$ a fcn of space alone (a steady grav. field)

$$m\ddot{\underline{r}} = -m\nabla\phi(\underline{r})$$
$$\ddot{\underline{r}} = -\nabla\phi(\underline{r})$$
$$\dot{\underline{r}}\cdot\ddot{\underline{r}} = -\dot{\underline{r}}\cdot\nabla\phi(\underline{r})$$

integrate w.r.t. time ~~[illegible] not a~~
important that $\phi(\underline{r})$ not a fcn of time

$$\tfrac{1}{2}\dot{r}^2 + \phi(\underline{r}) = \text{const}$$
$$\tfrac{1}{2}u^2 + \phi = \text{const.}$$

potential energy / unit mass of particle

Bernoulli theorem III (the most general case an obvious extension of this result an energy integral for individual mass elements moving on streamlines

$\frac{1}{2}u^2 + \phi + q =$ const along a streamline (i.e. for an individual material particle)

$\frac{1}{2}u^2 =$ kinetic energy / unit mass
$\phi =$ grav. pot. energy / unit mass
$q \equiv$ work function = some additional energy / unit mass

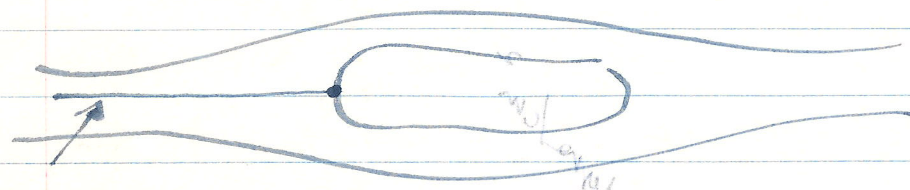
recall: incompressible $D_t \rho = 0$
$q = p/\rho = p\tau$

isothermal
$q = G = U + p\tau - S\theta$ Gibbs

isentropic
$q = H = U + p\tau$

Steady flow of a non-viscous fluid

At a certain point on the nose of the plane the air is at rest relative to the plane (stagnation point)



consider two cases: use Bernoulli III

case 1: the fluid is incompressible (hence perfect)

$$q = P/\rho$$

apply Bernoulli III to arrowed streamline

$$\frac{1}{2} v^2 + \frac{P_\infty}{\rho} = \frac{P_s}{\rho}$$

$$P_s - P_\infty = \frac{\rho}{2} v^2$$

this is for slow flow

one can thus use the Bernoulli theorem III to compute the pressure at the stagnation point, the stagnation pressure.

case 2: fluid is not incompr but is an isentropic ideal gas (a perfect fluid because the entropy per gram $S = S_0$ at all points) this is for supersonic flow

q = work function = $U + p\tau$, the enthalpy

for an ideal gas $\quad U = c_v\theta, \quad p\tau = \tilde{R}\theta$

$$U + p\tau = c_v\theta + \tilde{R}\theta$$
$$= c_p\theta$$

$$* \quad p_s = p_\infty\left(1 + \frac{\gamma-1}{\gamma}\,\frac{1}{2}\,\frac{\rho U^2}{p_\infty}\right)^{\frac{\gamma}{\gamma-1}}$$

again on the arrowed streamline

$$c_p\theta_\infty + \tfrac{1}{2}U^2 = c_p\theta_s$$

$$c_p(\theta_s - \theta_\infty) = \tfrac{1}{2}U^2 \qquad \text{temp. at stagnation point}$$

Note that for $\frac{1}{2}\rho U^2 \ll p_\infty$ this agrees with the incompressible result

now in an adiabatic change

$$p\tau^{\gamma} = \text{const}$$

$\boxed{\text{hottest temp occurs at stagnation point}}$

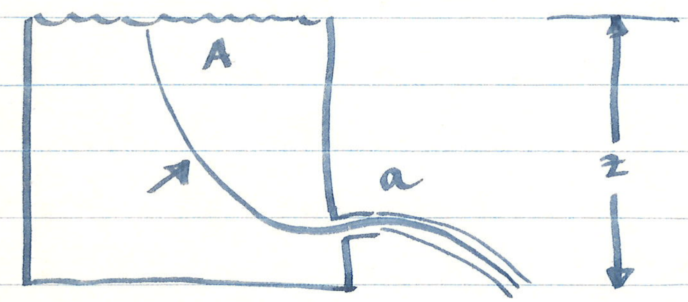
$$p\theta^{\frac{\gamma}{1-\gamma}} = \text{const}$$

$$p_\infty\theta_\infty^{\frac{\gamma}{1-\gamma}} = p_s\theta_s^{\frac{\gamma}{1-\gamma}} \qquad \text{pressure at stagnation point}$$

or equivalently *

note that we have only had to consider a single streamline. The outcome does not depend on how the atmosphere is stratified at levels other than the altitude of the airplane

Bernoulli's theorem may also be used to treat the problem (already considered) of H_2O flowing out of a very large tank.



very large tank ($a \ll A$)
filled with incompr. fluid (perfect) of density ρ.
At $t = 0$ the plug is removed and shortly thereafter flow is steady
Assume that at the top $\underline{u} = 0$ (very large tank).
What is $\underline{u}$ in jet a depth z below level of H_2O in tank
Apply Bernoulli III to any streamline e.g. the arrowed one (or if one doesn't twist the plug upon removing, then $\exists$ no vorticity and flow is potential flow. In any case

assume the experiment occurs in a vacuum.

Bernoulli III

$$\phi(\underline{r}_1) + \frac{1}{2}u^2(\underline{r}_1) + \frac{p(\underline{r}_1)}{\rho} = \phi(\underline{r}_2) + \frac{1}{2}u_2(\underline{r}_2) + \frac{p(\underline{r}_2)}{\rho}$$

$p(\underline{r}_1) = 0$ top of tank
$p(\underline{r}_2) = 0$ in jet of H_2O

$\phi(\underline{r}_1) = gz + \text{const}$
$\phi(\underline{r}_2) = \text{const}$

$$gz = \frac{1}{2}U_{jet}^2$$

$$U(z) = \sqrt{2gz} \qquad \text{Torricelli result}$$

Cannot be applied if z is so large that the jet has broken into droplets (flow no longer steady).

12. Potential flow of homogeneous incompressible fluids.

The Bernoulli theorems are nice but we have not yet seen in any particular case how to solve for flow patterns

We now specialize even further We consider now potential flow of a fluid which is perfect by virtue of the fact that it has $\rho = \rho_0$. density a constant everywhere at all times

This is more specialized than mere incompressibility. Must satisfy two conditions:

1. $\rho = \rho_0$ a constant in the initial configuration
2. flow is incompressible $D_t \rho = 0$ hence $\rho_L(\underline{x}, t) = \rho_L(\underline{x}, 0) = \rho_0$ for all times

What are the conditions under which the flow of a real fluid may be considered to be incompressible?

We have already seen that in almost all geophysically interesting circumstances, L is suff. large and T^2 suff small (periods sufficiently short) compared to thermal diffusivity $L^2/T \gg \tilde{\kappa}$ then isentropic flow is an excellent approximation.

We rephrase the question. Under what circumstances may isentropic flow be approximated by incompressible flow?

The answer is readily obtained for steady flow by using the Bernoulli theorem

pressure changes in steady flow

$$\Delta p / \rho \sim \tfrac{1}{2} u^2 \quad \text{or}$$

$$\Delta p \sim \rho u^2$$

now when the pressure changes adiabatically by Δp, the density changes by

$$\Delta \rho = \left(\frac{\partial \rho}{\partial p}\right)_s \Delta p$$

You will show in your homework prob. that

$$(\partial p/\partial \rho)_s = c^2 = \text{velocity of sound in fluid}$$

$$\Delta\rho \sim \rho u^2/c^2$$

the fluid may be regarded as <u>incompressible</u> relative density changes are very small (of any parcel of fluid)

$$\Delta\rho/\rho \ll 1$$
$$u^2/c^2 \ll 1 \quad \text{or}$$

$u \ll c$ for steady isentropic flow

result: <u>steady</u> <u>isentropic</u> flow may be adequately approximated by incompressible flow so long as $u \ll c$, fluid velocities $\ll$ speed of sound in fluid

If then the initial configuration is $\sim$ of uniform density, then one may safely take $\rho = \rho_0$ for all t unless one wishes to discuss <u>hypersonic</u> <u>or</u> <u>acoustical</u> phenomena.

For a more careful discussion see Batchelor p. 167–171

v/c called <u>Mach number</u>

speed of sound

air $15°C$. 1 atm $c = 340.6$ m/sec

water $15°C$. $c = 1470$ m/sec

for bodies moving slower than say 100 m/sec in air will show little effect of the compressibility

normal steady flows in H_2O very unlikely to ever be influenced by compressibility.

hypersonic flow of interest to aeronautical engineers of course

subject usually called gas dynamics

concerned with flow $v \sim c$ or $v > c$

let's consider then potential flow of such a fluid.

This is the classical problem of 19^{th} century fluid mechanics
It is peculiar among probs. in cont. mech. in that the motion of the fluid is determined completely by the cont. eqn. The mom. eqn. serves merely to determine the pressure once the velocity field is known

Recall this ignores baroclinicity and viscosity.

potential flow $\underline{u} = \nabla \Psi$
$\Psi =$ velocity potential

incompressible fluid $\nabla \cdot \underline{u} = 0$ (cont. eqn.)

$\nabla \cdot (\nabla \Psi) = \nabla^2 \Psi = 0$

$\nabla^2 \Psi = 0$. The velocity potential in the potential flow of a perfect fluid of const. ρ is always a harmonic fcn. (satisfies Laplace's eqn.

Hence the problem of determining $\underline{u}(\underline{r}, t)$ for such a fluid can be reduced to determining solns to Laplace's eqn.

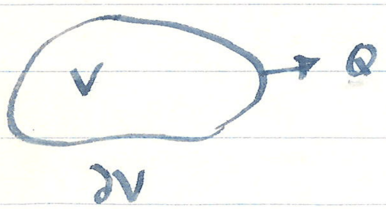
What are the relevant <u>boundary conditions</u> to be used in solving Laplace's equation? There are two cases:

I. <u>Interior problem</u>

The fluid is confined <u>inside</u> a cavity V. The walls ∂V can move in any way we please so long as we don't change the total volume

let $Q \equiv$ velocity of ∂V <u>normal to itself</u>

the b.c. is that on ∂V, $\hat{n} \cdot \underline{u} = Q$



$$\boxed{\begin{array}{ll} \hat{n} \cdot \nabla \phi = Q & \text{on } \partial V \\ \nabla^2 \phi = 0 & \text{in } V \end{array}}$$

this type of b.v. problem is called an <u>interior Neumann problem</u>

$$\nabla^2 \phi = 0 \quad \text{inside a volume } V$$
$$\hat{n} \cdot \nabla \phi = Q \quad \text{specified on bdry } \partial V$$

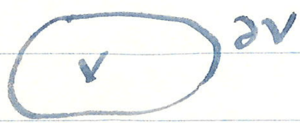
note: if the volume of the fluid is fixed we must have

$$\int_{\partial V} Q \, dA = 0 \qquad \text{or} \qquad \int_{\partial V} \hat{n} \cdot \nabla \phi \, dA = 0$$

<u>average value</u> of $Q = \hat{n} \cdot \nabla \phi$ on ∂V must be zero, or problem makes no physical sense

II. <u>Exterior problem</u>

e.g. body moving through an infinite extent fluid



$E - V \equiv$ exterior of V

say velocity of body normal to itself specified

$Q \equiv$ velocity of ∂V

$$\boxed{\begin{array}{l} \nabla^2 \phi = 0 \quad \text{in} \quad E - V \\ \hat{n} \cdot \nabla \phi = Q \quad \text{on} \quad \partial V \\ \text{growth cond. at } \infty \end{array}}$$

This type of b.v. problem called an <u>exterior Neumann problem</u>.

p. 54–56 of notes last term discuss the other common type of b.v. prob. for Laplaces eqn, internal and external <u>Dirichlet problem</u>

(ϕ specified on ∂V instead of $\hat{n} \cdot \nabla \phi$)

Will discuss Neumann problems later
First let's see how we determine the pressure

$\underline{u} = \nabla \psi$
$\nabla^2 \psi = 0$ in V or $E - V$
$\hat{n} \cdot \nabla \psi = Q$ on ∂V

1. solve Neumann problem for velocity potential $\psi(\underline{r}, t)$
This a well-posed problem.

2. Can then solve for $\underline{u} = \nabla \psi$

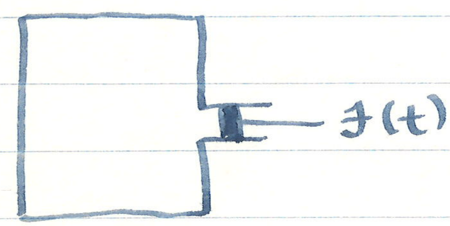
3. Then Bernoulli theorem I (not necessarily steady potential flow) can be used to obtain pressure variation

$$\boxed{\begin{aligned} &q = P/\rho_0 \\ &P/\rho_0 = -\frac{\partial \psi}{\partial t} - \frac{1}{2}|\nabla \psi|^2 - \phi + F(t) \end{aligned}}$$

reason for arbitrary fcn of time in eqn of pressure $F(t)$

Has <u>no effect on motion</u>, only on pressure

But for an incompr. fluid knowing motion is not in general sufficient to completely determine pressure



pushing on the piston produces <u>no motion</u> in the fluid, but we can alter the pressure in any way we like (<u>a fcn of time only</u>)

If $\exists$ a point where $p(\underline{r}, t)$ is known (e.g. a free surface) then $f(t)$ is known and $p(\underline{r}, t)$ is <u>uniquely</u> determined throughout the fluid.

Note: can trade $f(t)$ off against arb. const. in spec. of $\varphi(t)$

end here 13 Feb.

Lecture # 17 Review

The flow of an incompressible perfect fluid (a perfect fluid of constant density)

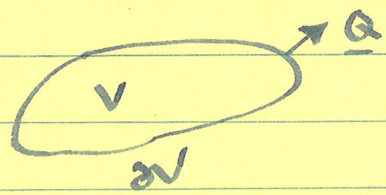
The flow is completely determined by the continuity eqn.

Potential or irrotational flow $\underline{u} = \nabla\psi$
ψ = velocity potential
Cont. eqn. for incompr. fluid

$\nabla^2\psi = 0$ Laplace's eqn

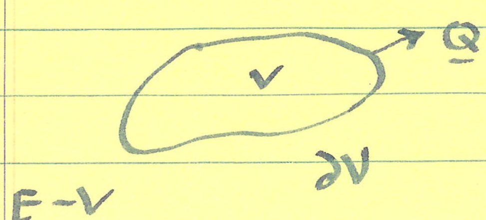
Leads commonly to two types of b.c. problems

Interior Neumann problem



$\nabla^2\psi = 0$ in V
$\hat{n}\cdot\nabla\psi = Q$ on ∂V
(normal fluid velocity ≡ that of wall)

Exterior Neumann problem



$\nabla^2\psi = 0$ in $E - V$ (outside of V)
$\hat{n}\cdot\nabla\psi = Q$ on ∂V

1. solve Neumann problem for velocity potential $\psi(\underline{r}, t)$
   This a well-posed problem

2. can then solve for $\underline{u}(\underline{r}, t) = \nabla\psi(\underline{r}, t)$

3. Then Bernoulli thm I can be used to obtain pressure variation $p(\underline{r}, t)$

$\underline{q} = P/\rho_0$

$$P/\rho_0 = -\frac{\partial \psi}{\partial t} - \frac{1}{2}|\nabla\psi|^2 - \phi + F(t)$$

arbitrary fcn of time.

Now examine the nature of Neumann b.v. problems.

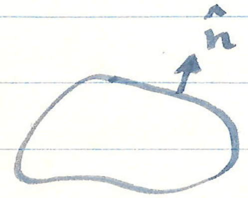
13. Neumann b.v. problems

I. Interior Neumann problem

Given a fcn ψ

$\nabla^2 \psi = 0$ in V (harmonic in V

$\hat{n} \cdot \nabla \psi = Q$ on ∂V



this a standard b.v. problem for Laplace's eqn

<u>Uniqueness:</u> claim ψ is uniquely determined in V

If $\exists$ two potential fcns ψ_1 and ψ_2 which satisfy $\nabla^2 \psi = 0$ in V and $\hat{n} \cdot \nabla \psi = 0$ on ∂V, then $\psi_1 - \psi_2 =$ const. in V.

proof: $\nabla^2 (\psi_1 - \psi_2) = 0$ in V

$\hat{n} \cdot \nabla (\psi_1 - \psi_2) = 0$ on ∂V

let $\psi = \psi_1 - \psi_2$

consider $\int_V |\nabla \psi|^2 \, dV$

but $|\nabla \psi|^2 = \nabla \psi \cdot \nabla \psi = \nabla \cdot (\psi \nabla \psi) - \psi \nabla^2 \psi$

$$\int_V |\nabla \psi|^2 \, dV = \int_V \nabla \cdot (\psi \nabla \psi) \, dV - \int_V \psi \underbrace{\nabla^2 \psi}_{\text{zero}} \, dV$$

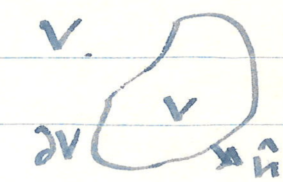
$$= \int_{\partial V} \hat{n} \cdot \psi \nabla \psi \, dA = \int_{\partial V} \psi \, \underbrace{\hat{n} \cdot \nabla \psi}_{\text{zero}} \, dA$$

$$= 0$$

Thus $\nabla \psi = 0$ or $\psi =$ constant

Thus $\psi_1 - \psi_2 =$ constant q.e.d.

Thus if at time t the velocity $\underline{u}$ is a incompressible potential flow in a region V and if the normal velocity $\hat{n} \cdot \underline{u}$ is specified everywhere on the boundary ∂V, then $\underline{u}$ is <u>uniquely</u> determined throughout V. (there is <u>no</u> dependence of $\underline{u}$ on an arbitrary constant)



Note the peculiar nature of incompr. pot. flow – completely determined by value of normal velocity on surface.

The thm asserts uniqueness but not existence. Existence is much harder to prove.

e.g. if $Q = 0$ for all of ∂V, <u>there can be no motion</u>. The flow is completely determined by the <u>instantaneous</u> value of $Q \equiv$ wall velocity

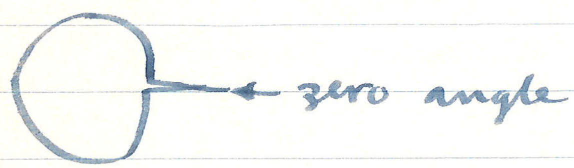
We just quote a result

<u>Theorem</u>: If ∂V is smooth except for some edges and corners with non-zero angles, and if Q is piecewise continuous and satisfies

$$\int_{\partial V} Q \, dA = 0$$

then $\exists$ a fcn ψ harmonic in V and which satisfies $\hat{n} \cdot \nabla \psi = Q$ everywhere on ∂V except possibly at edges or corners

<u>Remarks</u>: It is possible to invent surfaces for which no soln $\exists$. One can however allow corners with non-zero angles



a so-called Lebesgue spine

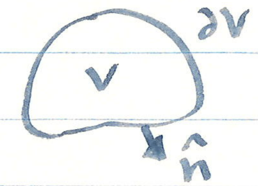
Reason for condition $\int_{\partial V} Q \, dA = 0$

Physically, to prevent changes in volume of our incompr. fluid

Mathematically, $\int_V \nabla^2 \psi \, dV = \int_V \nabla \cdot \nabla \psi \, dV$

$= \int_{\partial V} \hat{n} \cdot \nabla \psi \, dA$ clearly zero.

Exterior Neumann Problem



$E - V \equiv$ exterior of V

fluid flow around an obstacle

(i) Given $\nabla^2 \psi = 0$ in $E-V$

(ii) Given $\hat{n} \cdot \nabla \psi = Q$ on ∂V

(iii) For uniqueness, need a cond. at ∞

$\nabla \psi \to 0$ as $r \to \infty$

Then ψ is determined to within an additive constant.

For a simple proof, we make a stronger hypothesis than (iii)

(iiia) $r^2 |\nabla \psi|$ bounded as $r \to \infty$

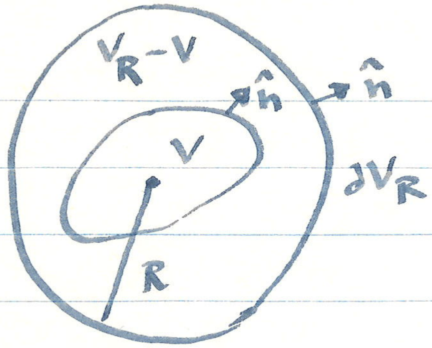
Theorem: Given two fcns ψ_1 and ψ_2 satisfying (i), (ii), and (iiia). Then $\psi_1 - \psi_2 = $ const. in $E - V$.

Proof: this time let $\psi = \psi_1 - \psi_2 +$ a const. $\ni$ $\psi \to 0$ at ∞

Then $\nabla^2 \psi = 0$ in $E - V$

$\hat{n} \cdot \nabla \psi = 0$ on ∂V

$r^2 |\nabla \psi|$ bd. as $r \to \infty$



let V_R be a ball of radius R completely enclosing V and let $\partial V_R \equiv$ surface of V_R

$$\int_{V_R - V} |\nabla\psi|^2 \, dV = \int_{V_R - V} [\nabla\cdot(\psi\nabla\psi) - \psi \underbrace{\nabla^2\psi}_{\text{zero}}] \, dV$$

$$= \int_{V_R - V} \nabla\cdot(\psi\nabla\psi) \, dV = \int_{\partial V_R} \psi \hat{n}\cdot\nabla\psi \, dA - \int_{\partial V} \psi \underbrace{\hat{n}\cdot\nabla\psi}_{\text{zero}} \, dA$$

so

$$\int_{V_R - V} |\nabla\psi|^2 \, dV = \int_{\partial V_R} \psi \hat{n}\cdot\nabla\psi \, dA$$

now we have chosen ψ so that $\psi(\infty) = 0$ (the reference level is arbitrary)

Assume $\psi \to 0$ as $r \to \infty$

$$\int_{V_R - V} |\nabla\psi|^2 \, dV = \int_{\partial V_R} \psi \hat{n}\cdot\nabla\psi \, dA$$

$$\leq 4\pi R^2 \underbrace{\max_{r=R} |\nabla\psi|}_{\text{bd.}} \underbrace{\max_{r=R} |\psi|}_{\to \text{ zero as } R \to \infty}$$

hence $\int_{V_R - V} |\nabla\psi|^2 \, dV \to 0$ as $R \to \infty$

$$\int_{E-V} |\nabla \Psi|^2 dV = 0 \quad \text{hence} \quad |\nabla \Psi| = 0$$

or $\quad \Psi_1 - \Psi_2 + \text{a const.} = \text{a const.}$

$$\Psi_1 - \Psi_2 = \text{const.} \qquad \text{q.e.d.}$$

For proof with weaker cond. at ∞ see Batchelor-116.
Existence can be shown for the three above conds. Do not need extra condition on $\int_{\partial V} \textcircled{a}\, dA = 0$

No physical reason for it either.

Once again we see the peculiar nature of incompr. potential flow. The flow of an incompr. fluid about a body is completely determined by the motion of the body.

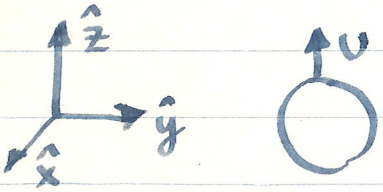
neither the acceleration or past history is relevant.

Important: Note that the flow field is determined at any time t by the instantaneous velocity of the body.
~~[illegible]~~
This strikingly peculiar result is true because the eqns of motion do not contain t explicitly, the b.c. which determine the solution uniquely contain only instantaneous quantities (no time derivatives)

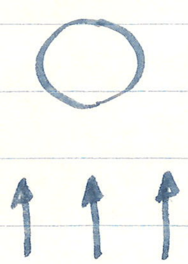
Familiar properties of fluids which are in conflict with these results are due to our neglect of <u>elastic</u> and <u>viscous</u> properties of fluids.

14. <u>Example: a uniformly moving spherical body, no gravity</u>

Consider a spherical body moving with velocity $\underline{U} = U\hat{z}$ through a homog. incompressible fluid



equivalent problem, flow of the fluid past a stationary spherical obstacle



$\underline{U} = U\hat{z}$ at ∞

We consider problem first from first point of view.

assume no motion at ∞ (fluid at rest at ∞ in fluid referance frame)

$$\left.\begin{array}{lr} \nabla^2 \psi = 0 & r \geq a \\ \hat{n} \cdot \nabla \psi = U \cos\theta & \text{on } r = a \\ \nabla \psi \to 0 & \text{at } r = \infty \end{array}\right\} \begin{array}{l} \text{consider at } t=0 \\ \text{center at } x, y, z = 0 \end{array} \quad \star$$

can consider separately at each instant t, because of the already noticed property of the flow (no dependence on past history or acceleration).

By above thm, the problem $\star$ has a unique soln.
Try the solution

$$\psi = -\frac{U}{2} \frac{a^3}{r^2} \cos\theta$$

$$\frac{\cos\theta}{r^2} = \frac{z}{r^3} = \frac{\partial}{\partial z}\left(\frac{1}{r}\right) \quad \text{and} \quad \nabla^2 \frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \frac{\partial}{\partial z} \nabla^2 \frac{1}{r} = 0$$ (except at $r = 0$, but

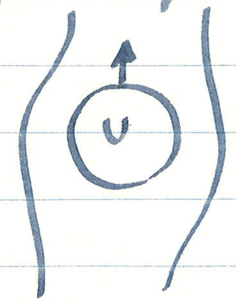
certainly outside $r \geq a$)

also $\psi \to 0$ as $r \to \infty$ and

$$\hat{n} \cdot \nabla \psi = \hat{r} \cdot \nabla \psi = \partial \psi / \partial r = U \frac{a^3}{r^3} \cos\theta$$

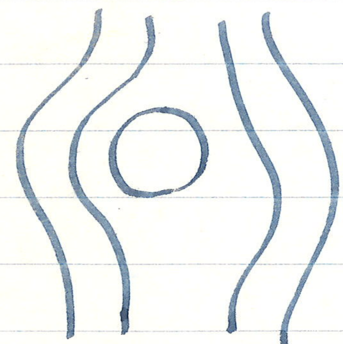
$= U \cos\theta$ on $r = a$

by uniqueness, this is <u>the</u> solution



streamlines

now to compute associated <u>pressure</u> fluctuations, go to <u>body</u> reference frame where motion is steady



in <u>body</u> reference frame $\underline{u}'(\underline{r},t) = \underline{u}(\underline{r},t) - U\hat{z}$

$\underline{u} = U\hat{z}$ at ∞ ($\mp\infty$)

$$\underline{u}' = \underline{u} - U\hat{z} = \nabla\psi - U\hat{z} = \nabla(\psi - Uz) = \nabla\psi'$$

$$\psi' = \psi - Uz = -U\cos\theta\left(r + \frac{1}{2}\frac{a^3}{r^2}\right)$$

now for <u>steady</u> motion, can use <u>Bernoulli theorem II</u>

$$\frac{p_\infty}{\rho_0} + \frac{1}{2}U^2 = \frac{p}{\rho_0} + \frac{1}{2}(u')^2 \qquad u' = \nabla\psi'$$

$$(u')^2 = U^2\left[1 + (\sin^2\theta - 2\cos^2\theta)\frac{a^3}{r^3} + (\cos^2\theta + \tfrac{1}{4}\sin^2\theta)\frac{a^6}{r^6}\right]$$

$$\frac{p - p_\infty}{\rho_0} = \frac{U^2}{2}\left(\frac{a}{r}\right)^3\left[(3\cos^2\theta - 1) - \frac{1}{4}\left(\frac{a}{r}\right)^3(3\cos^2\theta + 1)\right]$$

pressure on the sphere

$$\boxed{\frac{p_{\text{sphere}} - p_\infty}{\rho_0} = \frac{U^2}{2}\left[\frac{9}{4}\cos^2\theta - \frac{5}{4}\right]}$$

plot on surface of sphere

pressure the same in front & in back

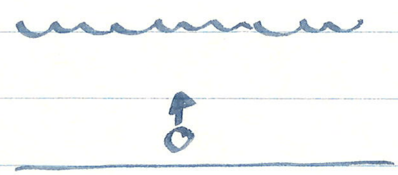
$-\frac{5}{4}$ (all times $\frac{U^2}{2}$)

reason: there is no dissipation of energy or flow of energy to ∞

clearly integrates to zero $\rightarrow$ perfect fluid theory predicts no drag force (D'Alembert's paradox)

No longer of course a paradox – drag forces are due to viscosity of fluids. The flow pattern we have predicted is in fact quite accurate except in a thin viscous boundary layer on surface of sphere. This gives rise to a viscous drag but affects the flow pattern very little. Energy dissipation (viscous).

Problem:



massless ping pong ball – what is the ensuing motion.

You may neglect the effect of the bottom.

end here 15 Mar '72

assign problem: to compute the eigenfrequencies of a self-gravitating sphere of perfect fluid, radius a

Comment on $l = 0, 1$.

Review:

Discussing the flow of an incompressible homogeneous, non viscous fluid in a grav. field

Governing equations are:

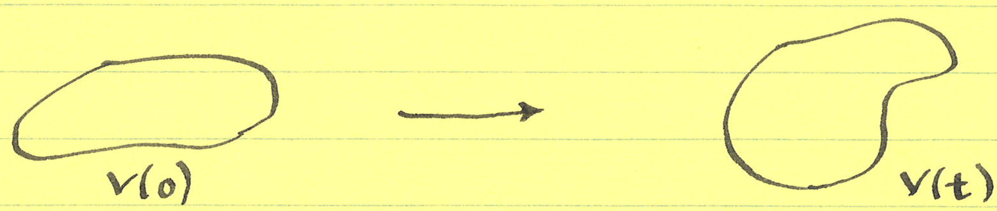
$$D_t \underline{u} = -\nabla\left(\frac{p}{\rho_0} + \phi\right)$$

$$\nabla \cdot \underline{u} = 0$$

$$\rho(\underline{r},t) \equiv \rho_0 \quad \text{constant}$$
$$\phi(\underline{r},t) = \text{external grav pot.}$$

4 eqns for four unknowns $\underline{u}(\underline{r},t)$, $p(\underline{r},t)$

By a consideration of the vorticity $\underline{\omega}(\underline{r},t) \equiv \nabla \times \underline{u}(\underline{r},t)$, we proved a fundamental result



Given a body of fluid (non-viscous, perfect) at time $t = 0$ with no vorticity $\underline{\omega}(\underline{r},0) = 0 \;\; \forall\, \underline{r} \in V(0)$

Then for all later times t, that body of fluid continues to have zero vorticity.

$$\underline{\omega}(\underline{r}, t) = 0 \quad \forall\, \underline{r} \in \mathcal{V}(t)$$

This fundamental result led us to pay particular attention to <u>potential flow</u>

If $\nabla \times \underline{u}(\underline{r}, t) = 0$, then $\underline{u}(\underline{r}, t)$ can be written in terms of a velocity potential

$$\underline{u}(\underline{r}, t) = \nabla \psi(\underline{r}, t)$$

Procedure for examining potential flow using eqns ★.

1. $\nabla \cdot \underline{u} = 0 \Rightarrow \nabla^2 \psi = 0$
   solve Neumann problem for $\psi(\underline{r}, t)$

2. find velocity $\underline{u}(\underline{r}, t)$ by $\underline{u} = \nabla\psi$
   motion thus <u>completely</u> determined by <u>cont. eqn.</u> alone

3. use $D_t \underline{u} = -\nabla\left(\frac{P}{\rho_0} + \phi\right)$ to deduce <u>Bernoulli thm</u>, use this to find pressure $p(\underline{r}, t)$

$$P/\rho_0 = -\partial_t \psi - \tfrac{1}{2}|\nabla\psi|^2 - \phi + f(t)$$

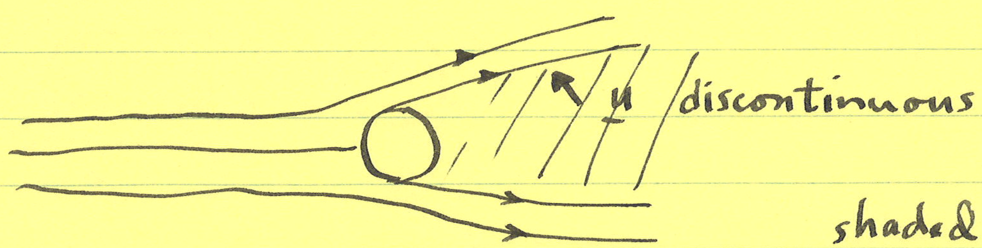
Potential flow problems have unique solutions.

We deduced, for example, the unique potential flow around a sphere.

Unfortunately, we are not entitled to say that we have found the unique solution for flow around a sphere

We have found the unique potential flow under the assumption that the flow is everywhere a potential flow

The eqns * also admit solns with separation at the surface of the body



shaded region not necessarily potential flow.

Note: such solutions are not in conflict with Kelvin's theorem. Irrotational upstream elements remain irrotational. But the tang. discont. of $\underline{u}$ allows the wake to be to be rotational. There are an ∞ no. of such solns with separation.

None of these solutions are physically significant, since such tang. disconts. in $\underline{u}$ are dynamically unstable (Kelvin-Helmholtz) and in practice turbulence would set in.

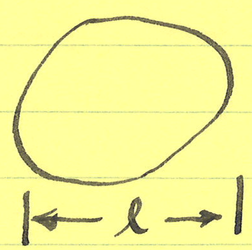
In practice, our solution (assuming pot. flow everywhere) is the correct soln for low Re; for higher Re the discont. solns are correct. Precisely why this is so cannot be learned from our non-viscous eqns.

When Re is low, actual flows are <u>observed</u> to be potential flows (except in thin bdry layers) and our theory is a good one. At higher Re our theory breaks down. This breakdown is <u>not</u> in conflict with Kelvin's theorem. <u>Except in the wake</u> the flow remains potential flow, in agreement with Kelvin. It is the possible existence of discont. solns that leads to the breakdown.

There is one circumstance for which we can assume that the flow of a non viscous fluid is <u>necessarily potential</u> flow in general

I.e., $\exists$ one circumstance in which discont. solns. can be shown <u>not to exist</u>. This is the circumstance of a body undergoing <u>small oscillations</u> in a fluid at rest.

In any such circumstance, <u>potential flow</u> is the only possible flow. The only effect of viscosity is to give rise to thin bdry layers which adhere to the body and <u>do not detach</u>. The proof does <u>not</u> depend on Kelvin's theorem.



Say body undergoes oscillations of amplitude $a \ll l$

We have $$D_t \underline{u} = -\nabla\left(\frac{p}{\rho_0} + \phi\right)$$

$$\partial_t \underline{u} + \underline{u} \cdot \underline{\underline{\nabla u}} = -\nabla\left(\frac{p}{\rho_0} + \phi\right)$$

Say the body has a velocity $\underline{v}$.
The fluid motion has a scale length $\approx l$

Hence
$$\partial_t \underline{u} \sim \omega v \qquad \omega \equiv \text{freq. of oscillation}$$
$$\underline{u} \cdot \underline{\underline{\nabla u}} \sim v^2/l$$

Now $\omega \sim v/a$ so $\partial_t \underline{u} \sim v^2/a$

If $a \ll l$ then $\underline{u} \cdot \underline{\underline{\nabla u}} \ll \partial_t \underline{u}$

Hence the non-linear term $\underline{u} \cdot \underline{\underline{\nabla u}}$ can be neglected if the oscillation is small
The mom. eqn. becomes

$$\partial_t \underline{u} = -\nabla \left(\frac{p}{\rho_0} + \phi \right)$$

Taking the curl of both sides

$$\nabla \times \partial_t \underline{u} = 0$$
$$\partial_t (\nabla \times \underline{u}) = 0$$
$$\nabla \times \underline{u} = \text{const. in time}$$

But in oscillatory motion $\langle \underline{u} \rangle = 0$, hence $\langle \nabla \times \underline{u} \rangle = 0$ and thus $\nabla \times \underline{u} = 0$

Thus the motion of a fluid executing small oscillations is a potential flow to first order in a/l.

This argument does not depend on Kelvin's theorem.

In general, we can be assured that potential flow is a very good approx. in any case of small oscillations.

We now apply this result to a discussion of the Slichter mode of the ⊙.

Lecture # 18

15. The Slichter Mode

Now good evidence that Earth has a solid inner core

Strong seismological evidence

PKiKP (Davies, Flinn)

PKJKP just recently observed (Davies)

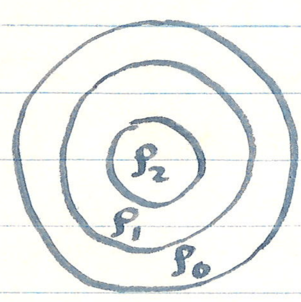
appears to be necessary to fit certain free oscillation data

Consider the following simple model of the Earth

1. rigid mantle and inner core
2. outer core incompr. fluid of constant density

Consider oscillations of inner core about c.o.m. of Earth. A possible mode of motion of such an ⊕ model.

Could be detected by a gravimeter at the surface



take $a_0 = 6400$ km

$a_1 = 3500$ km

$a_2 = 1200$ km

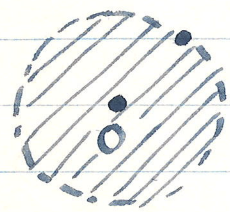
rounded off

I. first approximation

let the inner core be a <u>point mass</u>
liquid does not interfere
(can exert no buoyant forces, no energy is required to push water out of the way.

we have merely a pt. mass moving in a gravitational field

Fix coord. system to c.o.m. of $\oplus$
(fixed in space; an inertial frame)



move particle out a distance $\underline{r}$ from c.o.m.
only mass <u>inside</u> exerts a grav. attraction

force exerted on particle $= m\underline{g}$

$$\underline{g} = -\frac{\frac{4}{3}\pi r^3 \rho_1}{r^2}\,\hat{r} = -\frac{4}{3}\pi\rho_1 G\,\underline{r}$$

eqn. of motion of particle $\underline{F} = m\underline{a}$

$$m\underline{g} = m\ddot{\underline{r}}$$

$$m\underline{\ddot{r}} = -m\frac{4}{3}\pi G\rho_1 \underline{r}$$

$$\boxed{\underline{\ddot{r}} + \frac{4}{3}\pi G\rho_1 \underline{r} = 0}$$

eqn for a <u>simple harmonic oscillator</u>
frequency of oscillation

$$\boxed{\omega_I^2 = \frac{4}{3}\pi G\rho_1}$$

ρ_1 = density out fluid outer core
$\cong \cancel{11}\ 12$ gm / cm^3

$$\omega_I^2 \sim \frac{4}{3}(3.14)(6.67 \cdot 10^{-8})(\cancel{11}\ 12)$$
$$\sim 3.10 \cdot 10^{-6}$$
$$3.4 \cdot 10^{-6}$$

$\frac{2\pi}{\omega}$ = T ~ 3500 sec or T ~ 1 hr.

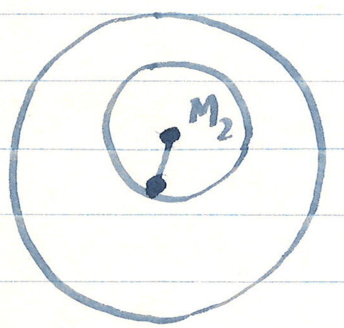
this gives a <u>very</u> rough idea of frequency of oscillation of this mode.

II. Second approximation

Take the inner core to have a finite volume but assume that at any time the force being exerted on the sphere is the same as if the sphere were at rest. We neglect the fact that the core must move water out of the way.

Furthermore let us treat the mantle as if it is of ∞ mass — does not move

We have then a rigid sphere moving inside a fluid contained within a spherical cavity.



let M_2 = mass of inner core

$\underline{s}$ = displacement of center of inner core

This produces an error of order mass of inner core / mass of mantle

$$\sim \frac{9 \cdot 10^{25} \text{ gm}}{4 \cdot 10^{27} \text{ gm}} \sim 2\%$$

We assume that the $\underline{\text{force}}$ exerted on displaced sphere is as if the displacement were completely static

$$\underline{F} = \underline{F}_{grav} + \underline{F}_{buoyant}$$

$\underline{F}_{grav}$ may be calculated as if there were no hole in the fluid since net force of fluid in M_2 would be zero by symmetry

Mantle has no grav. effect | your ass. of ∞ mass thus plays no role.

$\underline{F}_{grav}$ hence same as if the core were a pt. mass.

$$\underline{F}_{grav} = -\tfrac{4}{3}\pi G \rho_1 M_2 \underline{s}$$

$\underline{F}_b$ by Archimedes principle = the grav. force exerted on a sphere of fluid occupying the same volume

$$\underline{F}_{buoyant} = \left(\tfrac{4}{3}\pi G \rho_1 M_2 \underline{s}\right)\frac{\rho_1}{\rho_2}$$

Again Newton's second law of motion

$$\underline{F} = \underline{F}_{grav} + \underline{F}_{buoy} = M_2 \ddot{\underline{s}}$$

$$M_2 \ddot{\underline{s}} = -M_2 \left(\tfrac{4}{3}\pi G\rho_1\right)\left(1-\frac{\rho_1}{\rho_2}\right)\underline{s}$$

$$\boxed{\ddot{\underline{s}} + \left(\tfrac{4}{3}\pi G\rho_1\right)\left(1-\frac{\rho_1}{\rho_2}\right)\underline{s} = 0}$$

again simple harmonic oscillator eqn.

$$\boxed{\begin{aligned}\omega_{II}^2 &= \tfrac{4}{3}\pi G\rho_1\left(1-\rho_1/\rho_2\right) \\ &= \omega_I^2\left(1-\rho_1/\rho_2\right)\end{aligned}}$$

$\rho_1 \approx \cancel{11}\ 12$ gm/cm^3

$\rho_2 \approx \cancel{13}\ 13$ gm/cm^3 (density of solid inner core)

$$1-\rho_1/\rho_2 = 1-{}^{11}/_{13} = \frac{2}{13}$$

$$1-\frac{12}{13} = \frac{1}{13}$$

effect of ~~buoyant force~~ on solid inner core is to increase the period by $\sim \sqrt{13/2} \sim 2.5$

$\sqrt{13} \approx 3.6$ hrs

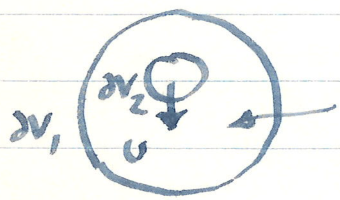
Period now increased to ~ ~~2.5~~ 3.6 hrs.

Physically, effect of buoyant force is to weaken the restoring force, thus decrease the frequency.

We have assumed that the reactive force of the liquid on the solid inner core is purely due to the static buoyancy

We have not considered dynamic pressure fluctuations due to Bernoulli theorem.

Straightforward way to do this is to solve



$$\nabla^2 \psi = 0 \quad \text{in fluid}$$

$$\hat{n} \cdot \nabla \psi = U \cos\theta \quad \text{on } \partial V_2$$

$$\hat{n} \cdot \nabla \psi = 0 \quad \text{on } \partial V_1$$

Then use Bernoulli thm to find the pressure on surface of inner core and integrate to find net force.

This may be difficult.

Let's adopt a simpler approach

System only has three degrees of freedom (position $\underline{s}$ of inner core)

Consider one-dimensional motion (clearly symmetric)

Said in another way: θ, ϕ are ignorable coordinates.

The dynamics of the system is completely described by giving the <u>Lagrangian</u> of the system $L(s, \dot{s})$

$$L = T - V \quad \text{conservative system}$$
$$T = K.E. = T(\dot{s})$$
$$V = P.E. = V(s)$$

small oscillations of a conservative system (Goldstein, Ch. 10)

Equation of motion is <u>Lagrange's eqn</u>

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0$$

Use Lagrangian (one-dimensional) approach on <u>second</u> <u>approximation</u>

K.E. we assumed that only the K.E. of the solid inner core was important — neglected K.E. of motion of fluid core

$$\boxed{T(\dot{s}) = \frac{1}{2} M_2 \dot{s}^2}$$

P.E. gravitational potential energy of sphere - fluid system

Recall force on sphere

$$\underline{F} = \underline{F}_{grav} + \underline{F}_{buoy} = -\frac{4}{3}\pi G\rho_1 M_2\left(1-\frac{\rho_1}{\rho_2}\right)\underline{s}$$

change in P.E. due to a displacement increment $\underline{ds}$

$$dV = -\underline{ds}\cdot\underline{F}$$

$$dV = \frac{4}{3}\pi G\rho_1 M_2\left(1-\frac{\rho_1}{\rho_2}\right) s\,ds$$

$$V = \frac{4}{3}\pi G\rho_1 M_2\left(1-\frac{\rho_1}{\rho_2}\right)\frac{s^2}{2} + \text{const}$$

This is the gravitational potential energy of the displaced configuration

take = zero ; P.E. = 0 for zero displacement

$$\boxed{V(s) = \frac{4}{3}\pi G\rho_1 M_2\left(1-\frac{\rho_1}{\rho_2}\right)\frac{s^2}{2}}$$

This is the gravitational potential energy.

$$L(s,\dot{s}) = T(\dot{s}) - V(s)$$

Lagrange's eqn $\quad \frac{d}{dt}(M_2\dot{s}) + \frac{4}{3}\pi G\rho_1 M_2\left(1-\frac{\rho_1}{\rho_2}\right)s = 0$

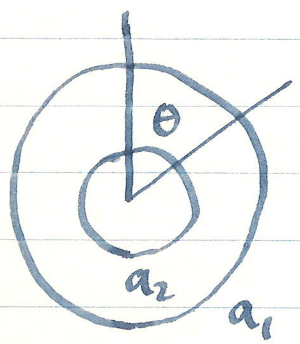
$$\boxed{\ddot{s} + \frac{4}{3}\pi G\rho_1\left(1-\frac{\rho_1}{\rho_2}\right)s = 0}$$

led to same eqn. as before

III. Third approximation

We see that the main ingredient we have neglected is the <u>K.E. of the fluid</u>.

Let's now assume the displacement s is <u>small</u> compared to $a_1 - a_2$



$s \ll a_1 - a_2$

flow of fluid outer core $\underline{u} = \nabla \psi$

$$\nabla^2 \psi = 0 \qquad a_2 \le r \le a_1$$
$$\partial \psi / \partial r = 0 \quad \text{on} \quad r = a_1$$
$$\partial \psi / \partial r = U \cos\theta \quad \text{on} \quad r = a_2$$
$$U = \dot{s}$$

Can easily establish a <u>uniqueness theorem</u> for this situation
$\exists$ a unique solution

Simultaneously interior and exterior.

Try solution $\psi = R(r) \cos\theta$

$$\nabla^2 = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2\right]$$

$$\nabla^2 R(r)\cos\theta = \left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right)\cos\theta - \frac{2}{r^2}R\cos\theta$$

$$\left[\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{2}{r^2}R\right]\cos\theta = 0$$

homog. linear eqn. for $R(r)$
equidemensional eqn
Try solution $R = r^n$

$$n(n-1) + 2n - 2 = 0$$
$$n^2 + n - 2 = 0$$
$$(n+2)(n-1) = 0$$
$$n = -2 \quad \text{or} \quad n = 1$$

$$R(r) = Ar + \frac{B}{r^2}$$

b.c. $\frac{dR}{dr} = 0$ at $r = a_1$

$\frac{dR}{dr} = U$ at $r = a_2$

determines A and B

$$A = -U\left[\frac{a_2^3}{a_1^3 - a_2^3}\right], \quad B = -\frac{U}{2}\left[\frac{a_1^3 a_2^3}{a_1^3 - a_2^3}\right]$$

This determines $\psi(r,\theta)$ the velocity potential

Now K.E. of moving fluid

$$T_{fluid} = \frac{1}{2}\rho_1 \int_{V_1-V_2} |\nabla\psi|^2 \, dV$$

$$\int_{V_1-V_2} |\nabla\psi|^2 dV = \int_{V_1-V_2} \nabla\cdot(\psi\nabla\psi)\,dV - \underbrace{\int_{V_1-V_2} \psi\nabla^2\psi\,dV}_{\text{zero}}$$

$$= \int_{\partial(V_1-V_2)} \psi\,\hat{n}\cdot\nabla\psi\,dA = \underbrace{\int_{\partial V_1} \psi\frac{\partial\psi}{\partial r}dA}_{\text{zero}} - \int_{\partial V_2} \psi\frac{\partial\psi}{\partial r}\,dA$$

$$= -U\int_{\partial V_2} \psi\cos\theta\,dA$$

$$T_{fluid} = -\frac{1}{2}\rho_1 U \int_{\partial V_2} \left(A a_2 + \frac{B}{a_2^2}\right)\cos^2\theta\,dA$$

$$= -\frac{1}{2}\rho_1 U\left(A + \frac{B}{a_2^3}\right)a_2^3 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\,\sin\theta\cos^2\theta$$

letting $U = \dot{s}$, and doing the integral

$$T_{fluid} = \frac{4}{3}\pi\rho_1 a_2^3 \frac{\dot{s}^2}{2} \underbrace{\left[\frac{\frac{1}{2} + \left(\frac{a_2}{a_1}\right)^3}{1 - \left(\frac{a_2}{a_1}\right)^3} \right]}_{\text{call this } \beta}$$

$$\boxed{T_{fluid} = \frac{1}{2} M_2 \dot{s}^2 \underbrace{\left[\frac{\frac{1}{2} + \left(\frac{a_2}{a_1}\right)^3}{1 - \left(\frac{a_2}{a_1}\right)^3} \right]}_{\text{call this } \beta} \frac{\rho_1}{\rho_2}}$$

reason for singularity at $a_2 = a_1$ is clear

As $a_2 \to a_1$, fluid must move very rapidly to get out of the way

$$T_{fluid} = \frac{1}{2} M_2 \dot{s}^2 \left(\frac{\rho_1}{\rho_2}\beta\right)$$

$$T_{total} = \frac{1}{2} M_2 \dot{s}^2 \left(1 + \beta\frac{\rho_1}{\rho_2}\right)$$

$$L(s, \dot{s}) = T_{total}(\dot{s}) - V(s)$$

$$\boxed{L(s, \dot{s}) = \frac{1}{2} M_2 \dot{s}^2 \left(1 + \beta\frac{\rho_1}{\rho_2}\right) - \frac{4}{3}\pi G \rho_1 M_2 \left(1 - \frac{\rho_1}{\rho_2}\right)\frac{s^2}{2}}$$

Put into Euler's equations

$$\ddot{s}\left(1+\beta\frac{\rho_1}{\rho_2}\right) + \frac{4}{3}\pi G\rho_1\left(1-\frac{\rho_1}{\rho_2}\right)s = 0$$

again S.H. oscillator eqn

$$\boxed{\begin{aligned} \omega_{III}^2 &= \frac{4}{3}\pi G\rho_1\left[\frac{1-(\rho_1/\rho_2)}{1+\beta\left(\frac{\rho_1}{\rho_2}\right)}\right] \\ &= \omega_I^2\left[\frac{1-(\rho_1/\rho_2)}{1+\beta(\rho_1/\rho_2)}\right] \end{aligned}}$$

Effect of K.E. of fluid is to further lower the char. freq. and to further increase the period

$a_1 = 3500$ km.
$a_2 = 1200$ km.

$$\frac{a_2}{a_1} = \frac{1200}{3500} = .343 \qquad \left(\frac{a_2}{a_1}\right)^3 = .040$$

$$\beta = \frac{\frac{1}{2}+\left(\frac{a_2}{a_1}\right)^3}{1-\left(\frac{a_2}{a_1}\right)^3} = \frac{.5+.04}{1-.04} = \frac{.54}{.96} = .55$$

$$\frac{1-12/13}{1+0.55\times 12/13} = \frac{1}{13}\,\frac{1}{1.5} = \frac{1}{13}\cdot\frac{2}{3} \approx \frac{1}{20}$$

$$\left(\frac{1-\rho_1/\rho_2}{1+\beta\,\rho_1/\rho_2}\right) = \left(\frac{1-\frac{11}{13}}{1+(.55)\frac{11}{13}}\right) = \frac{2}{13}\left(\frac{1}{1.5}\right) = 0.1$$

the period is increased by a factor of

$$\left[\frac{1-\left(\frac{\rho_1}{\rho_2}\right)}{1+\beta\left(\frac{\rho_1}{\rho_2}\right)}\right]^{-1/2} = \sqrt{10} \approx 3.2$$
$$\sqrt{20} \approx 4.5$$

PREM: Slichter mode 5.5 hrs or 325 min

$$\boxed{\text{period} \sim 3.2 \text{ hrs. } (4.5 \text{ hrs})}$$

Model DG579 gives $T \approx 4.3 \pm 0.4$ hrs (split triplet)

Smith, J.G.R. (1976) 81, 3055–3065. Translational inner core oscillations of a rotating, slightly elliptical Earth.

Such a motion would have a period ~ 3 hrs.

It could possibly be detected by a gravimeter at surface of Earth

Freeman Gilbert of La Jolla, using a much better model of the real Earth found a period of the so-called Slichter mode of ~ 4 hrs.

Suggested that it might possibly have a very high Q (most of energy is compr. energy in the solid inner core).

Has never been observed, but worth looking for in gravimeter records.

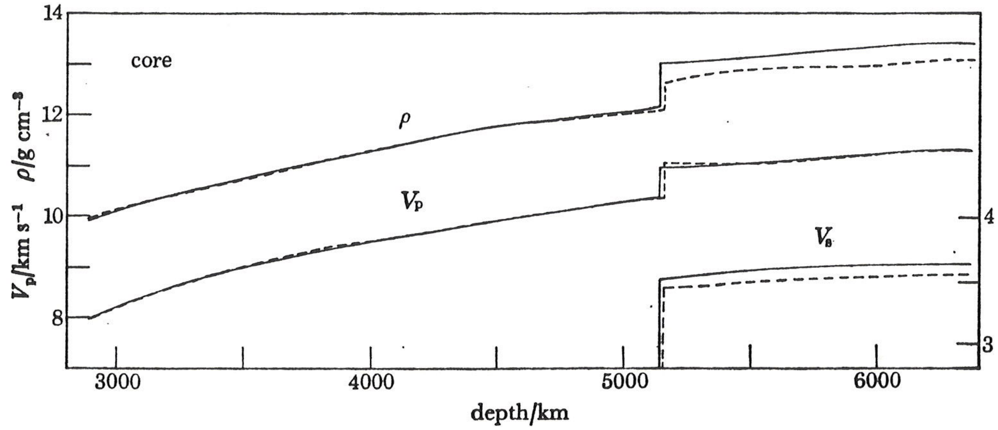


Figure 15. The Earth models obtained as a result of inverting 1463 gross Earth data, of which 1066 are distinct. model 1066A was derived from model 508 and model 1066B was derived from model B1.

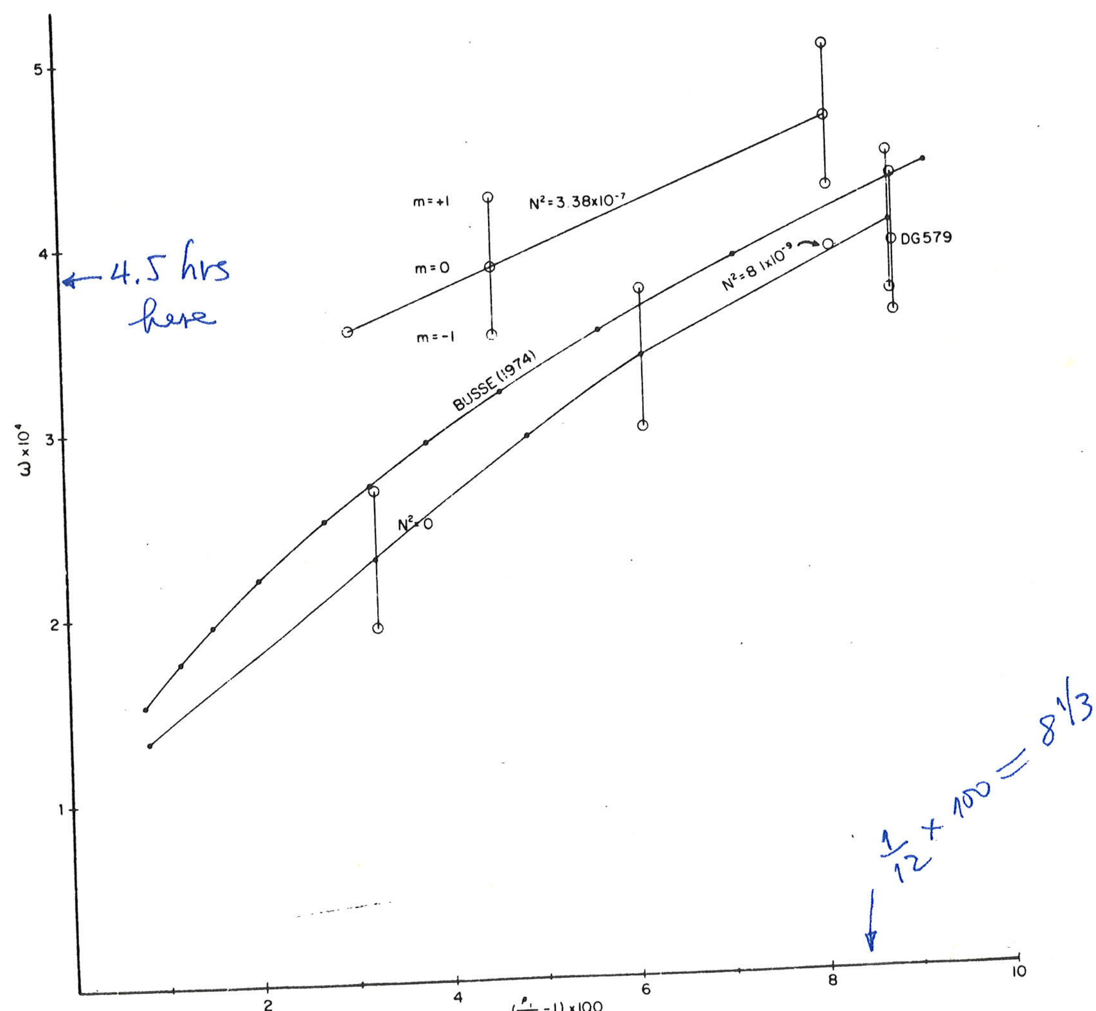


Fig. 9. Eigenfrequencies as a function of fractional density contrast between the inner core (ρ_i) and the outer core (ρ_o) computed in this study and those of *Busse* [1974]. The latter are for the m = 0 mode. The triplet for each model is joined by a vertical bar; $\omega^+ > \omega^0 > \omega^-$ in each case.

$$m\ddot{\underline{r}} + 2m\underline{\Omega}\times\dot{\underline{r}} + m\underline{\Omega}\times(\underline{\Omega}\times\underline{r}) + \underline{F}_B + \underline{F}_H = 0$$

ρ_2 ρ_1

$a_2/a_1 \to 0 \qquad \beta \to \frac{1}{2}$

$$\ddot{\underline{r}}\left(1+\frac{1}{2}\frac{\rho_1}{\rho_2}\right)$$

$$m = \frac{4}{3}\pi\rho_2 a^3$$

$$\ddot{s}\left(1+\frac{1}{2}\frac{\rho_1}{\rho_2}\right)$$

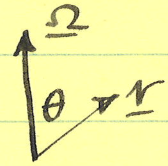
$$\ddot{\underline{r}}\left(1+\frac{1}{2}\frac{\rho_0}{\rho_i}\right) + 2\underline{\Omega}\times\dot{\underline{r}} + \underline{\Omega}\times(\underline{\Omega}\times\underline{r})$$
$$+ \frac{4}{3}\pi G\rho_0\left(1-\frac{\rho_0}{\rho_i}\right)\underline{r} = 0$$

along the rotation axis, $\underline{\Omega}=\underline{0}$ is applicable

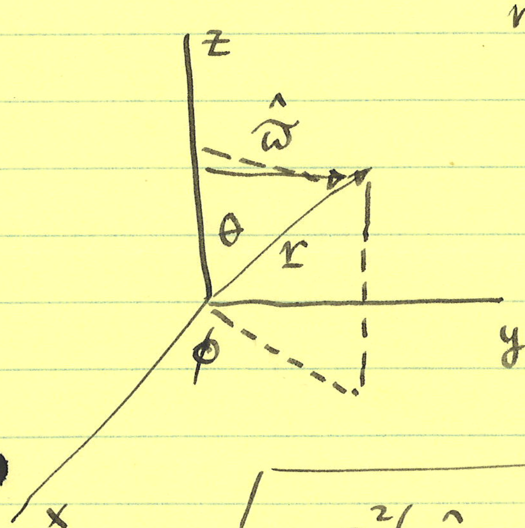
$$\ddot{\underline{r}}\left(1+\frac{1}{2}\frac{\rho_o}{\rho_i}\right) + 2\underline{\Omega}\times\dot{\underline{r}} + \underline{\Omega}\times(\underline{\Omega}\times\underline{r})$$

$$+ \frac{4}{3}\pi G\rho_o\left(1-\frac{\rho_o}{\rho_i}\right)\underline{r} = 0$$

now $\underline{\Omega}\times(\underline{\Omega}\times\underline{r}) =$



$$-\ \Omega^2 r\sin\theta\,\hat{\varpi}$$



$r\sin\theta = \sqrt{x^2+y^2}$

$\cancel{\hat{\varpi} = \frac{1}{\sqrt{2}}(\hat{x}+\hat{y})}$

$$\hat{\varpi} = \frac{x\hat{x}+y\hat{y}}{\sqrt{x^2+y^2}}$$

$$\boxed{-\Omega^2(x\hat{x}+y\hat{y})}$$

$$2\Omega\times\dot{r} = 2\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 0 & 0 & \Omega\\ \dot{x} & \dot{y} & \dot{z}\end{vmatrix} =$$

$$-2\hat{x}\,\Omega\dot{y} + 2\hat{y}(\Omega\dot{x})$$

$$\ddot{x}\left(1+\frac{1}{2}\frac{\rho_0}{\rho_i}\right)-2\Omega\dot{y}+\left[\frac{4}{3}\pi G\rho_0\left(1-\frac{\rho_0}{\rho_i}\right)-\Omega^2\right]x=0$$

$$\ddot{y}\left(1+\frac{1}{2}\frac{\rho_0}{\rho_i}\right)+2\Omega\dot{x}+\left[\frac{4}{3}\pi G\rho_0\left(1-\frac{\rho_0}{\rho_i}\right)-\Omega^2\right]y=0$$

$$\ddot{z}\left(1+\frac{1}{2}\frac{\rho_0}{\rho_i}\right)+\left[\frac{4}{3}\pi G\rho_0\left(1-\frac{\rho_0}{\rho_i}\right)\right]z=0$$

consider the z equation

SHM

$$\omega^2=\frac{4}{3}\pi G\rho_1\left[\frac{1-\rho_1/\rho_2}{1+\frac{1}{2}\frac{\rho_1}{\rho_2}}\right]$$

$$\boxed{\omega^2=\frac{4}{3}\pi G\rho_0\left[\frac{1-\rho_0/\rho_i}{1+\frac{1}{2}\rho_0/\rho_i}\right]}$$ as before

now the coupled eqns

$$a\ddot{x}-2\Omega\dot{y}+bx=0$$
$$a\ddot{y}+2\Omega\dot{x}+by=0$$

let $u=x+iy$

$v=x-iy$

$$a(\ddot{x}+i\ddot{y})-2\Omega(\dot{y}-i\dot{x})+b(x+iy)=0$$

$$\boxed{a\ddot{u}-2i\Omega\dot{u}+bu=0}$$

$$a(\ddot{x}-i\ddot{y})-2\Omega(\dot{y}+i\dot{x})+b(x-iy)=0$$

$$\boxed{a\ddot{v}+2i\Omega v+bv=0}$$

$$e^{i\omega t}$$

$$-a\omega^2 - 2i\Omega(i\omega) + b = 0$$

$$a\omega^2 - 2\Omega\omega - b = 0$$

$$\omega = \frac{2\Omega \pm \sqrt{4\Omega^2 + 4ba}}{2a}$$

$$\omega = \frac{1}{a}\left[\Omega \pm \sqrt{\Omega^2 + ba}\right]$$

$$\cancel{\omega = \Omega \pm \sqrt{\tfrac{4}{3}\pi G\rho_0\left(1-\tfrac{\rho_0}{\rho_i}\right)}}$$

$$ba = \left[\frac{4}{3}\pi G\rho_0\left(1-\frac{\rho_0}{\rho_i}\right) - \Omega^2\right]\left[1+\frac{1}{2}\frac{\rho_0}{\rho_i}\right]$$

$$\omega = \left(1+\frac{1}{2}\frac{\rho_0}{\rho_i}\right)^{-1}\left[\Omega \pm \sqrt{\Omega^2 + \left(1+\frac{1}{2}\frac{\rho_0}{\rho_i}\right)\left(\frac{4}{3}\pi G\rho_0\left(1-\frac{\rho_0}{\rho_i}\right)-\Omega^2\right)}\right]$$

$$\omega = \left(1+\frac{1}{2}\frac{\rho_0}{\rho_i}\right)^{-1}\left[\Omega \pm \sqrt{\frac{4}{3}\pi G\rho_0\left(1-\frac{\rho_0}{\rho_i}\right)\left(1+\frac{1}{2}\frac{\rho_0}{\rho_i}\right) - \frac{1}{2}\Omega^2\frac{\rho_0}{\rho_i}}\right]$$

$$\boxed{\omega = \left(1+\frac{1}{2}\frac{\rho_0}{\rho_i}\right)^{-1}\left[\Omega \pm \sqrt{\frac{4}{3}\pi G\rho_0\left(1-\frac{\rho_0}{\rho_i}\right)\left(1+\frac{1}{2}\frac{\rho_0}{\rho_i}\right) - \frac{1}{2}\Omega^2\frac{\rho_0}{\rho_i}}\right]}$$

reduces to previous result for $\Omega = 0$

$\rho_o < \rho_i$ for buoyancy to act

$(1 - \rho_o/\rho_i) > 0$

get complex root if

$$\frac{1}{2}\Omega^2 \frac{\rho_o}{\rho_i} > \frac{4}{3}\pi G \rho_o \left(1 - \frac{\rho_o}{\rho_i}\right)\left(1 + \frac{1}{2}\frac{\rho_o}{\rho_i}\right)$$

$$\boxed{\Omega^2 > \frac{8}{3}\pi G \rho_i \left(1 - \frac{\rho_o}{\rho_i}\right)\left(1 + \frac{1}{2}\frac{\rho_o}{\rho_i}\right)}$$

get instability

$$\Omega^2 > 2\frac{\rho_i}{\rho_o}\left(1 + \frac{1}{2}\frac{\rho_o}{\rho_i}\right)^2 \omega^2$$

for z-motion

$$2\frac{13}{11}\left(1 + \frac{1}{2}\frac{11}{13}\right)^2 \approx 2\left(\frac{3}{2}\right)^2 = 2\cdot\frac{9}{4} = \frac{9}{2}$$

$\Omega \gtrsim \frac{3}{\sqrt{2}}\omega$ or $\Omega \gtrsim 2\omega$

$\frac{2\pi}{\omega} \approx 3$ hrs okay for $\oplus$

whew!

$T <$

The effect of $\underline{\Omega}$ on the fluid flow has of course been neglected.