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Lecture # 19

III. Viscous Fluids

1. Review (non-viscous fluids)

Recall we have derived four general conservation laws valid for the deformation of any continuum.

$\partial_t \rho + \nabla \cdot (\rho \underline{u}) = 0$ cont. eqn.

$\rho D_t \underline{u} = \nabla \cdot \underline{\underline{T}} + f$ momentum eqn.

$\underline{\underline{T}} = \underline{\underline{T}}^T$ angular momentum eqn.

$\rho D_t U + \nabla \cdot \underline{H} = \mathrm{tr}(\underline{\underline{T}} \cdot \underline{\underline{\epsilon}}) + h$ energy eqn (first law of thermodynamics

We also have a general inequality valid for all continua

$\rho D_t S + \nabla \cdot (\underline{H}/\theta) - \frac{h}{\theta} \geq 0$ Clausius - Duhem inequality (second law of thermodynamics)

Recall defn of a fluid: 1. a continuum whose stress tensor is everywhere and always isotropic when the fluid is at rest.

$\underline{\underline{T}}(\underline{t}) = -p(\underline{t})\underline{\underline{I}}$ fluid pressure

2. If one also assumes that the fluid at rest is in a state of local thermodynamic equilibrium, then at any pt $\underline{r} \in$ fluid, the equations of state of the fluid can be used to deduce any thermodynamical variable $\rho(\underline{r})$, $p(\underline{r})$, $\theta(\underline{r})$, $S(\underline{r})$, $U(\underline{r})$ in terms of just two variables

Two aspects to defn of a fluid:

1. stress always isotropic when fluid at rest $\underline{\underline{T}}(\underline{r}) = -p(\underline{r})\underline{\underline{I}}$
2. always in local thermodynamic equilibrium when at rest $p(\underline{r}) = \tilde{p}(\theta(\underline{r}), \rho(\underline{r}))$, for example

not complete thermal equil. since $T(r,t) \neq$ const., so $\underline{H} \neq 0$

Note: fluid at rest need not be in complete thermodynamic equilibrium, i.e. isothermal. The hydrostatic equation only guarantees mechanical equilibrium.

Now recall defn of a non-viscous fluid: a fluid in local thermodynamic equilibrium even when moving thus even when moving

1. $\underline{\underline{T}}(\underline{r}, t) = -p(\underline{r}, t)\,\underline{\underline{I}}$ isotropic
2. eqns of state are locally satisfied
   $$p(\underline{r},t) = \tilde{p}\left(\theta(\underline{r},t), \rho(\underline{r},t)\right)$$
   $$S(\underline{r},t) = \tilde{S}\left(\theta(\underline{r},t), \rho(\underline{r},t)\right)$$

Recall that for a non-viscous fluid, since the entropy $S(\underline{r},t)$ is a *state function* (not a functional) related to the other variables by the *first law of thermodynamics*

$$\theta D_t S = D_t U + p D_t \tau$$

$$D_t u = \theta D_t S - p D_t \tau$$
$$\theta = \left(\frac{\partial u}{\partial S}\right)_\tau ; \quad p = -\left(\frac{\partial u}{\partial \tau}\right)_S$$

this can be used to write the *energy equation* [illegible] as a *conservation law for the entropy*

$$\rho D_t S + \nabla \cdot (\underline{H}/\theta) = h/\theta + H \cdot \nabla \frac{1}{\theta} \quad \text{or}$$
$$\partial_t(\rho S) + \nabla \cdot (\rho S \underline{u} + \underline{H}/\theta) = \frac{h}{\theta} + H \cdot \nabla \frac{1}{\theta}$$

current density of entropy in space

$$\underset{\text{advection}}{\rho S \underline{u}} + \underset{\text{heat flow}}{\underline{H}/\theta}$$

rate of production of entropy $\text{cm}^{-3}\,\text{sec}^{-1}$

$$\underset{\text{irreversible internal heating}}{h/\theta} + \underset{\text{irreversible heat flow}}{H \cdot \nabla \frac{1}{\theta}}$$

note: no entropy generation due to motion of fluid. Entropy generation only by heat flow and by internal heating. Same two sources as if the fluid were at rest.
This really a key aspect of the non-viscous fluid.

One has a complete system of 10 eqns and 10 unknowns for a non-viscous fluid.

continuity: $\partial_t \rho + \nabla \cdot (\rho \underline{u}) = 0$
momentum: $\rho D_t \underline{u} + \nabla p = \underline{f}$
entropy: $\rho D_t S + \nabla \cdot (\underline{H}/\theta) = h/\theta + H \cdot \nabla(\frac{1}{\theta})$
two eqns of
state: $p = \tilde{p}(\rho, \theta)$
$S = \tilde{S}(\rho, \theta)$

heat flow eqn: (also necessary to assume)
often use simple Fourier's Law $H = -K \nabla \theta$
K = thermal conductivity

This system still too hard for us to solve in general. We showed that there were geophysically interesting situations where the pressure p could be considered a fcn of the single variable ρ.

led to our discussion of pseudo-perfect fluids and perfect fluids.
This led to a complete system of equations in terms of mechanical variables alone.

2. Viscous fluids (Stokesian Generalized Newtonian)

Defn: in general we will call a viscous fluid any fluid which is not a non-viscous fluid.

We will eventually consider only one particularly simple mathematical model of a viscous fluid, namely a Newtonian viscous fluid. Whether this particular simple math. model has any counterparts among real fluids must then be decided by experiment. It turns out that most geophysical fluids (water, air, liquid Fe) are very well approximated as Newtonian viscous fluids
There are other fluids which are not so well approximated. They usually contain very complex molecules and are of interest to chemical engineers and physiologists (e.g., blood).
Also, however, ice and olivine, and magma.

Let us begin making assumptions about the nature of specialized viscous fluids.

Choose as the two fundamental thermodynamical variables the density $\rho(\underline{r}, t)$ and the internal energy $U(\underline{r}, t)$

Go to p. 210$\frac{1}{2}$

At every point $\underline{r}, t$ in the fluid, the density $\rho(\underline{r}, t)$ and the internal energy (per gram) $U(\underline{r}, t)$ are well-defined.

Now if the fluid were not moving, the equations of state would define completely the values of all other thermodynamical variables.

local thermo equil amts really to a defn of S, θ

$$\left\{ \begin{array}{l} p(U, \rho) \\ S(U, \rho) \\ \theta(U, \rho) \leftarrow \end{array} \right.$$

a system not in complete thermal equil. does not really have a well-defined temp. We define the temp θ by the eqn of state as if ∃ complete thermal equil. We work with this as if it were a real temp

Now when the fluid <u>is in motion</u>, it is not even in mechanical equilibrium.

We will still assume that both the temp. θ and the specific entropy s are given in the moving fluid by the equil values

Assume: 1. $\theta(\underline{r},t) \equiv \tilde{\theta}(v(\underline{r},t), \rho(\underline{r},t))$

2. $s(\underline{r},t) \equiv \tilde{s}(v(\underline{r},t), \rho(\underline{r},t))$

$\theta(\underline{r},t)$ and $s(\underline{r},t)$ are <u>defined</u>, even in the moving fluid by a <u>local</u> application of the eqn. of state.

The only quantitys which is not defined by the <u>local</u> <u>thermodynamic</u> value is the <u>stress tensor</u> $\underline{\underline{T}}(\underline{r},t)$ and the heat flow $\underline{H}$

~~[illegible]~~

Let's make a table

Such a fluid will be called a <u>generalized</u> <u>Newtonian viscous fluid</u>

<u>Real fluid</u> (<u>gen'lized Newtonian</u>)		<u>Local Thermodynamic equil. state</u>	
U	$=$	$U_{eq.}$	} <u>fundamental</u>
ρ	$=$	ρ_{eq}	
Θ	$=$	$\Theta_{eq}(U,\rho)$	} <u>defn.</u>
S	$=$	$S_{eq}(U,\rho)$	
$\underline{\underline{T}}$	$\neq$	$\underline{\underline{T}}_{eq} = -P_{eq}(U,\rho)\underline{\underline{I}}$	
$\underline{H}$	$\neq$	$\underline{H}_{eq} = \underline{0}$	

Assumptions 1. and 2. are <u>constitutive relations</u> for the generalized Newtonian <u>viscous fluid</u>.

We still need to discuss the constitutive relations for the two <u>non-equil quantities</u>

Usually one still assumes the simple linear <u>Fourier's law</u> for the heat flow vector
$\underline{H} = -\kappa\nabla\Theta = -\kappa\nabla\Theta_{eq}.$

The main constitutive relation to which we must pay attn is the one for the non-equilibrium stress.

A better approach to this:
Use as the two fundamental variables s and $\tau = \rho^{-1}$.

$$\left.\begin{array}{l} s(\underline{r},t) \\ \tau(\underline{r},t) \end{array}\right| \quad \text{fundamental equation of state } u = \tilde{u}(\tau, s)$$

In l.t.e. $\theta = \left.\dfrac{\partial\tilde{u}}{\partial s}\right|_{\tau}$, $-p = \left.\dfrac{\partial\tilde{u}}{\partial\tau}\right|_{s}$

We assume, for a viscous fluid

Viscous fluid		l.t.e. state
τ	$=$	τ_{eq}
s	$=$	s_{eq}
~~[illegible]~~	~~$=$~~	~~$\theta_{eq} \equiv \partial\tilde{u}$~~
u	$=$	$u_{eq} \equiv \tilde{u}(\tau, s)$
θ	$=$	$\theta_{eq} \equiv (\partial\tilde{u}/\partial s)_{\tau}$
$\underline{\underline{T}}$	$\neq$	$\underline{\underline{T}}_{eq} \equiv -p_{eq}\underline{\underline{I}}$, $-p_{eq} \equiv (\partial\tilde{u}/\partial\tau)_s$
$\underline{H} = -\kappa\nabla\theta$	$\neq$ ↑ this was true even for a non-viscous fluid.	$\underline{H}_{eq} \equiv \underline{0}$.

If the fluid were at rest, the stress at a point where the density was ρ and the internal energy U (or the temp. θ) would be the equil. stress

$$\underline{\underline{T}}_{eq.} = -p(\rho, U)\,\underline{\underline{I}} \qquad (*)$$

The stress tensor deviates from $(*)$ because the fluid is not at rest, but has a velocity $\underline{u}$.

Define $\underline{\underline{V}}(\underline{r}, t)$ the viscous stress tensor as the difference between the real stress tensor and the equil. stress tensor

$$\begin{aligned}\underline{\underline{T}} &= \underline{\underline{T}}_{eq} + \underline{\underline{V}} \\ &= -\underset{\substack{\uparrow \\ \text{thermo. equil. pressure}}}{p}(\rho, U)\,\underline{\underline{I}} + \underline{\underline{V}}\end{aligned}$$

The viscous stress $\underline{\underline{V}}(\underline{r}, t)$ depends on the motion $\underline{u}(\underline{r}, t)$ of the fluid. How does it depend on this motion?

We know that if $\underline{u} = 0$ then $\underline{\underline{V}} = 0$

But if $\underline{u} =$ const., then we can find an unaccelerated reference frame in which the fluid is at rest so that $\underline{\underline{T}} = \underline{\underline{T}}_{eq.}$, so $\underline{\underline{V}} = 0$.

$$\boxed{\text{This is very important. To make } \underline{\underline{V}} \neq 0, \text{ it is not enough to make } \underline{u} \neq 0, \text{ one must make } \underline{u} \neq \text{const.}}$$

This is known as invoking the principle of material frame indifference.

In general the viscous stress $V(\underline{r}, t)$ could conceivably depend on the past history of $\underline{u}(\underline{r}, t)$ at all pts. in the fluid (the fluid might have a memory).

3. Newtonian Viscous Fluids

The simplest assumption is that $V(\underline{r}, t)$ depends only on state of motion at $\underline{r}, t$.
Thus it might depend on $D_t \underline{u}$, $\underline{\nabla u}$, $D_t \underline{\nabla u}$, $\underline{\underline{\nabla \nabla u}}$, or higher derivatives.

Again we make the <u>simplest</u> assumption, that only first order derivatives are important.

$V(\underline{r},t) =$ fcn $[\, \underline{\underline{\nabla u}}(\underline{r},t),\ D_t \underline{u}(\underline{r},t)\,]$
a fcn only of local instantaneous rate of deformation and local instantaneous particle acceleration.

Now when $D_t \underline{u}$ and $\underline{\underline{\nabla u}} = 0$, we know
$\underline{u} =$ const and $\underline{\underline{V}} = 0$
If we expand $\underline{\underline{V}}$ in a Taylor series, the series will begin with the linear terms.
Again seeking simplicity we assume that the h.o.t. are negligible.

Thus we assume that for small $\underline{\underline{\nabla u}}$ and $D_t \underline{u}$, $\underline{\underline{V}}$ is a linear fcn of $\underline{\underline{\nabla u}}$ and $D_t \underline{u}$

$\exists$ two linear tensor valued fcns $\underline{\underline{F}}$ and $\underline{\underline{G}}$ such that

$$\underline{\underline{V}} = \underline{\underline{F}}(\underline{\underline{\nabla u}}) + \underline{\underline{G}}(D_t \underline{u})$$

$\underline{\underline{F}}$ and $\underline{\underline{G}}$ are <u>linear</u> tensor-valued fcns

$$\underline{\underline{F}}\,(a\,\underline{\underline{\nabla u}}_1 + b\,\underline{\underline{\nabla u}}_2) = a\,\underline{\underline{F}}(\underline{\underline{\nabla u}}_1) + b\,\underline{\underline{F}}(\underline{\underline{\nabla u}}_2)$$
$$\underline{\underline{G}}\,(a\,D_t\underline{u}_1 + b\,D_t\underline{u}_2) = a\,\underline{\underline{G}}(D_t\underline{u}_1) + b\,\underline{\underline{G}}(D_t\underline{u}_2)$$

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We have not quite achieved the specification of a <u>Newtonian</u> viscous fluid. We do have what might be called a <u>linearized</u> viscous fluid.

Summarizing assumptions to date:

1. local thermo. equil. except for $\underline{\underline{T}}(\underline{r},t)$ and $\underline{H}(\underline{r},t)$
2. $\underline{H}(\underline{r},t) = -\kappa\nabla\theta$, $\kappa(\tau,s)$
3. $\underline{\underline{T}}(\underline{r},t) = -p_{eq}(\underline{r},t)\,\underline{\underline{I}} + \underline{\underline{V}}(\underline{r},t)$ where
4. $\underline{\underline{V}}(\underline{r},t)$ depends only on the <u>local</u> deformation rate $\underline{\underline{\nabla u}}(\underline{r},t)$ and local particle acceleration $D_t\underline{u}(\underline{r},t)$, and furthermore
5. that dependence is linear

fcns of τ, s

$$\underline{\underline{V}} = \underline{\underline{F}}(\underline{\underline{\nabla u}}) + \underline{\underline{G}}(D_t\underline{u})$$

linear

6. If we wish the fluid to be a <u>Newtonian</u> viscous fluid, must further assume <u>isotropy</u> (to be discussed)

Lecture # 20 review

Discussing constitutive nature of a Newtonian viscous fluid

Assumptions:

1. local thermodynamic equil. except for $\underline{\underline{T}}(\underline{r},t)$ and $\underline{H}(\underline{r},t)$

$$\left.\begin{array}{l} \rho(\underline{r},t),\ U(\underline{r},t) \\ \theta(\underline{r},t) \equiv \theta(\rho,U) \\ S(\underline{r},t) \equiv S(\rho,U) \end{array}\right.\quad \begin{array}{l}\text{fundamental} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{defn of } \theta, S \text{ for fluid } \underline{\text{not}} \text{ in complete equil.}\end{array}$$

$\rho(\underline{r},t)$, $U(\underline{r},t)$, $\theta(\underline{r},t)$, $S(\underline{r},t)$ given by equil. values

2. $\underline{H}(\underline{r},t) = -K\nabla\theta$ (Fourier's law)

$\underline{H}(\underline{r},t) \neq \underline{H}_{eq}(\underline{r},t) \equiv 0$

3. $\underline{\underline{T}}(\underline{r},t) \neq \underline{\underline{T}}_{eq}(\underline{r},t)$

$$\underline{\underline{T}}_{eq}(\underline{r},t) = -p_{eq}(\rho,U)\,\underline{\underline{I}}$$

$$\underline{\underline{T}}(\underline{r},t) = -p(\rho,U)\,\underline{\underline{I}} + \overset{\text{viscous stress}}{\underline{\underline{V}}}(\underline{r},t)$$

$\underline{\underline{V}}(\underline{r},t)$ depends on motion of fluid

must depend on derivatives of $\underline{u}(\underline{r},t)$

4. we assume further $\underline{\underline{V}}(\underline{r},t)$ depends only on local defm rate $\underline{\underline{\nabla u}}(\underline{r},t)$ and local fluid acceleration $D_t \underline{u}(\underline{r},t)$

5. we assume further that dependence is linear

$$\underline{\underline{V}} = \underline{\underline{F}}(\underline{\underline{\nabla u}}) + \underline{\underline{G}}(D_t \underline{u})$$

$\underline{\underline{F}}, \underline{\underline{G}}$ linear tensor-valued fcns

6. for a Newtonian fluid, must make one further assumption

assume fluid isotropic

These constitutive assumptions allow us to achieve a complete system of equations.

Defn: a Newtonian viscous fluid is an isotropic (to be discussed) fluid which satisfies assumptions 1. thru 6. above. The most important is of course 5., the viscous stress $\underline{\underline{V}}(\underline{r},t)$ at any $\underline{r},t$ depends linearly on defm rate tensor $\underline{\underline{\nabla u}}(\underline{r},t)$ and local fluid accel. $D_t\underline{u}(\underline{r},t)$ at that $\underline{r},t$

Complete system of equations

$D_t\rho + \rho\nabla\cdot\underline{u} = 0$ cont.

$\rho D_t\underline{u} = -\nabla p_{eq.} + \nabla\cdot\underline{\underline{V}} + \underline{f}$ momentum

$\underline{\underline{V}} = \underline{\underline{V}}^T$ ang. momentum

not right → $\rho D_t S + \nabla\cdot(\underline{H}/\theta) = \frac{h}{\theta} + H\cdot\nabla\frac{1}{\theta}$ $+ \frac{1}{\theta}(\underline{\underline{V}}:\varepsilon)$ entropy

this makes it right

$\rho D_t U + \nabla\cdot\underline{H} = -p\nabla\cdot\underline{u} + \mathrm{tr}(\underline{\underline{V}}\cdot\underline{\underline{\varepsilon}}) + h$

$p_{eq} = p(U,\rho)$ two eqns of state for

$\theta = \theta(U,\rho)$ equil. variables

$\underline{H} = -\kappa\nabla\theta$ heat flow

$\underline{\underline{V}} = \underline{\underline{F}}(\underline{\underline{\nabla u}}) + \underline{\underline{G}}(D_t\underline{u})$ Newtonian viscous

The thermal conductivity $\kappa(\underline{r},t)$ is assumed to depend on $\underline{r},t$ only thru $U(\underline{r},t)$ and $\rho(\underline{r},t)$, i.e., $\kappa(\underline{r},t) = \tilde{\kappa}(U(\underline{r},t),\rho(\underline{r},t))$

unknowns $\rho, \mathcal{U}, \theta, p_{eq}, \underset{\overset{\uparrow}{3}}{\underline{u}}, \underset{\overset{\uparrow}{3}}{\underline{H}}, \underset{\overset{\uparrow}{6}}{\underline{\underline{V}}}$

16 unknowns
16 equations

4. The Newtonian viscous const. relationship.

$$\underline{\underline{V}} = \underline{\underline{F}}(\underline{\underline{\nabla u}}) + \underline{\underline{G}}(D_t \underline{u})$$

For any fixed $\underline{x}, t$
$\underline{\underline{F}}$ is a linear mapping which assigns a second order tensor to a second order tensor; $\underline{\underline{G}}$ is a linear mapping which assigns a second order tensor to a vector

$$\underline{\underline{F}} : R^3 \otimes R^3 \to R^3 \otimes R^3$$
$$\underline{\underline{G}} : R^3 \to R^3 \otimes R^3$$

An aside on linear mappings and tensors of order q.

Remark 1: Recall that a linear mapping $\underline{\underline{T}} : R^3 \to R^3$ generates a unique second order tensor $\underline{\underline{T}}$
$\underline{u} \cdot \underline{\underline{T}} \cdot \underline{v} = \underline{\underline{T}}(\underline{u}, \underline{v})$ for any two vectors $\underline{u}, \underline{v}$

Remark 2: Suppose T is a linear mapping $T : \otimes^n R^3 \to \otimes^m R^3$ which assigns an m^{th} order tensor to every n^{th} order tensor. We claim that T generates a unique tensor of order $m+n$ in a natural way, and that conversely every tensor of order $m+n$ generates in a natural way a linear mapping $T : \otimes^n R^3 \to \otimes^m R^3$.

Proof:

Introduce a Cartesian axis system $\hat{x}_1, \hat{x}_2, \hat{x}_3$. [redacted] Choose a tensor $S \in \otimes^n R^3$ and let R be the tensor $\in \otimes^m R^3$ which T assigns to S

$$R = T(S)$$

R, S may be written in terms of their components

$$R = R_{i_1 \ldots i_m} \hat{x}_{i_1} \ldots \hat{x}_{i_m}$$
$$S = S_{j_1 \ldots j_n} \hat{x}_{j_1} \ldots \hat{x}_{j_n}$$

$$R = T(S)$$
$$R_{i_1 \ldots i_m} \hat{x}_{i_1} \ldots \hat{x}_{i_m} = T(S_{j_1 \ldots j_n} \hat{x}_{j_1} \ldots \hat{x}_{j_n})$$
$$= S_{j_1 \ldots j_n} T(\hat{x}_{j_1} \ldots \hat{x}_{j_n})$$

now $T(\hat{x}_{j_1} \cdots \hat{x}_{j_n})$ is a tensor of order m. It too may be written in terms of its components. Let

$$T(\hat{x}_{j_1} \cdots \hat{x}_{j_n}) = T_{i_1 \cdots i_m j_1 \cdots j_n} \hat{x}_{i_1} \cdots \hat{x}_{i_m}$$

then

$$R = L(S) \qquad \text{is}$$

$$R_{i_1 \cdots i_m} \hat{x}_i \cdots \hat{x}_{i_m} = T_{i_1 \cdots i_m j_1 \cdots j_n} S_{j_1 \cdots j_n} \hat{x}_{i_1} \cdots \hat{x}_{i_m}$$

$$R_{i_1 \cdots i_m} = T_{i_1 \cdots i_m j_1 \cdots j_n} S_{j_1 \cdots j_n}$$

We have shown how to associate a <u>unique</u> set of 3^{m+n} numbers $T_{i_1 \cdots i_m j_1 \cdots j_n}$ with a Cartesian axis system by means of the linear mapping T

~~$T_{i_1 \cdots i_m j_1 \cdots j_n}$~~

$$\boxed{\overset{\hat{x}_{i_1} \cdots \hat{x}_{i_m}}{\wedge} T_{i_1 \cdots i_m j_1 \cdots j_n} \hat{x}_{i_1} \cdots \hat{x}_{i_m} = T(\hat{x}_{j_1} \cdots \hat{x}_{j_n})}$$

note this assignment is <u>independent</u> of R and S.

Claim that the 3^{m+n} numbers

$T_{i_1 \cdots i_m j_1 \cdots j_n}$ are the <u>components</u> w.r.t. to $\hat{x}_1, \hat{x}_2, \hat{x}_3$ of a tensor of order $m+n$.

$$T \in \otimes^{m+n} R^3$$
$$T = T_{i_1 \cdots i_m j_1 \cdots j_n} \hat{x}_{i_1} \cdots \hat{x}_{i_m} \hat{x}_{j_1} \cdots \hat{x}_{j_n}$$

To demonstrate this must demonstrate that they transform properly under a change of coord. system

Given another Cartesian axis system $\hat{x}_1', \hat{x}_2', \hat{x}_3'$, one can similarly associate another set of numbers $T'_{i_1 \cdots i_m j_1 \cdots j_n}$

We let $R'_{i_1 \cdots i_m}$ and $S'_{j_1 \cdots j_n}$ be the Cartesian comp. w.r.t primed system of the tensors R and S

$$R'_{i_1 \cdots i_m} = T'_{i_1 \cdots i_m j_1 \cdots j_n} S'_{j_1 \cdots j_n}$$

as before

We have two tensors R, S components $R_{i_1 \cdots i_m}$, $R'_{i_1 \cdots i_m}$ and $S_{i_1 \cdots i_n}$, $S'_{i_1 \cdots i_n}$ and two sets of numbers $T_{i_1 \cdots i_m j_1 \cdots j_n}$ and $T'_{i_1 \cdots i_m j_1 \cdots j_n}$.

Claim this implies ∃ a Tensor T of order m+n and that the two sets of numbers are its comp. w.r.t. $\hat{x}_1\hat{x}_2\hat{x}_3$ and $\hat{x}_1'\hat{x}_2'\hat{x}_3'$.

Let $S = x_{k_1}' \cdots x_{k_n}'$

Then

$$S_{j_1\cdots j_n} = (\hat{x}_{k_1}'\cdot\hat{x}_{j_1})\cdots(\hat{x}_{k_n}'\cdot\hat{x}_{j_n})$$

$$S_{j_1\cdots j_n}' = (\delta_{k_1 j_1})\cdots(\delta_{k_n j_n})$$

$$R_{i_1\cdots i_m}\hat{x}_{i_1}\cdots\hat{x}_{i_m} = T_{i_1\cdots i_m j_1\cdots j_n}(\hat{x}_{k_1}'\cdot\hat{x}_{j_1})\cdots(\hat{x}_{k_n}'\cdot\hat{x}_{j_n})\,\hat{x}_{i_1}\cdots\hat{x}_{i_m}$$

dot with $\hat{x}_{l_1}'\cdots\hat{x}_{l_m}'$

$$R_{i_1\cdots i_m}(\hat{x}_{l_1}'\cdot\hat{x}_{i_1})\cdots(\hat{x}_{l_m}\cdot\hat{x}_{i_m}) =$$

$$T_{i_1\cdots i_m j_1\cdots j_n}(x_{k_1}'\cdot\hat{x}_{j_1})\cdots(x_{k_n}'\cdot\hat{x}_{j_n})(\hat{x}_{l_1}'\cdot\hat{x}_{i_1})\cdots(\hat{x}_{l_m}'\cdot\hat{x}_{i_m})$$

but $R_{i_1\cdots i_m}(\hat{x}_{l_1}\cdot\hat{x}_{i_1})\cdots(\hat{x}_{l_m}\cdot\hat{x}_{i_m})$

$= R_{l_1\cdots l_m}'$

$= T_{l_1\cdots l_m j_1\cdots j_n}'\, S_{j_1\cdots j_n}'$

$= T_{l_1\cdots l_m j_1\cdots j_n}'(\delta_{k_1 j_1})\cdots(\delta_{k_n j_n})$

$$= T'_{l_1 \cdots l_m k_1 \cdots k_n}$$

$$\boxed{\begin{aligned} T'_{l_1 \cdots l_m k_1 \cdots k_n} = (\hat{x}'_{l_1} \cdot \hat{x}_{i_1}) \cdots (\hat{x}'_{l_m} \cdot \hat{x}_{i_m})(\hat{x}'_{k_1} \cdot \hat{x}_{j_1}) \\ \cdots (x'_{k_n} \cdot \hat{x}_{j_n}) T_{i_1 \cdots i_m j_1 \cdots j_n} \end{aligned}}$$

Hence the two sets of 3^{m+n} numbers transform like the components of a tensor T of order 3^{m+n}. q.e.d.

We have shown any <u>linear</u> mapping $T: \otimes^n R^3 \to \otimes^m R^3$ generates and may be completely described by a <u>unique</u> tensor T of order $m+n$. Conversely every tensor T of order $m+n$ generates a unique linear mapping $T: \otimes^n R^3 \to \otimes^m R^3$

Denote the components w.r.t. any $\hat{x}_1, \hat{x}_2, \hat{x}_3$ of T by $T_{i_1 \cdots i_m j_1 \cdots j_n}$, $\forall\ R \in \otimes^m R^3$ $S \in \otimes^n R^3$

$$R = T(S)$$

may be written in terms of the <u>comp.</u> of the tensor T

$$\boxed{R_{i_1 \cdots i_m} = T_{i_1 \cdots i_m j_1 \cdots j_n} S_{j_1 \cdots j_n}}$$

now $\underline{\underline{V}} = \underline{\underline{F}}(\underline{\underline{\nabla u}}) + \underline{\underline{G}}(D_t \underline{u})$

$\underline{\underline{F}} : R^3 \otimes R^3 \rightarrow R^3 \otimes R^3$
$\underline{\underline{G}} : R^3 \rightarrow R^3 \otimes R^3$

hence $\underline{\underline{F}}$ is completely described by a unique fourth order tensor for any fixed $\underline{r}, t$ The fourth order tensor F depends on $\underline{r}, t$.
Similarly $\underline{\underline{G}}$ is completely described by a third order tensor (which also may vary with $\underline{r}, t$.

At any fixed $\underline{r}, t$ the components of $\underline{\underline{V}}$ relative to any $\hat{x}_1, \hat{x}_2, \hat{x}_3$ are

$$V_{ij}(\underline{r}, t) = F_{ijkl}(\underline{r}, t)\, \partial_k u_l(\underline{r}, t) + G_{ijk}(\underline{r}, t)\, D_t u_k(\underline{r}, t)$$

The 81 numbers $F_{ijkl}(\underline{r}, t)$ are the components of a fourth order tensor $F(\underline{r}, t)$. The 27 numbers $G_{ijk}(\underline{r}, t)$ are the components of a third order tensor $G(\underline{r}, t)$

How do we obtain the 81 constants $F_{ijkl}(\underline{r},t)$ and the 27 constants $G_{ijk}(\underline{r},t)$?
We must measure them. They are *empirical* constants.

The 108 empirical constants
First lets discuss their dependence on $\underline{r}$ and t.

We make the further assumption that $F_{ijkl}(\underline{r},t)$ and $G_{ijk}(\underline{r},t)$ depend on $\underline{r},t$ only as fcns of the *equil pressure* $p(\underline{r},t)$ and *temperature* $\theta(\underline{r},t)$ at $\underline{r},t)$

$$\boxed{\begin{array}{l} F_{ijkl}\left(p(\underline{r},t),\ \theta(\underline{r},t)\right) \\ G_{ijk}\left(p(\underline{r},t),\ \theta(\underline{r},t)\right) \\ \\ \text{(or fcns of any two of } v,\rho,p,\theta,S) \end{array}}$$

Our task is therefore the measurement of 108 empirical fcns of temp. and pressure. Before we set about this we ask, how many of these are independent.

Must they all be measured separately?

First we know $V_{ij} = V_{ji}$ since the total stress is symmetric

if $\partial_k u_\ell = 0$ then

$$V_{ij} = G_{ijk} D_t u_k = V_{ji} = G_{jik} D_t u_k$$

if this is to be true for any $D_t u_k$ then

$$\boxed{G_{ijk} = G_{jik}}$$

if we choose $D_t u_k = 0$ then

$$V_{ij} = F_{ijk\ell} \partial_k u_\ell = V_{ji} = F_{jik\ell} \partial_k u_\ell$$

if this is to be true for any $\partial_k u_\ell$ then

$$\boxed{F_{ijk\ell} = F_{jik\ell}}$$

there are at most 18 independent G_{ijk} and 54 independent $F_{ijk\ell}$

We have reduced # of measurements required to $54 + 18 = 72$ from $81 + 27 = 108$.

Lecture #21 Review

Constitutive eqns for a Newtonian viscous fluid

In local thermodynamic equilibrium except for $\underline{\underline{T}}(\underline{r},t)$ and $\underline{H}(\underline{r},t)$

We assume:

1. Fourier's law: $$H(\underline{r},t) = -\underset{\substack{\uparrow \\ \text{thermal conductivity}}}{\kappa(\underline{r},t)}\,\nabla\Theta(\underline{r},t)$$

assume further $\kappa(\underline{r},t)$ depends on $\underline{r},t$ only thru $\rho(\underline{r},t)$, $\theta(\underline{r},t)$

$$\kappa(\underline{r},t) = \kappa(\rho(\underline{r},t), \theta(\underline{r},t))$$

2. Navier-Poisson law:

$$\underline{\underline{T}}(\underline{r},t) = -\underset{\substack{\uparrow \\ \text{equilibrium} \\ \text{pressure}}}{p(\underline{r},t)}\,\underline{\underline{I}} + \underset{\substack{\uparrow \\ \text{viscous stress} \equiv \text{deviation} \\ \text{from equil. pressure.}}}{\underline{\underline{V}}(\underline{r},t)}$$

equilibrium pressure given by fluid eqn. of state as if in complete equil

$p(\underline{r},t) = p(\rho(\underline{r},t), U(\underline{r},t))$

<u>viscous</u> <u>stress</u> $\underline{\underline{V}}(\underline{r},t)$ for a fixed $\underline{r},t$ assumed to depend <u>linearly</u> on the two first order derivatives $\underline{\underline{\nabla u}}(\underline{r},t)$ and $D_t\underline{u}(\underline{r},t)$

$$\underline{\underline{V}}(\underline{r},t) = \underline{\underline{F}}_{\underline{r},t}(\underline{\underline{\nabla u}}(\underline{r},t)) + \underline{\underline{G}}_{\underline{r},t}(D_t\underline{u}(\underline{r},t))$$

$\underline{\underline{F}}_{\underline{r},t}$ $\underline{\underline{G}}_{\underline{r},t}$ linear mappings depend on point $\underline{r},t$ (different mappings for different $\underline{r},t$)

$\underline{\underline{F}}: \otimes^2 R^3 \to \otimes^2 R^3$

$\underline{\underline{G}}: R^3 \to \otimes^2 R^3$

showed that can associate with such linear maps $\underline{\underline{F}}, \underline{\underline{G}}$ <u>unique</u> tensors F and G of fourth and third order

choose a Cartesian axis system $\hat{x}_1, \hat{x}_2, \hat{x}_3$

~~[illegible]~~ then the correspondence is:

$$\underset{\substack{\uparrow\\ \text{fourth order tensor}}}{F(\underline{r},t)} = \underset{\substack{\nwarrow\\ \text{81 comps.}}}{F_{ijkl}(\underline{r},t)}\hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_l = \underset{\substack{\uparrow\\ \text{linear map}}}{\underline{\underline{F}}_{\underline{r},t}}(\hat{x}_k\hat{x}_l)\hat{x}_k\hat{x}_l$$

$$G(\underline{r},t) = G_{ijk}(\underline{r},t)\hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_l = \underline{\underline{G}}_{\underline{r},t}(\hat{x}_k)\hat{x}_k$$

Thus at every point $\underline{r}, t$ $\exists$ two tensors $F(\underline{r}, t)$, $G(\underline{r}, t)$ with comps. $F_{ijkl}(\underline{r}, t)$, $G_{ijk}(\underline{r}, t)$ $\ni$

$$\boxed{\begin{aligned} V_{ij}(\underline{r}, t) &= F_{ijkl}(\underline{r}, t)\, \partial_k u_l(\underline{r}, t) \\ &\quad + G_{ijk}(\underline{r}, t)\, D_t u_k(\underline{r}, t) \end{aligned}}$$

We assume further, as for $\kappa(\underline{r}, t)$, that the dependence of F and G on $\underline{r}, t$ depends only on $p(\underline{r}, t)$, $\theta(\underline{r}, t)$

$$\boxed{\begin{aligned} F_{ijkl}(\underline{r}, t) &= F_{ijkl}(p(\underline{r}, t), \theta(\underline{r}, t)) \\ G_{ijk}(\underline{r}, t) &= G_{ijk}(p(\underline{r}, t), \theta(\underline{r}, t)) \end{aligned}}$$

Now we have $81 + 27 = 108$ *empirical* constants which define our fluid. We must measure them for any given fluid, just *as we must measure* $\kappa(\underline{r}, t)$.

Hence we have reduced the number of measurements required to $54 + 18 = 72$ from $81 + 27 = 108$.

end here 29 Mar '72

5. Isotropic Fluids and Isotropic tensors

To make further progress we must make further assumptions about the nature of our fluid.

To measure F_{ijkl} and G_{ijk} we must choose a basis $\hat{x}_1, \hat{x}_2, \hat{x}_3$.
Once the components of the tensors F and G are known in the chosen basis they are readily computed in any other basis.

Our measurements might be easier if we choose the proper basis.
We can line up the axes in the fluids preferred directions. How can a fluid have preferred directions.

Maybe it has dissolved in it some long thin molecules (e.g. proteins) which have been aligned.

We will avoid such fluids.

we consider only fluids with <u>no</u> <u>preferred directions</u>.

Isotropic fluids: fluids with no preferred directions

An isotropic fluid is indifferent to our choice of basis to measure $\hat{x}_1, \hat{x}_2, \hat{x}_3$.

Isotropic fluid: the tensors F and G will have the same components F_{ijkl} and G_{ijk} in <u>every</u> Cartesian axis system $\hat{x}_1, \hat{x}_2, \hat{x}_3$.
Such a tensor is called an isotropic tensor

[illegible]

This leads us then to a discussion of <u>isotropic tensors</u>.

Defn: A tensor of any order $\underline{q}$ is completely isotropic if its components do not change from one orthonormal basis to another.

Defn: A tensor of any order $\underline{q}$ is chirally isotropic if its components do not change from one right-handed orthonormal basis to another.

Defn: skew isotropic = chirally isotropic but not completely isotropic

Examples:

1. the identity tensor, completely isotropic

   $I_{ij} = \delta_{ij}$

2. the alternating tensor, chirally isotropic

   right-handed $A_{ijk} = \epsilon_{ijk}$

   left-handed $A_{ijk} = -\epsilon_{ijk}$

   hence skew isotropic

3. any tensor product of isotropic tensors is isotropic

   $I \otimes I$ completely isotropic

   $A \otimes I$ chirally isotropic

4. any contraction of an isotropic tensor is isotropic

   $\mathrm{tr}\, \underline{\underline{I}} = 3$

5. any tensor obtained by permuting indices of an isotropic tensor is isotropic

6. any lin. comb. of isotropic tensors is isotropic.

A catalogue of isotropic tensors

There are surprisingly few. Essentially only the ones in examples 1–6 exist. There are no others.

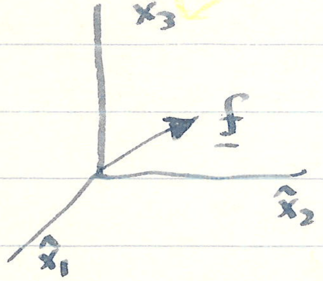
We prove this only for $\underline{q} \leq 4$.

0. isotropic tensors of order zero

any scalar is completely isotropic

1. isotropic tensors of order one

what vector f has the same components in all right-handed axis systems? Only the zero vector



There are ~~no~~ others

There are no isotropic vectors except zero.

$\underline{f} = \underline{0}$

$\underline{0}$ is completely isotropic

2. isotropic tensors of order two.
suppose $\underline{Q}$ is such a tensor
Let $\hat{x}_1, \hat{x}_2, \hat{x}_3$ and $\hat{x}_1', \hat{x}_2', \hat{x}_3'$ be two bases, <u>both right-handed</u>.

$$Q(\hat{x}_i, \hat{x}_j) = Q(\hat{x}_i', \hat{x}_j')$$

choose $\hat{x}_1' = -\hat{x}_1$
$\hat{x}_2' = -\hat{x}_2$
$\hat{x}_3' = +\hat{x}_3$

$Q(\hat{x}_1, \hat{x}_3) = Q(-\hat{x}_1, \hat{x}_3)$ $\quad Q_{13} = -Q_{13} = 0$
similarly $\quad Q_{31} = 0, \quad Q_{23} = Q_{32} = 0$

now choose $\hat{x}_1' = \hat{x}_1, \hat{x}_2' = -\hat{x}_2, \hat{x}_3' = -\hat{x}_3$

get $\quad Q_{12} = Q_{21} = 0$

now choose $\hat{x}_1' = \hat{x}_2, \hat{x}_2' = \hat{x}_3, \hat{x}_3' = \hat{x}_1$

$Q_{11} = Q_{22}$
$Q_{22} = Q_{33}$ $\quad \exists$ a const $q \ni$

$$Q_{11} = Q_{22} = Q_{33} = q$$

$$Q_{ij} = q\delta_{ij} \qquad \underline{\underline{Q}} = q\,\underline{\underline{I}}$$

The only chirally isotropic second order tensors are <u>multiples of the identity tensor</u>. Such tensors are not merely chirally but <u>completely isotropic</u>

3. isotropic tensors of order three

(a) chirally isotropic tensors
suppose $\underline{\underline{Q}}$ is such a tensor
the requirement

$$Q(\hat{x}_i, \hat{x}_j, \hat{x}_k) = Q(\hat{x}_{i'}, \hat{x}_{j'}, \hat{x}_{k'})$$

produces various relations among the components Q_{ijk} depending on how we choose the ′ system. The procedure is exactly the same as before.

new basis	relations among components
$1' = -1,\ 2' = -2,\ 3' = 3$	$Q_{111} = Q_{222} = 0,\ Q_{112} = Q_{121} = Q_{211} = 0,\ Q_{221} = Q_{212} = Q_{122} = 0$
$1' = -1,\ 2' = 2,\ 3' = -3$	$Q_{111} = 0,\ Q_{333} = 0,\ Q_{113} = Q_{131} = Q_{311} = 0,\ Q_{331} = Q_{313} = Q_{133} = 0$
$1' = 1,\ 2' = -2,\ 3' = -3$	$Q_{222} = 0,\ Q_{333} = 0,\ Q_{223} = Q_{232} = Q_{322} = 0,\ Q_{332} = Q_{323} = Q_{233} =$
$2' = 1,\ 3' = 2,\ 1' = 3$	$Q_{123} = Q_{231} = Q_{132},\quad Q_{213} = Q_{132} = Q_{321}$
$1' = 2,\ 2' = 1,\ 3' = -3$	$Q_{123} = -Q_{213}$

From the above table $Q_{ijk} = \underline{q}\,\epsilon_{ijk}$

Any <u>chirally</u> isotropic third order Tensor is a scalar multiple of the alternating tensor $\underline{\underline{A}}$

(b) completely isotropic
Suppose Q is such a tensor.
Then it is chirally isotropic
Hence $Q = \underline{q}\,A$.
But if $1' = -1$, $2' = -2$, $3' = -3$
$Q_{123} = -Q_{123}$
so $Q_{123} = 0$
thus $\underline{q} = 0$ so $Q = 0$

The only completely isotropic third order tensor is the zero tensor.

4. isotropic tensors of order four

for fourth order isotropic tensors, one proceeds in exactlly the same way

note that so far we have not even nearly exploited to the full the full power of the assumption that the tensor $\underline{q}$ has the same components for <u>all</u> right-handed coord systems. We have considered only bases obtainable from a given basis by <u>interchange</u> of axes and <u>sign changes</u> chosen to preserve right-handedness. (This the cubic group, NaCl xtals) To treat <u>fourth</u> order isotropic tensors, we must for the first time consider slightly more general basis changes (e.g. 45° degree rotations).

See the discussion by Wayne Pennington, or do it yourself. I only quote the result.

If Q is any <u>chirally</u> isotropic fourth order tensor, then $\exists$ three scalars p, q, r such that relative to any right-handed basis

$$* \quad Q_{ijkl} = p\,\delta_{ij}\delta_{kl} + q\,\delta_{ik}\delta_{jl} + r\,\delta_{il}\delta_{jk}$$

This tensor is <u>completely</u> isotropic as well, so $*$ is the most general form of completely isotropic as well as chirally isotropic fourth order tensors

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Summary: isotropic tensors

order	chirally isotropic	completely isotropic
0	q, arbitrary scalar	q
1	only $\vec{q} = \vec{0}$	"
2	$\underline{\underline{Q}} = q\,\underline{\underline{I}}$, q arbitrary scalar	"
3	$\underline{\underline{\underline{Q}}} = q\,\underline{\underline{\underline{A}}}$, $Q_{ijk} = q\,A_{ijk}$ (skew)	only zero
4	$Q_{ijkl} = p\,\delta_{ij}\delta_{kl} + q\,\delta_{ik}\delta_{jl} + r\,\delta_{il}\delta_{jk}$	"

Many of these results are readily extended

Any non-zero completely isotropic tensor necessarily has even order. Any non-zero chirally isotropic tensor which is not also completely isotropic has odd order ≥ 3. Any chirally isotropic tensor of $\begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$ order ≥ 3 is $\begin{Bmatrix}\text{completely}\\ \text{not completely}\end{Bmatrix}$ isotropic
(skew)

6. Newtonian Viscous Fluids: final form of equations.

Now we return to the viscous stress-strain relation and examine the consequences of the isotropy of our fluid. For a fixed $\underline{x}$, t

$$V_{ij} = F_{ijk\ell}\, \partial_k u_\ell + G_{ijk}\, D_t u_k$$

The tensor G is isotropic and of third order, hence it is necessarily zero

The tensor G must vanish for reasons of symmetry alone.

It is somewhat amusing that we can arrive at the same conclusion making slightly fewer assumptions

Assume only that G is <u>chirally</u> isotropic (perhaps the fluid is optically active - contains dissolved left-handed sugar)

Then $G_{ijk} = q A_{ijk}$

But since $V_{ij} = V_{ji}$ (the stress tensor is <u>symmetric</u>), we know that

$$G_{ijk} = G_{jik}$$

but $A_{ijk} = - A_{jik}$ hence $q = 0$ and

G must vanish for this case as well.

stop here 3/Apr.

Now consider the tensor F, F is isotropic (or again it is only necessary to assume <u>chiral</u> isotropy)

$$F_{ijkl} = p\delta_{ij}\delta_{kl} + q\delta_{ik}\delta_{jl} + r\delta_{il}\delta_{jk}$$

Lecture # 22

Linear stress- strain law for a Newtonian viscous fluid

$$V_{ij}(\underline{r},t) = F_{ijkl}(\underline{r},t)\, \partial_k u_l(\underline{r},t)$$

$$+ \; G_{ijk}(\underline{r},t)\, D_t u_k(\underline{r},t)$$

assumed linear dependence on first order derivs.

Now further assume isotropy of fluid.

Components $F_{ijkl}(\underline{r},t)$ and $G_{ijk}(\underline{r},t)$ independent of Cart. axis system

Then $G_{ijk}(\underline{r},t) = 0$ no dependence on local acceleration

Actually only need assume chiral isotropy e.g. dissolved laevo-rotory sugar.

Now for tensor F, isotropy (only need chiral isotropy

$$F_{ijkl} = p\,\delta_{ij}\delta_{kl} + q\,\delta_{ik}\delta_{jl} + r\,\delta_{il}\delta_{jk}$$

once again symmetry of $\underline{V}$ implies

$$F_{ijkl} = F_{jikl}$$

$$F_{jikl} = p\,\delta_{ij}\delta_{kl} + q\,\delta_{il}\delta_{jk} + r\,\delta_{ik}\delta_{jl}$$

$$= F_{ijkl} = p\,\delta_{ij}\delta_{kl} + q\,\delta_{ik}\delta_{jl} + r\,\delta_{il}\delta_{jk}$$

hence $F_{ijkl} = F_{jikl} \Rightarrow q = r$

$$F_{ijkl} = p\,\delta_{ij}\delta_{kl} + q\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Now we use the usual notation

denote $p = \gamma - \frac{2}{3}\eta$

$q = \eta$

$$\boxed{\begin{aligned} F_{ijkl} &= (\gamma - \tfrac{2}{3}\eta)\,\delta_{ij}\delta_{kl} + \eta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ G_{ijk} &= 0 \end{aligned}}$$

We are led to these forms of F and G by the assumption of <u>chiral</u> isotropy of our fluid

This means

$$V_{ij} = F_{ijkl}\,\partial_k u_l =$$

$$(\gamma - \tfrac{2}{3}\eta)\,\delta_{ij}\delta_{kl}\,\partial_k u_l + \eta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,\partial_k u_l$$

$$= (\gamma - \tfrac{2}{3}\eta)\,\nabla\cdot\underline{u}\,\delta_{ij} + \eta(\partial_i u_j + \partial_j u_i)$$

$$\underline{\underline{V}} = (\gamma - \tfrac{2}{3}\eta)(\nabla\cdot\underline{u})\,\underline{\underline{I}} + \eta\left[\underline{\underline{\nabla u}} + (\underline{\underline{\nabla u}})^T\right]$$

$$\underline{\underline{V}} = (\gamma - \tfrac{2}{3}\eta)(\nabla\cdot\underline{u})\,\underline{\underline{I}} + 2\eta\,\underline{\underline{\epsilon}}$$

$$\underline{\underline{\epsilon}} = \tfrac{1}{2}\left[\underline{\underline{\nabla u}} + (\underline{\underline{\nabla u}})^T\right]$$

$$\nabla\cdot\underline{u} = \operatorname{tr}\underline{\underline{\epsilon}}$$

deformation rate tensor

$$V_{ij} = (\gamma - \tfrac{2}{3}\eta)(\nabla\cdot\underline{u})\,\delta_{ij} + \eta(\partial_i u_j + \partial_j u_i)$$

The assumption of <u>chiral</u> <u>isotropy</u> allows one to show that $\underline{\underline{V}}(\underline{r},t)$ is necessarily independent of $D_t\underline{u}(\underline{r},t)$ the local fluid acceleration.

Remark 1:

Recall we have taken as the heat flow eqn (constitutive relation)

$$\underline{H}(\underline{r},t) = -k(\underline{r},t)\,\nabla\Theta(\underline{r},t)$$

We might have instead assumed the most general linear relationship

→ (if $\underline{\nabla\Theta} = \underline{0}$, then $\underline{H} = \underline{0}$, depends on derivative)

$\underline{H} = \underline{K}(\underline{\nabla\Theta})$ where $\underline{K}$ is a linear mapping

$\underline{K} : \mathbb{R}^3 \to \mathbb{R}^3$

$\underline{K}(a\underline{\nabla\Theta}_1 + b\underline{\nabla\Theta}_2) = a\underline{K}(\underline{\nabla\Theta}_1) + b\underline{K}(\underline{\nabla\Theta}_2)$

Then $\exists$ a second order tensor $\underline{\underline{K}}(\underline{r},t)$ such that

$$\boxed{\underline{H}(\underline{r},t) = \underline{\underline{K}}(\underline{r},t) \underset{\underset{\text{Gibbs dot product}}{\uparrow}}{\cdot} \underline{\nabla\Theta}(\underline{r},t)}$$

Then the assumption of fluid isotropy $\Rightarrow$ $\underline{K}$ is a second order isotropic tensor $\Rightarrow$ const $\cdot\ \underline{\underline{I}}$

$\underline{\underline{K}} = -\kappa\, \underline{\underline{\underline{I}}}$ and hence

$$\underline{H} = \underline{\underline{K}} \cdot \underline{\nabla}\theta = -\kappa\, \underline{\underline{\underline{I}}} \cdot \underline{\nabla}\theta = -\kappa \nabla\theta$$

Fourier's law follows from assumption of *linear* dependence on *first derivative*
NOTE: → (if $\underline{\nabla}\theta = \underline{0}$, $\underline{H} = 0$) + assumption of *fluid isotropy*.

Remark 2:

We might have considered, at the outset, the possibility that the *viscous* stress tensor $\underline{\underline{V}}(\underline{r}, t)$ depends linearly on $\underline{\underline{\nabla u}}$, $D_t \underline{u}$, $\underline{\nabla}\theta$, ∇p, perhaps an electric and magnetic field $\underline{E}$, $\underline{B}$

$$\underline{\underline{V}} = \underset{\substack{\uparrow \\ \underline{\text{linear}}}}{\underline{\underline{V}}}\left(\underline{\underline{\nabla u}}, D_t \underline{u}, \underline{\nabla}\theta, \underline{\nabla} p, \underline{E}, \underline{B}\right)$$

The assumption of *linearity* + *isotropy* would then allow us to prove exactly as before that $\underline{\underline{V}}$ is in fact independent of $\underline{\nabla}\theta$, ∇p, $\underline{E}$, $\underline{B}$, $D_t \underline{u}$

The point is that a non-zero second order symmetric tensor cannot depend <u>linearly</u> on a <u>vector</u> if the dependence is chirally isotropic

Most important is that there can be no dependence of $\underline{V}$ on $\underline{\nabla\Theta}$ for an isotropic, so long as that dependence is <u>linear</u>. Jim Roberts is looking at fluids of this type for which the dependence is non-linear.

[redacted]

Summary of eqns governing the mechanics and thermodynamics of a Newtonian viscous fluid

Constitutive assumptions:

1. local thermodynamic equil. except for $\underline{\underline{T}}(\underline{r},t)$, $\underline{H}(\underline{r},t)$

2. $H(\underline{r},t)$ linear fcn of $\underline{\nabla}\theta(\underline{r},t)$

3. $\underline{\underline{T}}(\underline{r},t) = -p(\underline{r},t)\,\underline{\underline{I}} + \underline{\underline{V}}(\underline{r},t)$

4. $\underline{\underline{V}}(\underline{r},t) =$ linear fcn of $\underline{\underline{\nabla u}}(\underline{r},t)$, $D_t\underline{u}(\underline{r},t)$

5. linear fcns depend on $\underline{r},t$ only thru $p(\underline{r},t)$, $\theta(\underline{r},t)$

6. isotropic (actually only need assume chirally isotropic)

Governing equations

$D_t\rho + \rho\nabla\cdot\underline{u} = 0$ continuity

$\rho D_t\underline{u} + \nabla p = \nabla\cdot\underline{\underline{V}} + \underline{f}$ momentum

$\underline{\underline{V}} = \underline{\underline{V}}^T$ ang. momentum

$\rho D_t U + p\nabla\cdot\underline{u} + \nabla\cdot\underline{H} = \mathrm{tr}(\underline{\underline{V}}\cdot\underline{\underline{\epsilon}}) + h$
internal energy

or $\rho = \rho(p,\theta)$, $U = U(p,\theta)$: p,θ most easily <u>measured</u>

$p = p_{eq} = p(U,\rho)$
$\theta = \theta_{eq} = \theta(U,\rho)$
} two eqns of state for equil. variables

$\underline{H} = -\kappa\underline{\nabla}\theta$ Fourier heat flow

$\underline{\underline{V}} = (\gamma - \frac{2}{3}\eta)(\nabla\cdot\underline{u})\underline{\underline{I}} + 2\eta\underline{\underline{\epsilon}}$ <u>Navier–Poisson viscous stress</u> law

The constitutive constants

$\kappa(\underline{r},t)$, $\gamma(\underline{r},t)$, $\eta(\underline{r},t)$ depend on $\underline{r},t$

only thru $\rho(\underline{r},t)$, $U(\underline{r},t)$ or, from the state eqns only thru $p(\underline{r},t)$, $\theta(\underline{r},t)$

$\kappa = \kappa(p,\theta)$ thermal conductivity
$\eta = \eta(p,\theta)$ shear viscosity
$\gamma = \gamma(p,\theta)$ bulk viscosity or second viscosity

16 eqns, 16 unknowns

$\rho, U, p, \theta, \underline{u}, \underline{H}, \underline{\underline{V}}$

We want to learn how to *interpret* these constitutive constants

Note that any *particular* Newtonian viscous fluid is defined in terms of

1. two eqns of state, e.g. taking $p(\underline{r},t)$ and $\theta(\underline{r},t)$ as the most readily measured independent variables

$$\rho = \rho(p,\theta)$$
$$U = U(p,\theta)$$

2. three constants, also *empirical* fcns of p, θ

$$\kappa = \kappa(p,\theta)$$
$$\eta = \eta(p,\theta)$$
$$\gamma = \gamma(p,\theta)$$

The most important constitutive relation is the viscous stress-strain law

7. The viscosity coefficients

note if we write

$$\underline{\underline{\nabla u}} = \underline{\underline{\epsilon}} + \underline{\underline{\Omega}}$$

$$\underline{\underline{\epsilon}} = \text{symmetric part} = \tfrac{1}{2}\left[(\underline{\underline{\nabla u}}) + (\underline{\underline{\nabla u}})^T\right]$$
$$\underline{\underline{\Omega}} = \text{antisymmetric part} = \tfrac{1}{2}\left[(\underline{\underline{\nabla u}}) - (\underline{\underline{\nabla u}})^T\right]$$

then

$$\boxed{\underline{\underline{V}} = \left(\gamma' - \tfrac{2}{3}\eta\right)(\mathrm{tr}\,\underline{\underline{\epsilon}})\underline{\underline{I}} + 2\eta\,\underline{\underline{\epsilon}}}$$

the viscous stress depends only on the local <u>strain rate</u> (the local rate of symmetric squeezing) and <u>not</u> on the local rate of <u>rotation</u> of the fluid.

Let us write $\underline{\underline{D}}_\epsilon$ as the <u>deviator</u> of the symmetric strain rate tensor $\underline{\underline{\epsilon}}$

$$\underline{\underline{\epsilon}} = \underbrace{\tfrac{1}{3}(\mathrm{tr}\,\underline{\underline{\epsilon}})\underline{\underline{I}}}_{\text{isotropic part}} + \underbrace{\underline{\underline{D}}_\epsilon}_{\text{deviatoric part}}$$

let's <u>interpret</u> this

say the motion is a <u>pure</u> <u>dilatation</u>

$\underline{u} = \omega \underline{r}$, ω = constant

then $\underline{\underline{\nabla u}} = \omega \underline{\underline{\nabla r}} = \omega \underline{\underline{I}}$ so

$\underline{\underline{\epsilon}} = \omega \underline{\underline{I}}$ and the <u>isotropic</u> part of $\underline{\underline{\epsilon}}$

gives the <u>dilation rate</u> $\nabla \cdot \underline{u} = 3\omega$

$\underline{\underline{D}}_{\epsilon} = 0$

now say the motion is incompressible,
no local volume changes

$\mathrm{tr}\, \underline{\underline{\epsilon}} = \nabla \cdot \underline{u} = 0$ so $\underline{\underline{\epsilon}} = \underline{\underline{D}}_{\epsilon}$

We have then

$$\boxed{\underset{\substack{\uparrow \\ \text{local deform-} \\ \text{ation rate}}}{\underline{\underline{\nabla u}}(\underline{r},t)} = \underset{\substack{\uparrow \\ \text{local rot-} \\ \text{ation rate}}}{\underline{\underline{\Omega}}(\underline{r},t)} + \underset{\substack{\uparrow \\ \text{local dilation} \\ \text{rate}}}{\tfrac{1}{3}\,\mathrm{tr}\,\underline{\underline{\epsilon}}(\underline{r},t)\,\underline{\underline{I}}} + \underset{\substack{\uparrow \\ \text{local rate} \\ \text{of volume-preserving} \\ \text{symmetric squeeze}}}{\underline{\underline{D}}_{\epsilon}(\underline{r},t)}}$$

substitute in:

$$\underline{\underline{V}} = \left(\kappa - \tfrac{2}{3}\eta\right)(\mathrm{tr}\,\underline{\underline{\epsilon}})\,\underline{\underline{I}} + 2\eta\left[\left(\tfrac{1}{3}\mathrm{tr}\,\underline{\underline{\epsilon}}\right)\underline{\underline{I}} + \underline{\underline{\mathcal{D}}}_{\epsilon}\right]$$

$$\boxed{\underline{\underline{V}} = \kappa\,(\mathrm{tr}\,\underline{\underline{\epsilon}})\,\underline{\underline{I}} + 2\eta\,\underline{\underline{\mathcal{D}}}_{\epsilon}}$$

the isotropic part of $\underline{\underline{V}}$ determined from dilation rate and bulk viscosity alone deviator of $\underline{\underline{V}}$ determined from that part of $\underline{\underline{\nabla u}}$ which describes a volume-preserving symmetric squeeze.

The total stress $\underline{\underline{T}}(\underline{r},t)$ is given by

$$\underline{\underline{T}} = -p\,\underline{\underline{I}} + \underline{\underline{V}}$$

$$\boxed{\underline{\underline{T}} = (\underset{\substack{\uparrow \\ \text{isotropic part} \\ \text{of } \underline{\underline{T}}}}{-p_{eq} + \kappa\,\mathrm{tr}\,\underline{\underline{\epsilon}}})\,\underline{\underline{I}} + 2\eta\,\underset{\substack{\uparrow \\ \text{deviatoric} \\ \text{part}}}{\underline{\underline{\mathcal{D}}}_{\epsilon}}}$$

The shear viscosity η gives rise to a [redacted] purely deviatoric addition to the equilibrium stress

The bulk viscosity γ gives rise to a difference between the real fluid pressure and the thermodynamic pressure $P_{eq}(\underline{r}, t)$

real pressure in fluid $= - P_{eq} + \gamma \, \mathrm{tr} \, \underline{\underline{\epsilon}}$

note: pressure meters are usually directional, but an ideal pressure meter would measure the above quantity.

If the bulk viscosity γ of a fluid is non-zero, then the real fluid pressure will not be the thermodynamic equilibrium pressure.

There has been a great deal of misunderstanding with regard to the <u>bulk viscosity</u> γ in the literature.

Maxwell showed that an <u>ideal gas</u> would have <u>zero</u> bulk viscosity but would have a shear viscosity

$$\eta\ \cancel{\mu} \approx \sqrt{mkT} \,/\, a^2$$

m = mass (monatomic ideal gas)
a = radius

$\boxed{\theta}$

$\mu \sim T^{1/2}$, <u>μ ind. of density</u>!

gets more viscous as it gets hotter!

$\gamma = 0$

This result of <u>kinetic theory of gases</u>.

It was for many years assumed that $\gamma = 0$ necessarily for <u>all fluids</u>

Reasons: 1. no experimental evidence to the contrary
2. a qualitative argument widely attributed to Stokes

Most textbooks do <u>not</u> try to justify the assertion $\gamma = 0$; it is merely attributed to Stokes and is sometimes called the

Stoke's condition

For example quote from Rayleigh II, p. 314 and Lamb p. 645.

Stokes himself however seems to be well aware of the deficiencies of his argument

Quote taken from Liebermann Phys. Rev. 75, p. 1416, (1949)

It is not surprising that this assumption should have persisted for so long. In almost all problems

$$\underline{\underline{V}} = \gamma (\mathrm{tr}\,\underline{\underline{\epsilon}})\,\underline{\underline{I}} + 2\eta\, \underline{\underline{D}}_{\epsilon}$$

the shearing or deviatoric part of the defm rate tensor ≫ the dilatation $\nabla \cdot \underline{u}$

Hence even if $\gamma \approx \eta$, the term involving γ will be negligible.

The main phenomena of interest for which this not true are:

1. dissipation of sound waves
2. propagation of shock waves.

It is now not difficult to measure γ by measuring the attenuation of sound waves in the fluid

This was done, e.g. by Liebermann, Phys. Rev. 75, 1415, 1969. for liquids

1 poise = 10 Pas

At 1 atm, room temp 20° C.

	η (poise)		γ (poise)	
water	.01	0.1 Pa s	.03	0.3 Pa s
ethyl alcohol	.012		.04	
[redacted]	[redacted]		[redacted]	

poise = cgs unit of viscosity $\left[\frac{m}{\ell t}\right]$

centipoise = 1/100 poise

[redacted] $[\eta] =$ [redacted]

physical interpretation of the viscosities:

<u>shear viscosity</u> corresponds to <u>sliding friction</u> of the fluid

see page 256.

<u>bulk viscosity</u> corresponds to <u>non-zero reaction time to attain equilibrium</u>

if dilatation rate is rapid enough compared to the relaxation time required to reach equil. the bulk viscosity may have a large effect.

A very interesting example is sea water

i.e. WWII

During the last war, when it was important, it was discovered that there was an anomalously high attenuation of sound waves in sea water above about 10^5 hz.

This corresponded to a large bulk viscosity

$$\boxed{\eta_{\text{seawater}} \sim 0.2 \text{ Pa s}}$$

sea water $\eta \sim .02$ poise

$\zeta \sim 1$ poise (Leibermann, Phys. Rev. 76, 1520-24, 1949)

$$\boxed{\zeta_{\substack{\text{sea}\\\text{water}}} \sim 10 \text{ Pa s}}$$

Reason: above 10^5 hz. a chemical reaction involving Mg^{++} in sea H_2O and the equil pt. for this reaction is pressure-dependent. The time required to adjust to a sudden change in pressure is $\sim 10^{-6}$. So for [redacted] sound waves with $f \gtrsim 10^5$ hz. this reaction can't keep up and hence $P \neq P_{eq}$.

References Liebermann Phys. Rev. 75, 1415-1422
76, 1520-24
Landau Lifshitz §78 (Fluid Mechanics)

Repeat: bulk viscosity only of importance in:
1. attenuation of sound waves
2. shock wave propagation

For all large scale flow situations, the deviatoric deformation rates ≫ dilation rates.

Recall that a fluid may be very well approximated as <u>incompressible</u> so long as $u \ll c$.

If the fluid flow may be modeled as <u>incompressible</u> $\nabla \cdot \underline{u}$ then the bulk viscosity does not enter.

$$\underline{\underline{V}} = 2\eta \underline{\underline{\epsilon}} \quad \text{(incompressible)}$$

It is for the above reason that discussion of the bulk viscosity $\varkappa$ of a fluid can usually be avoided.

To invoke the Stokes criterion and just legislate $p = p_{eq}$ is just wrong

end here 5 April

Lecture # 23:

Any <u>particular</u> Newtonian viscous fluid is determined by:

$$\left.\begin{aligned} \rho &= \rho(p,\theta) \\ u &= u(p,\theta) \end{aligned}\right\} \text{ two equil. eqns of state}$$

$$\left.\begin{aligned} \kappa &= \kappa(p,\theta) && \text{thermal cond.} \\ \eta &= \eta(p,\theta) && \text{shear viscosity} \\ \gamma &= \gamma(p,\theta) && \text{bulk viscosity} \end{aligned}\right\} \text{ three empirical coeff.}$$

Except for very special cases (e.g. ideal gases) these equations must be regarded as <u>empirically determined</u>

Together with the conservation laws and the <u>linear isotropic constitutive relations</u>, one has a complete set of eqns governing the mechanics and thermodynamics of a Newtonian viscous fluid.

Some confusion occasioned last time by counting of variables as I forgot to count κ, η, γ as <u>unknowns</u>.

$$\left\{\begin{array}{ll} \text{continuity} & 1 \\ \text{momentum} & 3 \\ \text{ang. momentum} & \rightarrow \text{ only six } V_{ij} \\ \text{energy} & 1 \end{array}\right.$$

$$\left\{\ \text{two equil. state} \quad 2\right.$$

$$\left\{\ \text{three empirical coeff.} \quad 3\right.$$

$$\left\{\begin{array}{l} \underline{H} = -\kappa \nabla \Theta \\ \underline{\underline{V}} = (\gamma - \tfrac{2}{3}\eta)(\operatorname{tr} \underline{\underline{\epsilon}})\,\underline{\underline{I}} + 2\eta\,\underline{\underline{\epsilon}} \end{array}\right\} \begin{array}{l} \text{linear isotropic} \\ \text{const.} \quad 3+6 = \\ \qquad\qquad 9 \end{array}$$

19 eqns

19 unknowns $\rho, p, \theta, U, \underline{u}, \underline{H}, \underline{\underline{V}}, \underbrace{\kappa, \eta, \gamma}_{\text{must be counted}}$

Extended discussion of viscous stress-strain law

$$\underline{\underline{V}} = (\gamma - \tfrac{2}{3}\eta)(\operatorname{tr} \underline{\underline{\epsilon}})\,\underline{\underline{I}} + 2\eta\,\underline{\underline{\epsilon}}$$

$$= \gamma(\operatorname{tr} \underline{\underline{\epsilon}})\,\underline{\underline{I}} + 2\eta\, \underset{\substack{\uparrow \\ \text{deviatoric strain} \\ \text{rate}}}{\underline{\underline{\mathcal{D}}}_{\epsilon}}$$

total stress

$$\underline{\underline{T}} = \underbrace{\left(-p_{eq} + \gamma\, \mathrm{tr}\, \underline{\underline{\epsilon}}\right)}_{\substack{\underline{\text{real}}\text{ fluid}\\ \text{pressure} \neq \text{equil}\\ \text{pressure}}} \underline{\underline{I}} + 2\eta \underset{\substack{\uparrow\\ \text{deviatoric part}\\ \text{of stress}}}{\underline{\underline{D}}_\epsilon}$$

Shear viscosity η, familiar, corresponds to internal friction in fluid, a purely deviatoric strain rate produces a purely deviatoric initial stress.

Bulk viscosity η, a purely dilatorial strain rate produces a pressure in the fluid ≠ ~~[illegible]~~ equil fluid pressure.
Mechanism: if local density suddenly increased, fluid has a finite <u>relaxation time</u> before equil pressure is achieved. Usually associated with <u>intramolecular</u> adjustments e.g. vibrational → rotational modes.

Sometimes involves fairly slow <u>chemical</u>

reactions, e.g. sea water Mg^{++} gives rise to anomalously large sound wave absorption, $\gamma' \sim 100$ cpoise

	η (cpoise)	γ' (cpoise)
sea H_2O	~ 2	~ 100
H_2O	1	~ 3

(at STP)

One last word (brief) about temp. and pressure dependence of viscosity η.

Shear viscosities of gases are found to be $\sim \theta^{1/2}$ and <u>independent</u> of <u>density</u> (!) in reasonable agreement with the prediction of the kinetic theory of gases.

For most liquids, the dependence on temp θ is more severe.

e.g. H_2O

θ°C.	η (centipoise at 1atm)
0	1.8
20	1.00
30	.80
50	.55
100	.28

e.g. glycerine

θ°	η (centipoise 1atm)
-42°	$6.7 \cdot 10^4$
30°	6.3

including silicate melts

For most fluids, a good repr. is

$$\eta(\theta) = \eta_0 e^{-\theta_0/\theta} . \qquad \eta_0, \theta_0 \text{ const.}$$

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Viscosity of a Standard Soda-Lime-Silica Glass

Albert Napolitano and Earl G. Hawkins

The viscosity of a soda-lime-silica glass has been measured at the National Bureau of Standards and seven other laboratories. Determinations were made in the range of 10^2 to 10^{15} poises. The rotating cylinder was used at the higher temperatures (800 to 1450 °C) and the fiber elongation method at the lower temperatures (520 to 658 °C). The results have been critically evaluated and the glass has been issued as Standard Sample No. 710. This glass is available from the National Bureau of Standards.

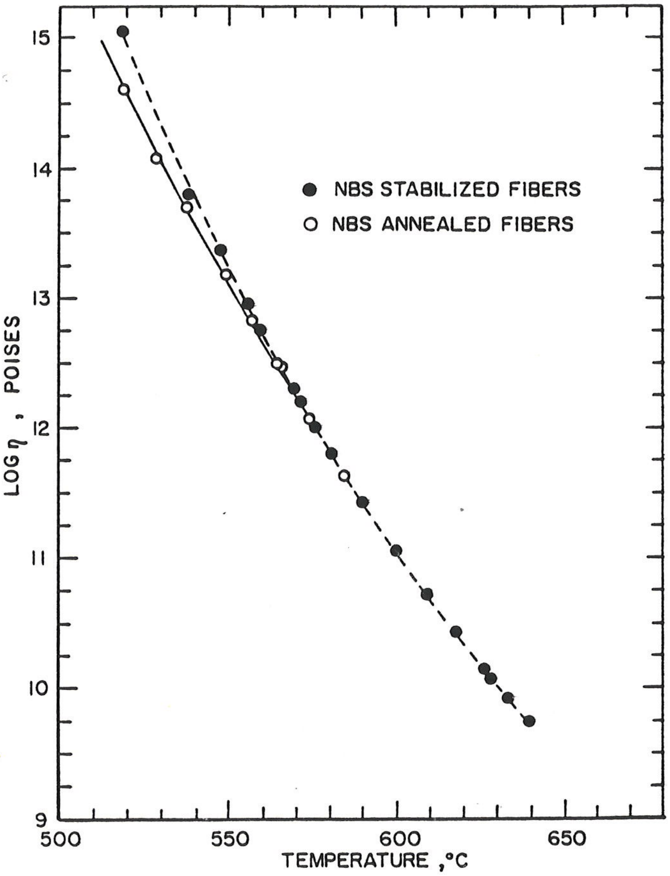


FIGURE 5. *Viscosity of glass after different heat treatments.*

In order to derive a viscosity curve for each of the five glasses, the data points from both the low temperature fiber elongation method and the high temperature rotating cylinder method were combined and fitted to the Fulcher equation [18] by a least squares calculation. This equation [6] has the form:

$$\log_{10} \eta = A + \frac{B}{T - T_0} \tag{4}$$

where T = temperature in °C
η = viscosity in poises
and A, B, and T_0 are constants.

[6] The fit of viscosity-temperature data for glasses to the Fulcher equation has been shown by previous investigators to be exceptionally good. For ease of manipulation, interpolation, and examining data for nonsystematic errors it has proven to be very useful. The use of data points from both the low temperature fiber elongation method and the high temperature rotating cylinder method are compatible within experimental error. Data to be published from this laboratory will show that, in the range 10^6 to 10^{10} P, the continuity of the two methods can be substantiated.

ABLE 8. *Fulcher equation constants from the data submitted by each participating laboratory*

Laboratory	Range	A	B	T₀
	°C			
A(1)	560–1400	−1.626	4,239	265.7
B	560–1300	−1.655	4,271	264.4
C	580–1300	−1.761	4,413	256.2
D	800–1350	−1.584	4,186	266.9
E	800–1300	−1.586	4,124	281.8
F	1025–1450	−1.727	4,580	226.8
G	790–1340	−1.473	4,112	268.7
H	840–1310	−1.211	3,711	309.0
Gen. eq (2)	560–1450	−1.626	4,236	266.0

[1] Data from samples 1–5 combined.
[2] Data from Laboratories A–E combined—see text.

$$\log \eta = -1.626 + \frac{4{,}236}{T\,°\mathrm{C} - 266}.$$

-*Comparison of temperatures for specified viscosities calculated from appropriate Fulcher equation table 8)*

Log viscosity	Temperature, °C								Value from combined equation
	Laboratories								
	A[1]	B	C	D	E	F	G	H	
P									
2.00	----------	----------	----------	----------	----------	1455.7	----------	----------	1434.3
2.10	1403.3	----------	----------	----------	----------	1423.6	----------	----------	1402.9
2.25	1359.3	----------	----------	1358.7	----------	1378.5	----------	----------	1358.9
2.50	1293.0	1292.4	1297.3	1291.9	1291.0	1310.4	1303.7	1308.9	1292.7
2.75	1234.3	1234.1	1236.4	1232.8	1232.8	1249.9	1242.4	1245.8	1234.0
3.00	1182.0	1182.0	1182.4	1180.1	1181.0	1195.7	1188.0	1190.2	1181.7
3.50	1092.6	1093.0	1090.7	1090.3	1092.6	1103.1	1095.6	1096.6	1092.4
4.00	1019.1	1019.7	1016.0	1016.5	1020.0	1026.6	1020.0	1021.1	1019.0
4.50	957.6	958.4	953.8	954.9	959.4	----------	957.1	958.7	957.5
5.00	905.4	906.2	901.3	902.6	907.9	----------	904.0	906.4	905.3
5.50	860.5	861.4	856.4	857.8	863.7	----------	858.4	861.9	860.5
6.00	821.5	822.4	817.5	818.8	825.4	----------	819.0	823.6	821.5
6.50	787.3	788.2	783.5	784.7	791.8	----------	----------	----------	787.3
7.00	757.1	757.9	753.6	----------	----------	----------	----------	----------	757.1
8.00	706.0	706.8	703.2	----------	----------	----------	----------	----------	706.1
9.00	664.6	665.3	662.5	----------	----------	----------	----------	----------	664.7
10.00	630.3	630.9	628.9	----------	----------	----------	----------	----------	630.4
11.00	601.4	601.9	600.7	----------	----------	----------	----------	----------	601.5
12.00	576.8	577.2	576.7	----------	----------	----------	----------	----------	576.9

[1] Results from Lab. A are the combined results from five samples.

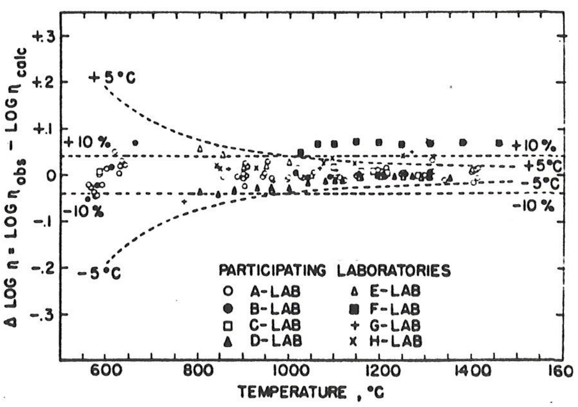


FIGURE 6. *Differences in observed and calculated viscositi of participating laboratories on Standard Glass No. 710.*

8. Entropy generation

Let us investigate the process of entropy generation in a viscous fluid

The entropy per gram will be assumed to be a state fcn obtainable from the eqn of state
(everything is l.t.e. except $\underline{H}$ and $\underline{V}$)

Use U and $\tau = \rho^{-1}$ as ind. variables
From the first law of thermo

$$\theta\, dS = dU + p\, d\tau$$

$$\theta\, D_t S = D_t U + p\, D_t \tau$$

by application to a parcel of fluid

This just as for a non-viscous fluid. Now as we did there combine with energy eqn.

The fact that entropy is a state fcn is used to convert energy eqn into a cons. law for entropy.

$$\rho D_t U + \nabla\cdot\underline{H} = \mathrm{tr}(\underline{\underline{T}}\cdot\underline{\underline{\epsilon}}) + h$$

$$\rho D_t U + p\,\mathrm{tr}\,\underline{\underline{\epsilon}} + \nabla\cdot\underline{H} = \mathrm{tr}\,(\underline{\underline{V}}\cdot\underline{\underline{\epsilon}}) + h$$

now $\quad p\,\mathrm{tr}\,\underline{\underline{\epsilon}} = p\nabla\cdot\underline{u}$

but from cont. eqn. $\quad \rho D_t \tau = \nabla\cdot\underline{u}$

$$p\,\mathrm{tr}\,\underline{\underline{\epsilon}} = \rho\, p D_t \tau$$

$$\rho\,(D_t U + p D_t \tau) + \nabla\cdot\underline{H} = \mathrm{tr}\,(\underline{\underline{V}}\cdot\underline{\underline{\epsilon}}) + h$$

$$\rho\theta D_t S + \nabla\cdot\underline{H} = \mathrm{tr}\,(\underline{\underline{V}}\cdot\underline{\underline{\epsilon}}) + h$$

$$\boxed{\rho D_t S + \nabla\cdot\left(\underline{H}/\theta\right) = \frac{h}{\theta} + \underline{H}\cdot\nabla\frac{1}{\theta} + \frac{1}{\theta}\mathrm{tr}\,(\underline{\underline{V}}\cdot\underline{\underline{\epsilon}})}$$

This is a conservation law for the entropy. Can also be written

$$\boxed{\partial_t(\rho S) + \nabla\cdot\left(\rho S\underline{u} + \underline{H}/\theta\right) = \frac{h}{\theta} + \underline{H}\cdot\nabla\frac{1}{\theta} + \frac{1}{\theta}\,\mathrm{tr}\,(\underline{\underline{V}}\cdot\underline{\underline{\epsilon}})}$$

the usual form of a conservation law

<u>current</u> <u>density</u> of entropy in space =

$$\rho S \underline{u} + \underline{H}/\theta$$

advection

current of entropy due to heat flow

rate of production of entropy per cm^3 per sec =

$$\frac{h}{\theta} + \underline{H}\cdot\nabla\frac{1}{\theta} + \frac{1}{\theta}\,\mathrm{tr}\,(\underline{\underline{V}}\cdot\underline{\underline{\epsilon}})$$

entropy increase due to internal heating

entropy generation by flow against viscosity

entropy increase due to irrev. heat flow down a temp. grad.

Looks exactly like a general non-viscous fluid except for the entropy production term $\frac{1}{\theta}\,\mathrm{tr}\,(\underline{\underline{V}}\cdot\underline{\underline{\epsilon}})$

entropy production due to non-equil. stresses_ lets examine this further

Lets consider now a <u>Newtonian</u> viscous fluid (linear isotropic const. relation)

For that case

$$\mathrm{tr}\,(\underline{\underline{V}}\cdot\underline{\underline{\epsilon}}) = \mathrm{tr}\,\underline{\underline{V}}\cdot\left[\left(\tfrac{1}{3}\mathrm{tr}\,\underline{\underline{\epsilon}}\right)\underline{\underline{I}} + \underline{\underline{\mathcal{D}}}_{\epsilon}\right]$$

$$= \tfrac{1}{3}(\mathrm{tr}\,\underline{\underline{\epsilon}})(\mathrm{tr}\,\underline{\underline{V}}) + \mathrm{tr}(\underline{\underline{V}}\cdot\underline{\underline{\mathcal{D}}}_{\epsilon})$$

now $\underline{\underline{V}} = \gamma(\mathrm{tr}\,\underline{\underline{\epsilon}})\underline{\underline{I}} + 2\eta\,\underline{\underline{\mathcal{D}}}_{\epsilon}$

so

$$\mathrm{tr}\,\underline{\underline{V}} = 3\gamma(\mathrm{tr}\,\underline{\underline{\epsilon}}) \qquad \text{and}$$

$$\mathrm{tr}\,\underline{\underline{V}}\cdot\underline{\underline{\mathcal{D}}}_{\epsilon} = 2\eta\,\mathrm{tr}\,(\underline{\underline{\mathcal{D}}}_{\epsilon}\cdot\underline{\underline{\mathcal{D}}}_{\epsilon})$$

thus

$$\mathrm{tr}\,(\underline{\underline{V}}\cdot\underline{\underline{\epsilon}}) = \gamma\,(\mathrm{tr}\,\underline{\underline{\epsilon}})^2 + 2\eta\,\mathrm{tr}\,(\underline{\underline{\mathcal{D}}}_{\epsilon}\cdot\underline{\underline{\mathcal{D}}}_{\epsilon})$$

entropy generation by viscosity

$$\frac{1}{\theta}\mathrm{tr}(\underline{\underline{V}}\cdot\underline{\underline{\epsilon}}) = \frac{1}{\theta}\gamma(\mathrm{tr}\,\underline{\underline{\epsilon}})^2 + \frac{1}{\theta}2\eta\,\mathrm{tr}\,(\underline{\underline{\mathcal{D}}}_{\epsilon}\cdot\underline{\underline{\mathcal{D}}}_{\epsilon})$$

now consider entropy generation due to irreversible heat flow

$$\underline{H}\cdot\nabla\frac{1}{\theta} \qquad \underline{H} = -\kappa\nabla\theta$$

$$\underline{H}\cdot\nabla\frac{1}{\theta} = -\frac{1}{\theta^2}\underline{H}\cdot\nabla\theta = \frac{\kappa}{\theta^2}|\nabla\theta|^2$$

Entropy production rate for a Newtonian viscous fluid

$$\boxed{\frac{h}{\theta} + \frac{\kappa}{\theta^2}|\nabla\theta|^2 + \frac{\gamma}{\theta}(\mathrm{tr}\,\underline{\underline{\epsilon}})^2 + \frac{2\eta}{\theta}\mathrm{tr}(\underline{\underline{D}}_\epsilon\cdot\underline{\underline{D}}_\epsilon)}$$

$\frac{2\eta}{\theta}\mathrm{tr}(\underline{\underline{D}}_\epsilon\cdot\underline{\underline{D}}_\epsilon)$ entropy generation due to irreversible internal fluid sliding friction

$\frac{\gamma}{\theta}(\mathrm{tr}\,\underline{\underline{\epsilon}})^2$ entropy generation due to expansions or contractions → viscous pressure

e.g. a sudden contraction sets off <u>irreversible</u> reactions which tend toward equil.

Return for a moment to the <u>energy eqn</u>

$$\rho D_t U + \nabla\cdot\underline{\underline{H}} = -p\nabla\cdot\underline{u} + \mathrm{tr}(\underline{\underline{V}}\cdot\underline{\underline{\epsilon}}) + h$$

~~For a Newtonian viscous fluid~~

can rewrite

$$\partial_t(\rho U) + \nabla\cdot(\underbrace{\rho U\underline{u} + \underline{H}}_{\text{current density}}) = -p\nabla\cdot\underline{u} + \mathrm{tr}(\underline{\underline{V}}\cdot\underline{\underline{\epsilon}}) + h$$

production rate of internal energy

$$-p\nabla\cdot\underline{u} + \mathrm{tr}(\underline{\underline{V}}\cdot\underline{\underline{\epsilon}}) + h$$

For Newtonian viscous fluid

$$\boxed{\underbrace{-p\nabla\cdot\underline{u}}_{\substack{\text{heating by}\\ \text{adiabatic}\\ \text{compression}}} + \underbrace{\gamma(\mathrm{tr}\,\underline{\underline{\epsilon}})^2 + 2\eta\,\mathrm{tr}(\underline{\underline{D}}_\epsilon\cdot\underline{\underline{D}}_\epsilon)^2}_{\text{viscous heating}} + \underbrace{h}_{\substack{\text{heating by}\\ \text{sunlight or}\\ \text{radioactivity}}}}$$

Now let us make use of <u>Clausius-Duhem inequality</u>

second law of thermodynamics

$$\rho D_t S + \nabla \cdot \underline{H}/\theta - \frac{h}{\theta} \geqslant 0$$

Comparing with entropy cons. law (for case where entropy is a state fcn)

$$\rho D_t S + \nabla \cdot (\underline{H}/\theta) = \frac{h}{\theta} + \underline{H} \cdot \nabla \frac{1}{\theta} + \frac{1}{\theta} \mathrm{tr}(\underline{\underline{V}} \cdot \underline{\underline{\epsilon}})$$

Clausius – Duhem: rate of generation of entropy per cm^3 per sec by causes other than internal heating h/θ always & everywhere $\geqslant 0$.

$$\underline{H} \cdot \nabla \frac{1}{\theta} + \frac{1}{\theta} \mathrm{tr}(\underline{\underline{V}} \cdot \underline{\underline{\epsilon}}) \geqslant 0$$

true for any generalized Newtonian viscous fluid

Now for a Newtonian viscous fluid

$$\boxed{\frac{\kappa}{\theta^2}|\nabla\theta|^2 + \frac{\lambda}{\theta}(\mathrm{tr}\,\underline{\underline{\epsilon}})^2 + \frac{2\eta}{\theta}\mathrm{tr}(\underline{\underline{D}}_\epsilon\cdot\underline{\underline{D}}_\epsilon) \geq 0 \quad \star}$$

This equation must be true under all circumstances. It is form of second law of thermo for a Newtonian viscous fluid.

Clearly $\frac{1}{\theta^2}|\nabla\theta|^2 \geq 0$, $(\mathrm{tr}\,\underline{\underline{\epsilon}})^2 \geq 0$, and

$\mathrm{tr}(\underline{\underline{D}}_\epsilon\cdot\underline{\underline{D}}_\epsilon) \geq 0$

we can always choose i.c. such that any two terms in $\star$ vanish. Hence the third, non-vanishing term ≥ 0.

$$\boxed{\text{Hence} \quad \kappa \geq 0, \quad \lambda \geq 0, \quad \eta \geq 0 \qquad (\star\star)}$$

Direct consequences of second law
Have the same rigorous character
(for N. viscous fluids) as the
eqns of motion.
This is a typical use of the Clausius-Duhem inequality to place restrictions

on the const. relations

The actual values of $\kappa(p,\theta)$, $\gamma(p,\theta)$, $\eta(p,\theta)$ must still be measured in the lab, but if the measurements ever violate $(**)$ we have discovered a perpetual motion machine.

9. The Navier - Stokes equations

We have seen that we have a complete system of 19 eqns in 19 unknowns. Let's proceed to reduce this to a more tractable set of equations. Consider the momentum eqn

$$\rho D_t \underline{u} = -\nabla p + \nabla \cdot \underline{\underline{V}} + \underline{f}$$

Let's consider only motion in a gravitational field $\underline{f} = \rho \underline{g}$

$$\rho D_t \underline{u} = -\nabla p + \nabla \cdot \underline{\underline{V}} + \rho \underline{g}$$

Now for a <u>Newtonian</u> fluid

$$\nabla \cdot \underline{\underline{V}} = \nabla \left[\left(\gamma - \tfrac{2}{3}\eta\right)(\nabla \cdot \underline{u}) \right] + \nabla \cdot \left[\eta \left(\underline{\underline{\nabla u}} + \underline{\underline{\nabla u}}^T \right) \right]$$

$$= \left[\nabla \left(\gamma - \tfrac{2}{3}\eta\right) \right] \nabla \cdot \underline{u} + \left(\gamma - \tfrac{2}{3}\eta\right) \nabla (\nabla \cdot \underline{u}) + \nabla\eta \cdot \underline{\underline{\nabla u}} + \eta\, \nabla \cdot (\underline{\underline{\nabla u}}) + \nabla\eta \cdot (\underline{\underline{\nabla u}})^T + \eta\, \nabla \cdot (\underline{\underline{\nabla u}})^T$$

Now in a <u>Cartesian</u> axis system

$$[\nabla \cdot (\underline{\underline{\nabla u}})]_i = \partial_j (\partial_j u_i) = \partial_j^{\,2} u_i$$

so $\nabla \cdot (\underline{\underline{\nabla u}})$ is often written $\nabla^2 \underline{u}$

Note: this means $(\nabla^2 u_i)\, \hat{x}_i$ <u>only</u> in a Cartesian system

Also in a Cartesian system

$$[\nabla \cdot (\underline{\underline{\nabla u}})^T]_i = \partial_j (\partial_i u_j) = \partial_i (\partial_j u_j)$$

$$= [\nabla (\nabla \cdot \underline{u})]_i$$

hence $\nabla\cdot(\underline{\underline{\nabla u}})^T = \nabla(\nabla\cdot\underline{u})$

finally

$\nabla\eta\cdot(\underline{\underline{\nabla u}})^T = (\underline{\underline{\nabla u}})\cdot\nabla\eta$ so

$$\nabla\cdot\underline{\underline{V}} = \left[\nabla(\gamma - \tfrac{2}{3}\eta)\right](\nabla\cdot\underline{u}) + (\gamma + \tfrac{1}{3}\eta)\nabla(\nabla\cdot\underline{u}$$
$$+ \nabla\eta\cdot\underline{\underline{\nabla u}} + \underline{\underline{\nabla u}}\cdot\nabla\eta + \eta\nabla^2\underline{u}$$

Momentum eqn for a Newtonian viscous fluid in a grav. field

$$\rho D_t\underline{u} = -\nabla p + \rho\underline{g} + \left[\nabla(\gamma - \tfrac{2}{3}\eta)\right](\nabla\cdot\underline{u})$$
$$+ \nabla\eta\cdot\underline{\underline{\nabla u}} + \underline{\underline{\nabla u}}\cdot\nabla\eta + (\gamma + \tfrac{1}{3}\eta)\nabla(\nabla\cdot\underline{u})$$
$$+ \eta\nabla^2\underline{u}$$

Eqn involves the <u>bulk</u> viscosity γ as well as the <u>shear</u> viscosity η

For most situations of geophysical interest, fluid can be considered to be essentially <u>incompressible</u>

$\nabla \cdot \underline{u} = 0$

Reasons: 1. recall <u>main</u> criterion for incompr. $u \ll c$ (valid N. viscous also)
2. bulk viscosity unimportant except in sound & shock propagation)

For large scale flow situations $u \ll c$, can approximate as <u>incompr. flow</u>. This is <u>always</u> done in geophysics

<u>Incompressible</u> Newtonian viscous fluid

$$\boxed{\rho D_t \underline{u} = -\nabla p + \rho \underline{g} + \eta \nabla^2 \underline{u} + \nabla \eta \cdot \underline{\underline{\nabla u}} + \underline{\underline{\nabla u}} \cdot \nabla \eta}$$

<u>If</u> we are considering a phenomenon in which the variation of viscosity with temp is [redacted] important, then the above is the eqn with which we must deal

Often one considers the viscous flow of a fluid whose shear viscosity is a <u>constant</u> <u>independent</u> of <u>position</u>

$$\boxed{\rho D_t \underline{u} = -\nabla p + \rho \underline{g} + \eta \nabla^2 \underline{u}}$$

This eqn is called the Navier-Stokes equation by most people, and the Stokes-Navier eqn by English writers

The assumptions that have gone into it are:

1. Newtonian viscous fluid
2. incompressible $D_t \rho = 0$, $\nabla \cdot \underline{u} = 0$
3. constant viscosity η.

One usually in such a situation also takes $\rho =$ const

The fluid is then completely characterized by its density ρ and shear viscosity μ.

one has a complete system of eqns involving only mechanical variables:

continuity $\nabla \cdot \underline{u} = 0$

Navier - Stokes $\rho D_t \underline{u} = - \nabla p + \rho \underline{g} + \eta \nabla^2 \underline{u}$

4 eqns, 4 unknowns $\underline{u}$, p.

This a considerably simplified system of eqns:

Warning: the modified pressure

if the fluid is at rest, the Navier - Stokes eqn for a fluid of const ρ becomes just the hydrostatic eqn

$$-\nabla p_0 + \rho \underline{g} = 0$$

$$\underset{\uparrow}{p_0} = \rho \underline{g} \cdot \underline{r}$$

pressure if at rest

often in the full NS eqn one writes

$$\boxed{\begin{aligned} p &= p_0 + P \\ &= \rho_0 \underline{g}\cdot\underline{r} + P \end{aligned}}$$

P is called the <u>modified pressure</u>, and in terms of it the NS eqn is

$$\boxed{\begin{aligned} &\rho D_t \underline{u} = -\nabla P + \eta \nabla^2 \underline{u} \\ &\nabla\cdot\underline{u} = 0 \end{aligned}}$$

The <u>actual</u> fluid pressure is $= \rho \underline{g}\cdot\underline{r}$ + modified pressure P

Gravity has disappeared from the eqns. If it does not appear in the b.c., then it has no effect on the motion, only adds a term $p_0 = \rho \underline{g}\cdot\underline{r}$ to the pressure, the modified pressure is the pressure due to the motion.

In practice, $\underline{g}$ enters the b.c. at a free surface or a fluid – fluid bdry.

It is important to realize when this has been done. If you look up a solution to a particular problem be sure to check whether the pressure is P or $p_0 + P$.

Another warning: <u>the kinematic viscosity</u>

The NS eqn often written in the form (by ÷ by ρ)

$$D_t \underline{u} = -\frac{1}{\rho}\nabla p + \underline{g} + \frac{\mu}{\rho}\nabla^2 \underline{u}$$

or

$$\boxed{\begin{aligned} D_t \underline{u} &= -\frac{1}{\rho}\nabla p + \underline{g} + \nu \nabla^2 \underline{u} \\ D_t \underline{u} &= -\frac{1}{\rho}\nabla P + \nu\nabla^2 \underline{u} \end{aligned}}$$

$\nu = \frac{\mu}{\rho}$ called the <u>kinematic</u> <u>viscosity</u>

reason: $[\nu] = \frac{\ell^2}{t}$ = purely kinematical units

Important to know which viscosity is being used. You can presumably tell this by checking the units.

10. Boundary conditions

Recall for a non-viscous fluid b.c.

$\hat{n} \cdot \underline{u}$ cont normal comp. of velocity
p cont

It is observed that viscous fluids adhere to the wall of a rigid body
Difficult to give a convincing molecular explanation of this fact.
We shall merely accept it. Viscosity ⇒ it "adheres to itself." Does not make sence that it would not also adhere to rigid walls.

Otherwise strain rate would be ∞ near the walls and bounded everywhere else.

Kinematic b.c. at a fluid-solid bdry for a N. viscous fluid is the no-slip condition

$\underline{u} = \underline{0}$ at a fluid solid bdry

very important observation (*)

Dynamic b.c. is unchanged

$\hat{n} \cdot \underline{\underline{T}}$ cont

$$\boxed{\begin{array}{ll} -p\hat{n} + \hat{n}\cdot\underline{\underline{V}} \text{ cont.} & \text{dynamical b.c.} \\ & \text{at any interface} \\ \quad (**) & \\ -p\hat{n} + 2\eta\, \hat{n}\cdot\underline{\underline{\nabla u}} \text{ cont} & \end{array}}$$

In practice (*) is the only b.c. ever used at a rigid bdry (the concept of stress in the rigid body has no meaning) and (**) is most important at a free surface

$$\boxed{\begin{array}{ll} -p\hat{n} + \hat{n}\cdot\underline{\underline{V}} = 0 & \text{at a free surface} \\ -p\hat{n} + 2\eta\hat{n}\cdot\underline{\underline{\nabla u}} = 0 & \end{array}}$$

Note: the free surface b.c. contains $p = p_0 + P$. Hence the expedient of introducing the <u>modified pressure</u> P is a problem involving a free surface does no good.

11. <u>Final remarks</u>

Summarizing the mechanical eqns governing the motion of an incompr. N. viscous fluid of const viscosity

$\nabla\cdot\underline{u} = 0$ incompr.

$\rho D_t \underline{u} = -\nabla p + \underline{g} + \eta\nabla^2\underline{u}$ N.S.

b.c. (most important cases)

$\underline{u} = \underline{0}$ at a rigid bdry

$-\hat{n}p + 2\eta\,\hat{n}\cdot\underline{\underline{\nabla u}} = 0$ at a <u>free</u> surface

Even this fairly simple set of eqns is very difficult to obtain solutions for.

VERY IMPORTANT: There are <u>no</u> <u>uniqueness</u> <u>thms</u> for this set of eqns.

Reason for difficulty is non-linearity of NS eqns

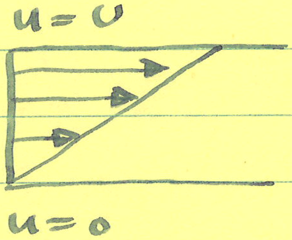
$$D_t\underline{u} = \partial_t\underline{u} + \underset{\uparrow}{\underline{u}\cdot\underline{\underline{\nabla u}}}$$

this is the non-linear term.

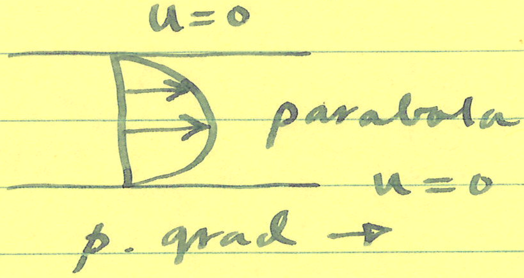
<u>All</u> the easy well-known <u>exact</u> solutions, e.g. Couette and Poiseuille flow, can be obtained because the non-linear term vanishes

identically.

Example: plane Couette flow *



plane Poiseuille flow



etc. These are exact solutions to the full non-linear equations.

If one does the expt. * & makes the U of top plate too large, R_{crit} is exceeded and motion becomes turbulent. Some think reason is b.v. problem for plane Couette flow not unique, and that more than the one laminar soln ∃. It may however be that NS eqns cannot describe highly turbulent motion. The answer not known at this time.

Our problem now one of solving the NS eqn, a math. problem. A great deal is known about this system of eqns

But unfortunately some of the things we would most like to know, esp. about turbulence, are not known.

Best book on this subject is Batchelor. Almost exclusively devoted to the system of eqns we have obtained

Batchelor, An Intro. to Fluid Dynamics, C.U.P. 615 pp. highly recommended.

Doesn't consider more complicated phenomena, e.g. convection.

12. The Reynolds Number and Dynamical Similarity

The motion of a uniform incompressible Newtonian viscous fluid is governed by

$$\rho D_t \underline{u} = -\nabla P + \eta \nabla^2 \underline{u}$$
$$\nabla \cdot \underline{u} = 0$$

here $P \equiv$ modified pressure

Let us consider the effect on the flow of changes in the constants ρ and η. To do this we will write these eqns in terms of dimensionless variables

No parameters with $[L]$ or $[\mathcal{U}]$ occur in these eqns, We must look to the boundary conditions, if we wish to scale $\underline{r}$ and $\underline{u}$.

Let $L =$ typical length scale of bdry
$\mathcal{U} =$ typical velocity of bdry, say speed of a body of dimension L thru the fluid

$$\left.\begin{aligned} \text{We write}\quad & \underline{u}' = \underline{u}/\mathcal{U} \\ & \underline{r}' = \underline{r}/L \\ & t' = t\,\mathcal{U}/L \\ \text{and}\quad & P' = P/\rho\,\mathcal{U}^2 \end{aligned}\right|\ \text{all dimensionless}$$

$$\nabla = \frac{1}{L}\nabla'$$

$$D_t = \frac{\mathcal{U}}{L} D'_{t'}$$

Substitute into the eqns.

$$\nabla' \cdot \underline{u}' = 0 \quad \text{or}$$

$$\partial u_i' / \partial x_i' = 0 \quad \text{where} \quad \underline{r}' = (x_1', x_2', x_3')$$

$$D'_{t'}\,\underline{u}' = -\nabla' P' + \frac{1}{R}\nabla'^2 \underline{u}' \quad \text{or}$$

$$\frac{\partial u_i'}{\partial t'} + u_j' \frac{\partial u_i'}{\partial x_j'} = -\frac{\partial P'}{\partial x_i'} + \frac{1}{R}\frac{\partial^2 u_i'}{\partial x_j' \partial x_j'}$$

$$\text{Where} \quad R \equiv \frac{\rho L \mathcal{U}}{\eta} = \frac{L\mathcal{U}}{\underset{\nwarrow\ \text{kinematic viscosity}}{\nu}}$$

The dimensionless eqns now contain explicitly only the dimensionless parameter R, called the Reynolds number.

The solution $\underline{u}'$, P' can depend only on

1. the ind. variables $\underline{r}', t'$
2. the Reynolds number R
3. the geometry of the configuration of the bdries.

Once these eqns have been solved in dimensionless form, it really corresponds to a triply infinite family of solutions (change $(\rho, L, \mathcal{U}, \eta)$ to keep R fixed).

All such flows are said to be dynamically similar.

The principle of dynamic similarity is widely used in "model tests".

For example, hydraulic engineers often wish to know settling times for small solid particles in suspension. Direct observation is difficult; too small to see.

But the flow is dynamically similar to that about a much larger sphere in a more viscous fluid

Oil: $\nu \approx 400$ times that of H_2O

$$R \equiv \frac{L\mathcal{U}}{\nu}.$$

The magnitude of the Reynolds number R may be regarded, in a crude sense, as an estimate of the relative importance of the non-viscous and viscous forces acting on unit volume of fluid.

$$\underbrace{\rho D_t \underline{u}}_{\substack{\text{inertia}\\\text{force}\\\text{(per unit}\\\text{volume)}}} = \underbrace{-\nabla P}_{\substack{\text{pressure}\\\text{force}}} + \underbrace{\eta \nabla^2 \underline{u}}_{\substack{\text{viscous}\\\text{force}}}$$

These three forces per unit volume must balance on any parcel. The relative magnitudes can be given by a ratio of any two. Pressure usually plays a passive role.

Customary to characterize flow by relative magnitudes of inertia and viscous forces.

$$\text{Ratio} = \frac{|\rho D_t \underline{u}|}{|\eta \nabla^2 \underline{u}|} = R \frac{|D'_{t'} \underline{u}'|}{|\nabla'^2 \underline{u}'|}$$

4

at any pt. in fluid

Hence if $D_t^{*} \underline{u}^{*}$ and $\nabla^{*2} \underline{u}^{*}$ are both $O(1)$ (this is likely if the flow is simple, and if L and U are really representative), then loosely speaking:

R measures the ratio of inertia to viscous forces.

If $R \ll 1$, inertia is negligible, viscous and pressure forces dominate (example: Stokes flow, a small particle falling at terminal velocity)

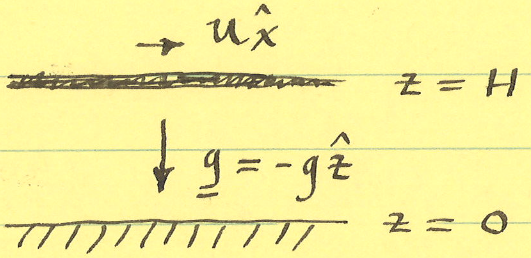
e.g. mantle convection

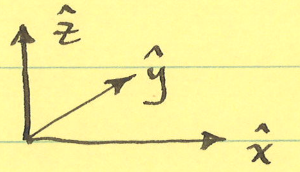
If $R \gg 1$, viscous forces are negligible, inertia and pressure forces dominate.

13. Plane Couette Flow ; an exact solution

Consider a simple plane viscometer

We assume NS eqns are applicable





Top plate moves horizontally with constant velocity $U\hat{x}$.

We assume that the flow is steady and that all measurable quants. depend only on z, not on x or y.

~~Then $\nabla\cdot\underline{u} = 0 \Rightarrow \partial_x u_x + \partial_y u_y + \partial_z u_z = 0$~~

~~or $\partial_z u_z = 0$~~

$$\underline{u} = (u(z),\ v(z),\ w(z))$$

$$P = P(z) \qquad (\text{recall } p = \rho g z + P)$$

$$\nabla\cdot\underline{u} = \partial w/\partial z$$

$$\nabla^2\underline{u} = (\partial^2 u/\partial z^2,\ \partial^2 v/\partial z^2,\ \partial^2 w/\partial z^2)$$

$$\underline{u}\cdot\underline{\nabla} = w\frac{\partial}{\partial z}$$

1. from continuity $\frac{\partial w}{\partial z} = 0$ $\quad w = \overset{\substack{\text{since steady}\\ \downarrow}}{\text{constant}}$

but $w = 0$ at bdry, hence $w = 0$

Now $\partial_t \underline{u} = 0$ and $D_t\underline{u} = \underline{u}\cdot\nabla\underline{u} = 0$

$$-\hat{z}\frac{\partial P}{\partial z} + \eta\left(\hat{x}\frac{\partial^2 u}{\partial z^2} + \hat{y}\frac{\partial^2 v}{\partial z^2}\right) = 0$$

Equate components

$$\partial P/\partial z = 0$$

$$\partial^2 u/\partial z^2 = 0$$

$$\partial^2 v/\partial z^2 = 0$$

$P =$ constant $= P_0$, say

$u(z) = Az + B$

$v(z) = Cz + D$

Actually $P_0(t)$
$\exists$ usual arbitrariness
in incompressible
situations

Now we make use of the <u>no-slip b.c.</u>

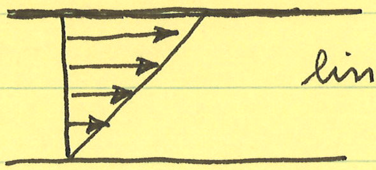
$u(0) = 0 \qquad u(H) = \mathcal{U}$

$v(0) = 0 \qquad v(H) = 0$

$C = D = 0$

$B = 0 \ , \quad A = \dfrac{\mathcal{U}}{H}$

Final solution $\quad \underline{u}(z) = \hat{x}\, z\, \dfrac{\mathcal{U}}{H}$



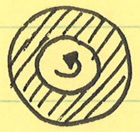
linear flow profile.

The <u>stress</u> on the top boundary is

$$\hat{z} \cdot \underline{\underline{V}} = \eta \frac{\partial u}{\partial z} = \eta \frac{\mathcal{U}}{H} .$$

This is a way to measure viscosity η. Measure the force F necessary to pull a plate of area A with a steady velocity $\mathcal{U}$.

Actual viscometers work on exactly this principle, but in cylindrical geometry



<u>Cylindrical</u> <u>couette</u> <u>flow</u>.

Measure torque required to rotate at a steady angular velocity.

Points to note:

1. This is an <u>exact</u> solution to NS eqns. It was easy because non-linear term was identically zero.

2. We obtained the solution by virtue of an assumed laminar form (steady flow independent of x, y)

3. If one actually does the experiment,

one finds, that if one pulls slowly enough, the solution agrees with observation.

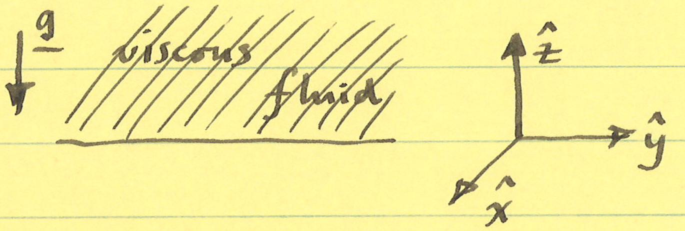
If one pulls too fast, and exceeds a certain $R_{crit} = \rho H U / \eta$, then the *assumption* is violated and turbulence sets in.

Most people think that the reason is that the boundary value problem is not unique and that more than the one *laminar* solution to NS eqns exists.

This is quite generally the case. In many geometries, if one *assumes* a simple laminar solution to NS, then one can find a solution. In an expt, that is *the* solution at low Reynolds number, but at some critical Reynolds number, turbulence sets in and the laminar assumption is no good.

14. The semi-infinite fluid (Rayleigh)

Consider a semi-infinite fluid with a rigid lower bdry. in a uniform grav. field.



Consider the motion of this fluid due to an arbitrary horizontal motion of the lower boundary.

Boundary conditions: $\underline{u}(\underline{r},t) \to 0$ as $z \to \infty$

$\underline{u} = U(t)\hat{x} + V(t)\hat{y}$ on $z=0$

Consider the fluid to be homogeneous, incompressible, Newtonian

$\rho =$ const density

$\eta =$ const shear viscosity

$$\rho D_t \underline{u} = -\nabla p + \rho \underline{g} + \eta \nabla^2 \underline{u}$$

$$\nabla \cdot \underline{u} = 0$$

use the <u>modified</u> <u>pressure</u>

$$\underset{\substack{\uparrow \\ \text{total pressure}}}{p(\underline{r}, t)} \equiv P(\underline{r}, t) + \underset{\substack{\uparrow \\ \text{hydrostatic pressure}}}{\rho g z}$$

$$\partial_t \underline{u} = -\frac{1}{\rho} \nabla P + \underset{\uparrow}{\nu} \nabla^2 \underline{u}$$

$$\nabla \cdot \underline{u} = 0$$

kinematic viscosity $\nu = \frac{\mu}{\rho}$.

$[L^2 T^{-1}]$

We will look only for a <u>laminar</u> <u>solution</u> (no x, y dependence)

We <u>assume</u> a soln of form $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$

i.e. $P(\underline{r}, t) = P(z, t)$

$$\underline{u}(\underline{r}, t) = u(z,t)\hat{x} + v(z,t)\hat{y} + w(z,t)\hat{z}.$$

$\nabla \cdot \underline{u} = 0$ reduces to $\frac{\partial w}{\partial z} = 0$, so

$w = f(t)$ only

But $w = 0$ on $z = 0$

Hence $w = 0$ everywhere

$$D_t \underline{u} = \partial_t \underline{u} + w \, \partial \underline{u} / \partial z$$
$$= \partial_t \underline{u}$$

non-linear term identically zero in our assumed laminar soln.

$$\nabla^2 \underline{u} = \partial^2 \underline{u} / \partial z^2$$

$$\partial_t \underline{u} = - \hat{z} \frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \, \partial^2 \underline{u} / \partial z^2$$

this is <u>exact</u> momentum <u>eqn</u> for assumed laminar soln.

In component form

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2}$$

$$\frac{\partial P}{\partial z} = 0$$, hence $P = f(t)$ only, say $P = P_0(t)$

Final solution for pressure $p(\underline{r}, t) = -\rho g z + P_0(t)$

Recall we always have this arbitrariness of pressure in an <u>incompressible</u> <u>fluid</u>. Pressure is essentially hydrostatic.

Problem for $\underline{u}(\underline{r}, t)$ reduces to:

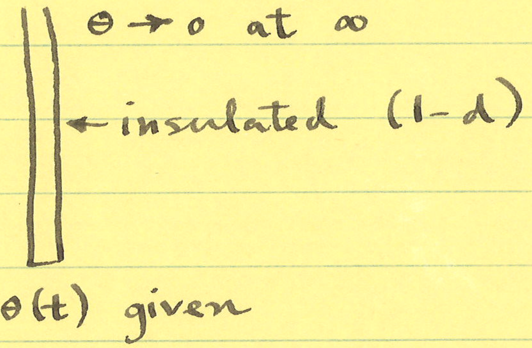
$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} \qquad \begin{aligned} u(0,t) &= \mathcal{U}(t) \\ u(\infty,t) &= 0 \end{aligned}$$

Similarly for $v(z,t)$. From now on just consider $u(z,t)$

Note: this the 1-d diffusion equation

Same eqn describes the heat flow in an infinite 1-d rod

If ν is the thermal diffusivity.



Consider now two cases of interest

Case 1: $\mathcal{U}(t) = \mathcal{U}_0 \cos \omega t = \mathrm{Re}\, \mathcal{U}_0 e^{i\omega t}$

Oscillating lower boundary

Look for steady state solution after dying away of transients.

Let
$$\begin{aligned} u(z,t) &= \mathrm{Re}\, \tilde{u}(z,t) \\ \tilde{u}(0,t) &= \mathcal{U}_0 e^{i\omega t} \\ \tilde{u}(\infty,t) &= 0 \end{aligned}$$

$\tilde{u}(z,t) = \tilde{\tilde{u}}(z)\, e^{i\omega t}$ Steady state

$\frac{\partial \tilde{u}}{\partial t} = \nu \frac{\partial^2 \tilde{u}}{\partial z^2}$ becomes

$\frac{\partial^2 \tilde{\tilde{u}}}{\partial z^2} = \frac{i\omega}{\nu} \tilde{\tilde{u}}$

Thus $\tilde{\tilde{u}}(z) = A e^{\alpha z} + B e^{-\alpha z}$

where $\alpha^2 = i\omega/\nu$ or $\alpha = \sqrt{\frac{\omega}{\nu}}\, e^{i\frac{\pi}{4}}$

or $\alpha = \sqrt{\frac{\omega}{2\nu}}\,(1+i)$

The b.c. $\left\{ \begin{array}{l} \tilde{\tilde{u}}(0) = U_0 \\ \tilde{\tilde{u}}(\infty) = 0 \end{array} \right\}$ imply that

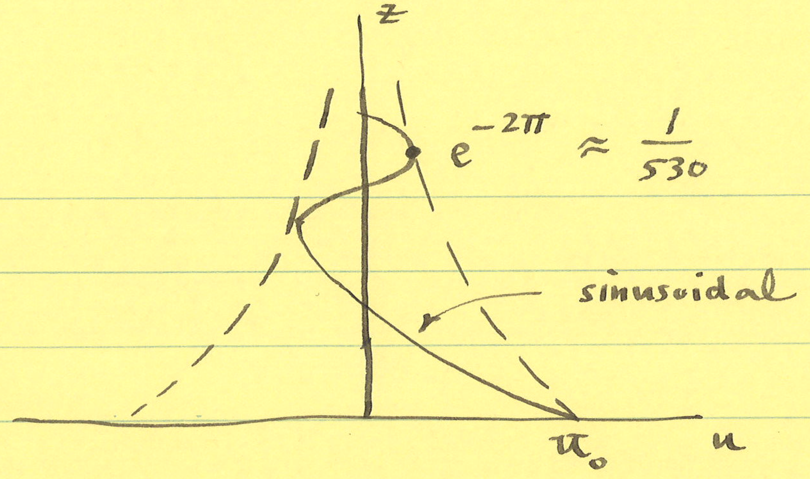
$A = 0$ and $B = U_0$

Thus $\tilde{\tilde{u}}(z) = U_0\, e^{-\sqrt{\frac{\omega}{2\nu}}\, z\,(1+i)}$

$u(z,t) = \text{Re}\; \tilde{\tilde{u}}(z)\, e^{i\omega t}$

$$\boxed{u(z,t) = U_0\, e^{-z\sqrt{\frac{\omega}{2\nu}}} \cos\left(\omega t - z\sqrt{\frac{\omega}{2\nu}}\right)}$$

Plot at $t = 0$



Note that wavelength and exponential decay are same

$\delta = \sqrt{\frac{2\nu}{\omega}}$ = length for ampl. to drop to 1/e.

δ is a good measure of <u>bdry layer thickness</u>. The no-slip b.c. forces the fluid near the bdry to move with it. A few δ above $z = 0$ there is essentially no motion.

For water at 20°C, $\nu = 10^{-2}\ \text{cm}^2/\text{sec}$ ‖ $\eta = 1.004$ cp, $\rho = 1.0\ \frac{\text{gm}}{\text{cm}^3}$

$\delta = \sqrt{\frac{\nu T}{\pi}}$, T = period

note: ~same as K_{rock} thermal diffusivity

T (sec)	δ (cm)
$10^{-2}\pi$	10^{-2}
π	10^{-1}
$10^2\pi \approx 5$ min	1
$10^4\pi \approx 8$ hr.	10

Boundary layer very thin

Note: precisely the same solution may be used to estimate the depth penetration of the diurnal solar heating at the Earth's surface

$\delta = \sqrt{\frac{kT}{\pi}}$ $\quad k =$ thermal diffusivity

≈ 0.01 cm^2/sec for soil

$T = 24$ hr $\quad \uparrow$ same as ν_{H_2O} !

$\delta \approx 17$ cm.

The viscous bdry layer above an oscillating plate is small, change in $u(z,t)$ within a few δ is large. Tremendous amount of shear and lots of energy dissipation, mostly in boundary layer.

Rate of energy dissipation per unit volume per second is

$$\dot{\varepsilon} \equiv \text{tr}\,(\underline{\underline{v}} \cdot \underline{\underline{\varepsilon}}) = \eta \left(\frac{\partial u}{\partial z}\right)^2$$

$$\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial \tilde{u}}{\partial z} + \frac{\partial \tilde{u}^*}{\partial z}\right) \quad , \quad \text{so}$$

$\uparrow$

this is $\text{Re}\left(\frac{\partial \tilde{u}}{\partial z}\right)$

$$\dot{\varepsilon} = \frac{\eta}{4}\left[\left(\frac{\partial\tilde{u}}{\partial z}\right)^2 + \left(\frac{\partial\tilde{u}^*}{\partial z}\right)^2 + 2\,\frac{\partial\tilde{u}}{\partial z}\,\frac{\partial\tilde{u}^*}{\partial z}\right]$$

Consider the time average dissipation $\langle\dot{\varepsilon}\rangle$

since $\tilde{u} \sim e^{i\omega t}$

$$\left\langle\left(\frac{\partial\tilde{u}}{\partial z}\right)^2\right\rangle = \left\langle\left(\frac{\partial\tilde{u}^*}{\partial z}\right)^2\right\rangle = 0 \quad , \text{ so}$$

$$\langle\dot{\varepsilon}\rangle = \frac{\eta}{2}\left|\frac{\partial\tilde{u}}{\partial z}\right|^2$$

$$= \frac{\eta}{2}\,U_0^2\left|-(1+i)\sqrt{\frac{\omega}{2\nu}}\;e^{-(1+i)\sqrt{\frac{\omega}{2\nu}}\,z}\right|^2$$

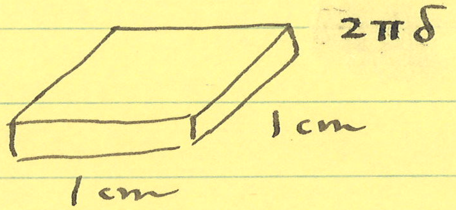
$$= \eta\,U_0^2\,\frac{\omega}{2\nu}\,e^{-2\sqrt{\frac{\omega}{2\nu}}\,z}$$

$$\boxed{\langle\dot{\varepsilon}\rangle = \left(\tfrac{1}{2}\rho U_0^2\right)\omega\,e^{-\frac{2z}{\delta}} \qquad \text{erg cm}^{-3}\,\text{sec}^{-1}}$$

For the total dissipation above 1 cm^2 of plate

$$\boxed{\langle\dot{E}\rangle = \int_0^\infty \langle\dot{\varepsilon}\rangle\,dz = \tfrac{1}{2}\left(\tfrac{1}{2}\rho U_0^2\right)\delta\omega \qquad \text{erg cm}^{-2}\,\text{sec}^{-1}}$$

i.e. in one cycle, amount of energy dissipated = mean kinetic energy of a parcel of fluid moving with velocity Re $\mathcal{U}_0 e^{i\omega t}$



$$\boxed{\begin{aligned}\langle \dot{E} \rangle &= \tfrac{1}{2}\left(\tfrac{1}{2}\rho\,\mathcal{U}_0^2\right)\delta\omega \quad \text{erg cm}^{-2}\ \text{sec}^{-1} \\ &= \pi\delta\left(\tfrac{1}{2}\rho\,\mathcal{U}_0^2\right) \quad \text{erg cm}^{-2}/\text{cycle} \\ &= \tfrac{1}{2}(2\pi\delta)\left(\tfrac{1}{2}\rho\,\mathcal{U}_0^2\right)\ (\text{erg}/\text{cm}^2)/\text{cycle}\end{aligned}}$$

Most of this dissipation takes place <u>near the plate</u>.

Another way to compute $\langle \dot{E} \rangle$

$\langle \dot{E} \rangle$ is energy necessary to move lower boundary. Let $F/A \equiv$ force/area on plate

$$\langle \dot{E} \rangle = \left\langle \left(\frac{F}{A}\,u_{\text{plate}}\right)_{z=0} \right\rangle \qquad \frac{F}{A} = -\eta\frac{\partial u}{\partial z}$$

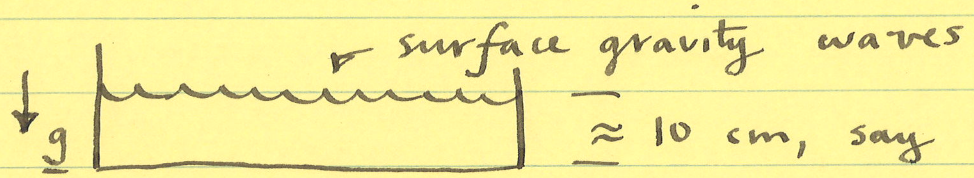
$$\dot{E} = -\eta u \frac{\partial u}{\partial z}\bigg|_{z=0} = -\frac{1}{2}\eta\,\frac{\partial}{\partial z}u^2\bigg|_{z=0}$$

gives same as above.

As an application of this problem, consider, e.g. a wave tank (or waves in the ocean or a lake)

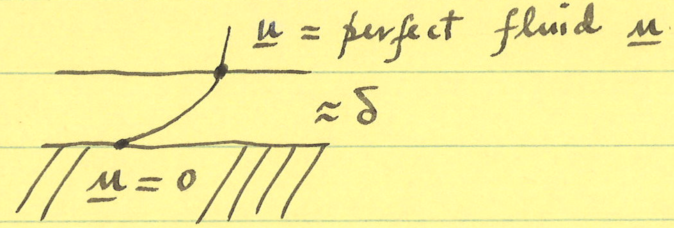
NOTE:

A better example: the Slichter mode problem



At wave frequencies for water $\nu = 10^{-2}$
$\delta = \sqrt{\frac{2\nu}{\omega}} \approx$ few mm bdry layer thickness
$\delta \ll 10$ cm.

Apply above boundary layer solution to a region large compared to δ but small compared to vertical pressure gradient due to waves.
Outside of this region motion is that of a perfect fluid.



From point of view boundary layer $\underline{u}$ goes from $\underline{u} = 0$ at $z = 0$ to $\underline{u} = \underline{u}_{\text{perfect}}$ at ∞

Slichter mode : $\delta/L \sim R^{-1/2}$

$\delta \sim \sqrt{\nu T} \approx$ typical of Rayleigh boundary layer

If $\nu_{core} \sim \nu_{H_2O}$ then $\delta \sim 10$ cm

which is $\ll$ $L \sim$ inner core radius ~ 1000 km

$\sim 10^8$ cm

$\delta/L \sim 10^{-7}$ ⇒ $R \sim 3000$

But from point of view of waves, $\underline{u}$

$\underline{u}_{\text{perfect}}$ at bottom.

This is how bdry layer solution is fit to internal perfect fluid motion.

The technique for doing this [redacted] rigorously is called the <u>method</u> of <u>matched asymptotic expansions</u>.

The NS eqn <u>inside</u> the wave tank is

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{\underline{\nabla u}} = -\frac{1}{\rho} \nabla (p + \rho g z) + \nu \nabla^2 \underline{u}$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$

$O(\omega \underline{u})$ $\quad$ $O(k^2 \underline{u})$ $\quad$ $O\left(k\left(\frac{p}{\rho} + g\right)\right)$ $\quad$ $O(\nu k^2 \underline{u})$

$O(k^2 \underline{u})$: neglect in linearized treatment

ω, k are ang. freq. & wavenumber of wave

Scaling valid away from walls or bottom or top.

<u>If</u> $\nu \equiv 0$, fluid is perfect

We have solved for wave motion using a bottom kinematic b.c. which <u>allows</u> <u>tang. slip</u>.

If $\nu \equiv 0$, term 1 balances term 3.
Viscosity will affect motion only if term 4 $\approx$ term 1 or term 3.

$\nu k^2 \ll \omega$ for $\approx$ no effect

$$\boxed{\text{or} \quad \frac{\lambda^2}{\nu T} = R \gg 1}$$

$\nu = 10^{-2}$

$\omega \cong \sqrt{gk}$

As long as $T \gg 10^{-2}$ sec can neglect viscosity in internal motion
But still must satisfy <u>no slip</u> boundary condition, so <u>u</u> must $\to 0$ in a thin bdry layer.

We see now why the study of <u>perfect</u> fluids is valuable. In the above problem, the inviscid or internal theory can be used to determine all the interesting wavelike features.

Most wave <u>damping</u> in wave tanks is bdry layer dissipation on bottom & sides (esp. on sides near free surface where <u>u</u> perfect is largest).

Exp. evidence also seems to indicate a need for a bdry layer on top (free) surface.

Reason: surface contamination → appears to be a flexible sheet on surface.

Case 2: $U(t) = U_0$

pull bottom plate at constant velocity

Corresponds to $\omega = 0$ in last case.

Note: bdry layer thickness $\delta = \sqrt{\frac{2\nu}{\omega}}$

$\delta = \infty$. This indicates there may be trouble

Consider a steady state solution $u(z)$

$$\nu \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} = 0$$

$$u = Az + B$$

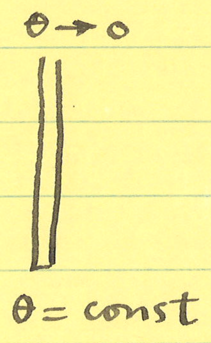
$$u(0) = U_0 \Rightarrow B = U_0$$

Cannot fit upper b.c. $u(\infty) \to 0$

There is no steady solution.

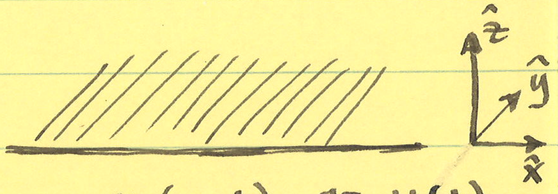
Same for an ∞ 1-d rod

No steady solution exists.



We must instead look at the initial value problem. Say at $t = 0$ we commence pulling lower bdry at a const. velocity U_0.

$$u(\underline{r}, t) = u(z,t)\,\hat{x}$$



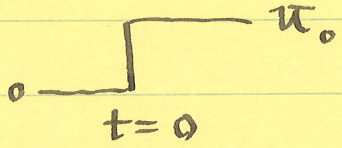
$u(0,t) = U_0 H(t)$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2}$$

i.c. $u(z,0) = 0$

b.c. $u(\infty, t) = 0$

$u(0,t) = U_0 H(t)$



See Lamb p. 590 for solution

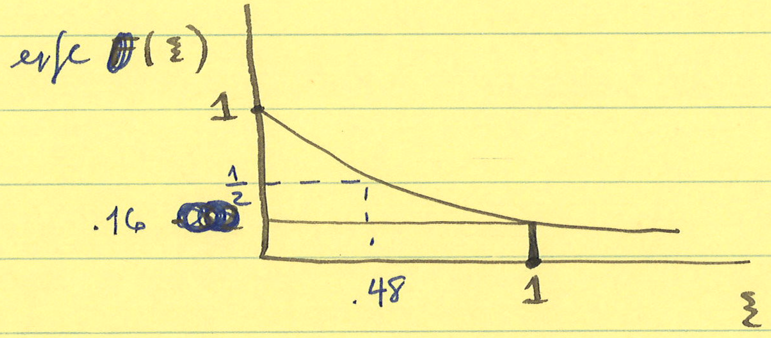
Answer is:

$$u(z,t) = U_0 \, \mathrm{erfc}\left(\frac{z}{\sqrt{4\nu t}}\right) \quad \text{where}$$

$$\mathrm{erfc}\,\xi \equiv \frac{2}{\sqrt{\pi}} \int_{\xi}^{\infty} e^{-x^2}\,dx$$

Can also be written in terms of error function integral.

The function $\overset{\text{erfc}}{\cancel{\theta}}(\xi)$ looks like



At time t, at what height δ is

$u(\delta, t) = .16\, u_0$?

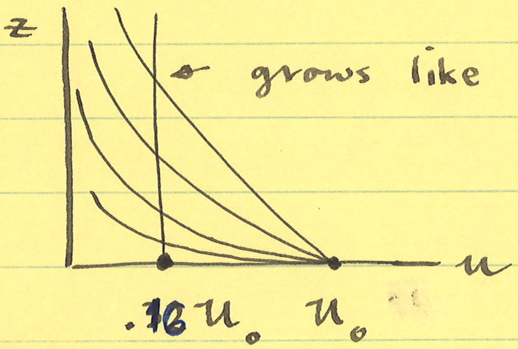
Answer: $\delta = \sqrt{4 \nu t}$

④

typical diffusion result.

↓

Thickness of bdry layer grows like $\sim \sqrt{t}$

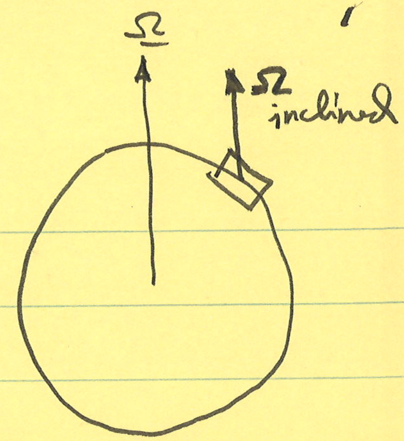


Growth is slower for fluids with smaller ν.

For water: in 1 sec, $\delta \approx 1$ mm.

in 3 hr, $\delta \approx 14$ cm.

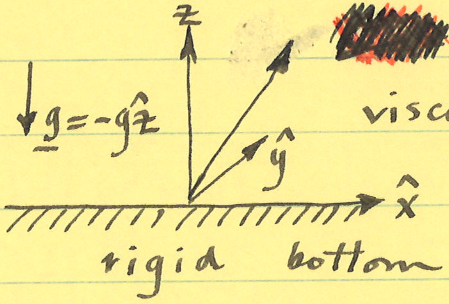
Boundary layer diffuses outward very slowly.



A final lecture on viscous fluids

The Ekman boundary layer

Let us consider the same problem for a body of fluid which possesses a net rotation.



$\underline{\Omega}$ (inclined rot. axis)

viscous fluid const. ρ

const. η

Say that the fluid is initially in a state of uniform rigid rotation

We shall use this uniformly rotating frame as a frame of reference

In the state of uniform rotation $\underline{u}(\underline{r}, t) = 0$

Momentum equation is

$$\rho\left[D_t\underline{u} + 2\underline{\Omega}\times\underline{u} + \underline{\Omega}\times(\underline{\Omega}\times\underline{r})\right] = \nabla\cdot\underline{\underline{T}} + \underline{f}\ .$$

$$\rho\underline{\Omega}\times(\underline{\Omega}\times\underline{r}) = -\nabla p - \rho g\hat{z}$$

or $$\nabla\left[p + \rho g z - \tfrac{1}{2}\Omega^2\varpi^2\right] = 0$$

The hydrostatic pressure $p_0(\underline{r}, t)$ is given by

$$p_0(\underline{r}, t) = -\rho g z + \frac{1}{2}\Omega^2 \varpi^2 + \underset{\nearrow}{f(t)}$$

usual arbitrariness
assoc. with incompr.
fluid

The hydrostatic pressure is affected both by gravity and by centrifugal force.
This defines the initial state of the fluid.

Now consider the motion induced in this body of fluid by a horizontal motion of the bottom bdry

Boundary conditions : $\underline{u}(\underline{r}, t) = U(t)\hat{x} + V(t)\hat{y}$
on $z = 0$
$\underline{u}(\underline{r}, t) \to 0$ as $z \to \infty$

The latter b.c. stipulates that the state of uniform rotation persists at $z \to \infty$.

Equations of motion are:

continuity $\nabla \cdot \underline{u} = 0$

momentum $\rho \left[D_t \underline{u} + 2 \underline{\Omega} \times \underline{u} + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}) \right]$
$= - \nabla p - \rho g z + \eta \nabla^2 \underline{u}$

We introduce a <u>modified</u> <u>pressure</u> $P(\underline{r}, t)$ by

$$P(\underline{r}, t) = p(\underline{r}, t) - \underset{\substack{\uparrow}}{p_0} (\underline{r}, t)$$

hydrostatic pressure (in the absence of motion)

$$D_t \underline{u} + 2 \underline{\Omega} \times \underline{u} = - \frac{1}{\rho} \nabla P + \underset{\substack{\uparrow}}{\nu} \nabla^2 \underline{u}$$
$$\nabla \cdot \underline{u} = 0$$

kinematic $\nu \equiv \frac{\eta}{\rho}$ $[L^2 T^{-1}]$

Once again we will <u>assume</u> the existence of a <u>laminar</u> solution, with no x, y dependence, $\partial / \partial x = \partial / \partial y = 0$

$$\underline{u}(\underline{r}, t) = u(z, t) \hat{x} + v(z, t) \hat{y} + w(z, t) \hat{z}$$
$$P(\underline{r}, t) = P(z, t)$$

As before $\nabla \cdot \underline{u} = 0 \rightarrow \frac{\partial w}{\partial z} = 0$

$w = f(t)$ only

But $w(0,t) \equiv 0$, hence $w(z,t) = 0$

No vertical motion

$$D_t \underline{u} = (\partial_t + \underline{u}\cdot\underline{\nabla})\underline{u} = \partial_t \underline{u} + w\frac{\partial \underline{u}}{\partial z}$$
$$= \partial_t \underline{u}$$

Non linear term vanishes identically.

For assumed laminar soln, mom. eqn. is (exactly)

$$\partial_t \underline{u} + 2\underline{\Omega} \times \underline{u} = -\frac{1}{\rho}\hat{z}\frac{\partial P}{\partial z} + \nu\frac{\partial^2 \underline{u}}{\partial z^2}$$

In component form

$$\frac{\partial u}{\partial t} - 2\Omega_z v = \nu\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} + 2\Omega_z u = \nu\frac{\partial^2 v}{\partial z^2}$$

$$2(\Omega_x v - \Omega_y u) = -\frac{1}{\rho}\frac{\partial P}{\partial z}$$

Boundary cond: $u(0,t) = U(t)$ $v(0,t) = V(t)$

$u(\infty,t) = 0$ $v(\infty,t) = 0$

Reduces to previous eqns for $\underline{\Omega} = 0$

Real problem is to solve first 2 eqns for $u(z,t)$ and $v(z,t)$; then use 3 to find $P(z,t)$

Note: if $\underline{\Omega}$ is vertical $\Omega_x = \Omega_y = 0$, then $P(z,t) = f(t)$ only, and pressure is essentially hydrostatic.

Now consider the special case of a sinusoidal oscillation of the lower boundary

$\mathcal{U}(t) = \mathcal{U}_0 \cos \omega t$

$V(t) = 0$

As before, easier to work with $e^{i\omega t}$

Write $u(z,t) = \operatorname{Re} \tilde{u}(z,t)$

$v(z,t) = \operatorname{Re} \tilde{v}(z,t)$

B.c. are: $\tilde{u}(0,t) = \mathcal{U}_0 e^{i\omega t}$ $\quad \tilde{v}(0,t) = 0$

$\tilde{u}(\infty,t) = \tilde{v}(\infty,t) = 0$

$$\left.\begin{aligned} \frac{\partial \tilde{u}}{\partial t} - 2\Omega_z \tilde{v} &= \nu \frac{\partial^2 \tilde{u}}{\partial z^2} \\ \frac{\partial \tilde{v}}{\partial t} + 2\Omega_z \tilde{u} &= \nu \frac{\partial^2 \tilde{v}}{\partial z^2} \end{aligned}\right\Vert$$

coupled eqns.

A useful trick in all Coriolis problems is the following

Define: $\tilde{u}_+ = \tilde{u} + i\tilde{v}$

$\tilde{u}_- = \tilde{u} - i\tilde{v}$

Use these as dependent variables

Note: $\tilde{u} \neq \operatorname{Re} \tilde{u}_+$

$$\partial \tilde{u}_+ / \partial t + 2i\Omega_z \tilde{u}_+ = \nu \, \partial^2 \tilde{u}_+ / \partial z^2$$

$$\partial \tilde{u}_- / \partial t - 2i\Omega_z \tilde{u}_- = \nu \, \partial^2 \tilde{u}_- / \partial z^2$$

Bdry cond. $\tilde{u}_+(0,t) = \tilde{u}_-(0,t) = U_0 e^{i\omega t}$

$\tilde{u}_+(\infty,t) = \tilde{u}_-(\infty,t) = 0$

The trick leads to <u>uncoupled equations</u>.

Let us look only for the <u>steady</u> solution after the decay of any initial transients associated with the onset of sinusoidal oscillation.

$\tilde{u}_\pm(z,t) = \tilde{\tilde{u}}_\pm(z) \, e^{i\omega t}$ <u>steady</u> solution

This leads to $$d^2\tilde{\tilde{u}}_+ / dz^2 = i\left(\frac{\omega + 2\Omega_z}{\nu}\right)\tilde{\tilde{u}}_+$$

$$d^2\tilde{\tilde{u}}_- / dz^2 = i\left(\frac{\omega - 2\Omega_z}{\nu}\right)\tilde{\tilde{u}}_-$$

Similar to previous solution

$$\tilde{\tilde{u}}_+(z) = Ae^{n_+ z} + Be^{-n_+ z} \quad \text{where}$$
$$n_+^2 = i(\omega + 2\Omega_z)/\nu$$

Now $$n_+^2 = i \operatorname{sgn}(\omega + 2\Omega_z)\left|\frac{\omega + 2\Omega_z}{\nu}\right|$$
Take positive square root.

$$n_+ = \left|\frac{\omega + 2\Omega_z}{\nu}\right|^{1/2} e^{i\frac{\pi}{4}\operatorname{sgn}(\omega + 2\Omega_z)}$$

$$n_+ = \left|\frac{\omega + 2\Omega_z}{2\nu}\right|^{1/2}\left[1 + i\operatorname{sgn}(\omega + 2\Omega_z)\right]$$

To satisfy b.c. must take $A = 0$
$B = U_0$

Hence
$$\tilde{u}_+(z,t) = U_0 e^{i\omega t}\exp\left[-z\left|\frac{\omega + 2\Omega_z}{2\nu}\right|^{1/2}\left(1 + i\operatorname{sgn}(\omega + 2\Omega_z)\right)\right]$$

For $\tilde{\tilde{u}}_-(z)$, $\tilde{u}_-(z,t)$, same analysis

Get

$$\tilde{u}_\pm(z,t) = \mathcal{U}_0 e^{i\omega t} \exp\left[-z\left|\frac{\omega \pm 2\Omega_z}{2\nu}\right|^{1/2}\left(1+i\,\mathrm{sgn}\,(\omega \pm 2\Omega_z)\right)\right]$$

$$u(z,t) = \mathrm{Re}\ \tfrac{1}{2}\left[\tilde{u}_+(z,t) + \tilde{u}_-(z,t)\right]$$

$$v(z,t) = \mathrm{Re}\ \frac{1}{2i}\left[\tilde{u}_+(z,t) - \tilde{u}_-(z,t)\right]$$

This is the complete solution

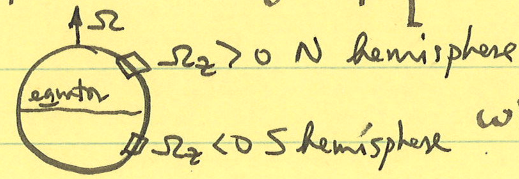
Let us consider in particular the case $\omega = 0$

A steady horizontal pull of the bottom bdry at a velocity $\mathcal{U}_0 \hat{x}$.

Recall that with <u>no rotation</u>, there was no steady soln to this problem.

In this case

$$\tilde{u}_\pm(z,t) = \mathcal{U}_0 \exp\left[-z\left|\frac{\Omega_z}{\nu}\right|^{1/2}(1 \pm i\sigma)\right]$$



where $\sigma \equiv \mathrm{sgn}\,\Omega_z$

Get

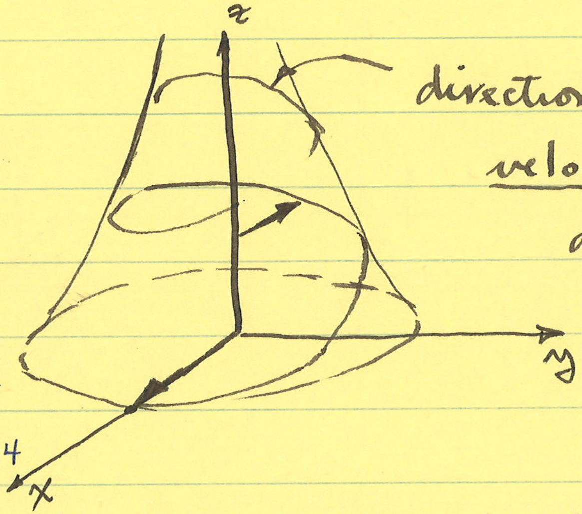
$$u(z,t) = \mathcal{U}_0 e^{-z/\delta} \cos(z/\delta)$$

$$v(z,t) = \sigma\,\mathcal{U}_0 e^{-z/\delta} \sin(z/\delta)$$

$$\delta \equiv \sqrt{\frac{\nu}{|\Omega_z|}}$$

With rotation, there is a steady solution to the problem of pulling a plate.

Note the spiral character of the solution.



direction of fluid velocity vector a spiral

oppositely directed at $z = \pi\delta$, diminished by $e^{-\pi} \sim \frac{1}{23} \sim .04$

Ekman (1902), (1910) } two papers

Thickness of this Ekman spiral boundary layer $\delta = \sqrt{\nu / |\Omega_z|}$

If $\Omega = 2\pi/24$ hrs, and $\nu \approx 10^{-2}$ (water)

then $\delta \approx 11$ cm, so $\pi\delta \sim 35$ cm

This steady flow exists because of a balance between viscous and Coriolis forces. The spiral allows the viscous force to be always ⊥ to the local fluid velocity, so that it can balance the Coriolis force.

This solution is of geophysical interest

Suppose we have a free stream velocity $U_0\hat{x}$ above a layer. How does that velocity → 0 at a rigid bottom?

Answer: the above solution is in the fluid frame of reference. We must simply add the constant velocity. Thus

$$u(z,t) = U_0\left[1 - e^{-z/\delta}\cos(z/\delta)\right]$$

$$v(z,t) = -\,\underset{\substack{\uparrow \\ \mathrm{sgn}\,\Omega_z}}{\sigma}\, U_0\, e^{-z/\delta}\sin(z/\delta)\,, \qquad \delta = \sqrt{\frac{\nu}{|\Omega_z|}}$$

This the way a free stream velocity in the atmosphere must → 0 at the surface. For air $\nu \sim 0.15\ \mathrm{cm}^2/\mathrm{s}$ at 20°C. and $\pi\delta \sim 150$ cm.

Actually eddy diffusion dominates and $\nu_{eddy} \gg \nu$. Variations of magnitude and direction of flow above ground used to estimate ν_{eddy}. Find $\pi\delta$ actually $\sim 500 - 1000$ m.

Ekman type boundary layers exist at the atmosphere - ocean bdry, atmosphere - land bdry, and ocean-bottom bdry. In real life they are generally turbulent.

Boundary layers in general

We have seen that in several problems where we were able to obtain exact solutions, the form of the solution involved a relatively narrow boundary layer near a moving rigid surface.

There is a very simple reason why we expect this type of situation to prevail even in more geometrically complicated situations.

Recall the dimensionless form of the Navier - Stokes eqns.

$L \equiv$ repr. length of bdry
$\mathcal{U} \equiv$ repr. velocity of bdry

$\underline{u}' = \underline{u} / \mathcal{U}$
$\underline{r}' = \underline{r} / L$
$t' = t\mathcal{U} / L$
$P' = P / \rho\mathcal{U}^2$

Then
$$\nabla' \cdot \underline{u}' = 0$$
$$\partial_{t'} \underline{u}' + \underline{u}' \cdot \underline{\underline{\nabla' \underline{u}'}} = -\nabla' P' + \frac{1}{R} \nabla' \nabla' \underline{u}$$

Where $R = \dfrac{\rho L \mathcal{U}}{\eta} = \dfrac{L\mathcal{U}}{\nu}$ Reynolds number

For most flow systems of geophysical or laboratory interest, the Reynolds number R is quite large $R \gg 1$

Air at 20°C: $R = 10^3$ if $\mathcal{U}L = 150$ cm^2/sec
Water: $R = 10^3$ if $\mathcal{U}L = 10$ cm^2/sec

One might be tempted to proceed with the study of fluids by simply using the approximation $R \to \infty$.

$$\nabla' \cdot \underline{u}' = 0$$

$$\partial_{t'} \underline{u}' + \underline{u}' \cdot \underline{\underline{\nabla' u'}} = - \nabla' P' + O\left(\frac{1}{R}\right)$$

These are the eqns which govern the flow of an <u>incompressible perfect fluid</u>

One might think that the large Reynolds number flow of a viscous fluid can be determined as just a slightly perturbed flow of an inviscid fluid. This is <u>not</u> the case except in rather special circumstances.

The reason is that this is the classical example of what is called a <u>singular perturbation</u>

A real fluid satisfies the <u>no-slip b.c.</u> at a <u>rigid bdry</u>.
Note that the $O(\frac{1}{R})$ term which one would like to neglect is the term with the highest order of differentiation.

<u>If</u> one tries to first solve the problem by simply setting $1/R \to 0$, the

resulting zeroeth order problem is overdetermined

The natural b.c. for an inviscid fluid allow tangential slip at a rigid bdry.

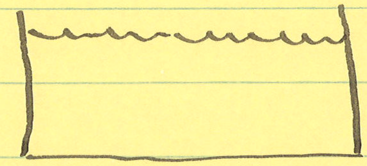
This is a typical singular perturbation problem The distinctive criterion is that the term which one would like to neglect has the highest order of differentiation

Small viscosity is not a simple small perturbation, it is a singular small perturbation. It is this fact that makes the flow of viscous fluids so much different from the flow of non-viscous fluids.

A technique has been developed which allows one to deal successfully with singular perturbations, at least in certain simple cases.

This technique is called the method of matched expansions. It works, in fluid mechanics, in all situations where boundary layer separation does not occur. i.e. whenever the flow does really consist of an inviscid interior flow + a thin viscous bdry layer at rigid bdries.

Example: a wave tank

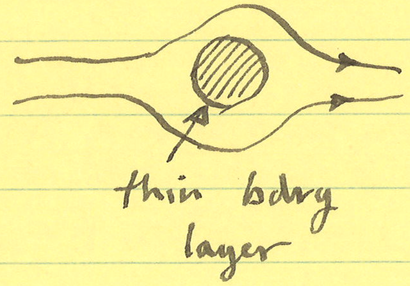


Another example: the Slichter mode.

One cannot determine by any kind of analysis when a bdry layer soln will be valid.

However when it is valid, the method of matched asymptotic expansions allows one to determine the flow: how to match the inviscid interior flow into the bdry layer flow

Example: flow past a sphere



For $R \lesssim 35$, bdry layer soln works well

For $R \gtrsim 35$, bdry layer separation is observed to occur. Flow changes character completely

An example of singular perturbation theory to show the mathematical origin of boundary layers.

We consider a simple ordinary differential eqn which however has the distinctive characteristics of a singular perturbation problem.

Given: $\epsilon > 0$, but $\varepsilon \ll 1$

Find $y_\varepsilon(x)$ on $0 \le x \le 1$, from the information

1. $y(0) = y(1) = 0$
2. y twice cont. diff. on $0 \le x \le 1$ and

$$\underset{\nearrow}{\epsilon} \frac{d^2y}{dx^2} + \frac{dy}{dx} = 1$$

highest order of diff.

This is a nice example since it has an exact solution, namely

$$y_\varepsilon(x) = x - \frac{1 - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}}$$

This easily demonstrated by substitution

Note: in most of $0 \le x \le 1$, $x \gg \varepsilon$ since $\varepsilon \ll 1$; $x/\varepsilon \gg 1$ and $1/\varepsilon \gg 1$

so

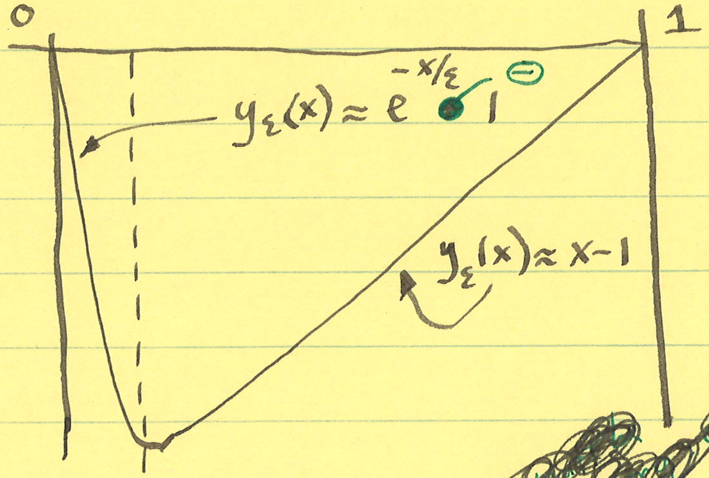
$$e^{-x/\varepsilon} \ll 1 \quad \text{and} \quad e^{-1/\varepsilon} \ll 1 \quad \text{except when}$$
$$x \approx \text{few times } \varepsilon$$

Thus

$$(\ast) \quad y_\varepsilon(x) \approx x-1 \quad \text{except for} \quad 0 \le x \le N\varepsilon$$

N is some number which determines the error in $(\ast)$. If $N = 10$, the error is $O(e^{-N}) \approx 4.5 \cdot 10^{-5}$.

When $0 \le x < N\varepsilon$, the exp. terms become important and bring $y_\varepsilon(x)$ smoothly to 0 at $x = 0$.



min at $x \approx \varepsilon \ln \frac{1}{\varepsilon}$

value of $y_\varepsilon(x)$ at $x_{\min} \approx -1 + \varepsilon \ln \frac{1}{\varepsilon}$

Now let us develop a method to approximate this exact solution. The method we develop may be generalized to more complicated problems for which no exact soln is known.

We assume the existence of bdry layers near $x = 0$ and $x = 1$ (we do not yet know there will not be one near $x=1$)

Outside these bdry layers (in the interior) we assume that the scale of variation in $y_\varepsilon(x)$ is $\approx L$, in this case 1.

$$\underset{\substack{\uparrow \\ \sim \varepsilon y / L^2 \\ = \varepsilon y}}{\epsilon \frac{d^2 y}{dx^2}} + \underset{\substack{\uparrow \\ \sim y / L \\ = y}}{\frac{dy}{dx}} = 1 \qquad \star$$

In the interior soln we can neglect the first term and we have

$$\frac{dy}{dx} \simeq 1 \qquad y_\varepsilon(x) \approx x + A,$$

A an arbitrary const.

Now we examine solution <u>inside</u> bdry layers. Let δ be their as yet unknown thickness. The bdry layers will be

$$0 \le x \le N\delta$$
$$1 - N\delta \le x \le 1$$

where N determines the error in our approximation.

We expect that the scale length of variations in $y_\varepsilon(x)$ in the bdry layer $\approx \delta$. We introduce a new bdry layer variable

$$x = \xi\delta \qquad \text{if} \quad x \text{ near } 0$$
$$x = 1 + \xi\delta \qquad \text{if} \quad x \text{ near } 1$$

Note: ξ means different things in the two bdry layers; we are treating them separately. Substitute into $\star$

$$\frac{\varepsilon}{\delta^2}\frac{d^2y}{d\xi^2} + \frac{1}{\delta}\frac{dy}{d\xi} = 1$$

$$\frac{d^2y}{d\xi^2} + \frac{\delta}{\varepsilon}\frac{dy}{d\xi} = \frac{\delta^2}{\varepsilon}$$

Now <u>assume</u> that $\delta = \varepsilon$ (we will come back to this later)

Then when $\varepsilon \ll 1$, $\star$ becomes

$$\frac{d^2y}{d\xi^2} + \frac{dy}{d\xi} \approx 0$$

This has solution $y(\xi) = B + Ce^{-\xi}$ $\star\star$

Now at $x = 1$, ξ becomes negative as we enter the fluid so $\star\star$ blows up unless $C = 0$
But at $\xi = 0$, $x = 1$, and $y = 0$, so $B = 0$. The bdry layer soln at $x = 1$ is $y = 0$
We know that

$y_\varepsilon(x) = x + A$ must satisfy $y(1) = 0$

Thus $A = -1$. This determines the constant in the interior solution

Hence $y_\varepsilon(x) \approx x - 1$ unless $0 \leq x \leq N\varepsilon$
no bdry layer near $x = 1$.

At $x = 0$, ξ becomes positive as we enter the fluid. Thus $y(\xi) \to B$ as $\xi \to \infty$

But for $\varepsilon \ll 1$, ξ can be very large while $x = \epsilon\xi$ is still small. Thus $**$ will $\to B$ at a value of x so small that $y_\xi(x) \approx -1$ in the interior soln. Hence we must choose $B = -1$ to match. Since $y(\xi)$ in $**$ must vanish at $\xi = 0$, we must take $C = 1$

Thus the bdry layer solution near $x = 0$ is

$$y_\xi(x) = -1 + e^{-x/\varepsilon} \qquad \text{for} \quad 0 \leq x \leq \varepsilon$$

bdry layer: $\xi = x/\delta$
interior $\xi = x/L \equiv x/1 \equiv x$.

The essence of the method involves a different scaling in the interior and in the bdry layer, and neglect of different terms in the eqns. Then no-slip b.c. + matching conds. are used to determine arbitrary consts, which arise in integration.

Question: why choose $\delta = \varepsilon$?

Answer:

$$\frac{d^2y}{d\xi^2} + \frac{\delta}{\varepsilon}\frac{dy}{d\xi} = \frac{\delta^2}{\varepsilon}$$

We seek an approx. that makes one term negligible; otherwise we might as well seek the exact soln.

If we make 2 terms negl. or if we make any term other than δ^2/ε negl., then we cannot find a bdry layer soln

If we make the two terms on the left comparable and δ^2/ε negl. as $\varepsilon \to 0$ then δ must = some const $\cdot\ \varepsilon$

No matter what const one chooses one gets the same bdry layer soln

$$y_\varepsilon(x) = -1 + e^{-x/\varepsilon}$$

Scale of bdry layer thickness automatically determined by bdry layer soln

In viscous flow one has typically $\delta/L \sim R^{-1/2}$ as $R \to \infty$. See the Rayleigh problem as an example.

$\delta = \sqrt{\frac{\nu T}{\pi}}$ for oscillation of plate with period T, amplitude $\mathcal{U}$

In this case $L = \mathcal{U}T$, $\delta = (\pi \mathcal{U}/\nu L)^{-1/2}$

or $\delta/L = (\pi \mathcal{U} L/\nu)^{-1/2} \sim R^{-1/2}$.

Final lecture on convection: Boussinesq approx.

Recall, first, the general equations for a Newtonian viscous fluid.

$$D_t \rho + \rho \nabla \cdot \underline{u} = 0 \qquad \text{continuity eqn.}$$

$$\rho D_t \underline{u} + \nabla p = \nabla \cdot \underline{\underline{V}} + \rho \underline{g} \qquad \text{momentum eqn.}$$

in grav. field

$$\rho D_t \mathcal{U} + p \nabla \cdot \underline{u} + \nabla \cdot \underline{H} = \underline{\underline{V}} : \underline{\underline{\varepsilon}} + h$$

energy eqn.

Choose $p \equiv p_{eq}$ and θ as the 2 independent thermodynamic variables. Then need 2 eqns of state

$$\rho = \rho(p, \theta)$$
$$\mathcal{U} = \mathcal{U}(p, \theta)$$

Alex
sound
attenuation

Also 2 const. relations:

$$\underline{H} = -\kappa \underline{\nabla \theta}, \qquad \kappa = \kappa(p, \theta) \qquad \text{Fourier's "law"}$$

$$\underline{\underline{V}} = (\gamma - \tfrac{2}{3}\eta)(\nabla \cdot \underline{u})\,\underline{\underline{I}} + 2\eta\,\underline{\underline{\varepsilon}}, \qquad \text{viscous stress}$$

$$\eta = \eta(p, \theta), \quad \gamma = \gamma(p, \theta)$$

As we've seen 16 eqns for 16 unknowns:

$$\rho,\ \underset{\substack{\uparrow\\3}}{\underline{u}},\ p,\ \theta,\ \mathcal{U},\ \underset{\substack{\uparrow\\3}}{\underline{H}},\ \underset{\substack{\uparrow\\6}}{\underline{\underline{V}}}$$

$$\left.\begin{aligned} \rho &= \rho(p,\theta)\\ \mathcal{U} &= \mathcal{U}(p,\theta)\\ \kappa &= \kappa(p,\theta)\\ \eta &= \eta(p,\theta)\\ \gamma &= \gamma(p,\theta) \end{aligned}\right\}\ \text{characterize the particular fluid.}$$

Approximations are necessary if progress is to be made.

To be specific, lets consider convection in a <u>planar</u> geometry, with g, the gravity vector constant. This is not, however, a <u>necessary</u> restriction.

~~Let $\rho_0 \equiv$ constant be a "typical" density, e.g. the density at some point (perhaps the bottom) in the fluid.~~

The approximation usually employed in convection problems is known collectively as the <u>Boussinesq approximation</u>. It has several ingredients. It is, sort of, the extension to convective flows, of the incompressibility approximation we have employed hitherto.

1. The flow is considered to be <u>incompressible</u> in the sence $D_t \rho = 0$. Thus the cont. eqn. becomes $\boxed{\nabla \cdot \underline{u} = 0}$

2. Now let $\rho_0 =$ const. be a "typical" density, say the density at the bottom of the convecting region. Since density variations are responsible for free convection, we must take them into account. Write

$$\rho = \rho_0 + \Delta\rho$$

"modified" pressure

and introduce $\underset{\swarrow}{P} \equiv p - \rho_0 \underline{g} \cdot \underline{r}$

Momentum equation becomes

$$(\rho_0 + \Delta\rho) D_t \underline{u} + \nabla P = \nabla \cdot \underline{\underline{V}} + \Delta\rho \, \underline{g}$$

$$\left(1 + \frac{\Delta\rho}{\rho_0}\right) D_t \underline{u} + \frac{1}{\rho_0} \nabla P = \frac{1}{\rho_0} \nabla \cdot \underline{\underline{V}} + \frac{\Delta\rho}{\rho_0} \underline{g}$$

Density fluctuations due to temperature variations are fairly small since $\alpha \sim 10^{-3}$ to 10^{-4} : coefficent of thermal expansion, so $\Delta\rho/\rho_0 \sim$ at most 1%, so we neglect this

Cannot, however, be neglected in the buoyancy term $\Delta\rho\, \underline{g}$, only in the inertia term $\rho D_t \underline{u}$.

Second part of Boussinesq approx. thus to neglect density variations except in buoyancy term (since this is what drives convection).

3. Let θ_0 be the constant reference temperature at the point where the density is ρ_0. Next, we ignore density variations due to pressure fluctuations, and linearize the dependence on temperature. Writing

$$\theta = \theta_0 + \Delta\theta ,$$

$$\Delta\rho = -\rho_0 \underset{\uparrow}{\alpha} \Delta\theta \longleftarrow$$

this the Boussinesq form of $\rho = \rho(p, \theta)$

const. coeff. of thermal expansion (volumetric)

4. Next we ignore variations of η (temp. dependence not taken into account). Note γ does not enter because $\nabla \cdot \underline{u} = 0$.

We've seen that when $\eta =$ const., $\nabla \cdot \underline{\underline{V}}$ reduces to $\eta \nabla^2 \underline{u}$.

With all these approximations the <u>momentum</u> equation becomes

$$\boxed{\frac{D}{t}\underline{u} = -\frac{1}{\rho_0}\nabla P + \nu \nabla^2 \underline{u} - \underline{g}\,\alpha\,\Delta\theta}$$

where $\nu = \mu/\rho_0$, the kinematic viscosity.

5. Now, for the energy eqn., we shall not consider internal heating, but keep it for now.

We write $U = c_v\,\theta$ — internal energy depends only on temp.

c_v: specific heat at const. volume (heat required to raise 1g by 1°)

better: $U = U_0 + c_v\,\Delta\theta$
where U_0 at same reference point

This the form of $U = U(p, \theta)$.

~~[illegible] of Boussinesq, replace ρ by the constant ρ_0.~~

use ρ_0 here

Then $\rho D_t \mathcal{U}$ becomes $\underset{\substack{\uparrow \\ \text{ignore dependence of this} \\ \text{i.e. take to be const.}}}{\overset{\downarrow}{\rho_0}\, c_v}\, D_t \overset{\swarrow \text{ replace by } \Delta\theta}{\theta}$

Further, ignore dependence of κ on anything.

Then $\nabla \cdot \underline{H} = -\kappa \nabla\theta$ ← replace by $\Delta\theta$

6. Finally, last part of Boussinesq approx. is to ignore viscous heating $\underline{\underline{V}} : \underline{\underline{\varepsilon}}$.

With these provisos, the energy equation becomes

$$\rho_0 c_v D_t \theta = \kappa \nabla^2 \theta + h \qquad \text{or}$$

write in terms of $\Delta\theta$

$$\boxed{D_t \theta = \Big(\underset{\substack{\uparrow \\ \text{thermal diffusivity, call this } k}}{\frac{\kappa}{\rho_0 c_v}}\Big) \nabla^2 \theta + h / \rho_0 c_v}$$

$[k] = \text{cm}^2/\text{sec}$

better (for later) to call this $\mathcal{K}$

Note that since θ appears differentiated everywhere in this eqn, can replace by $\Delta\theta$.

If we wish to consider convection due to internal heating, must keep h.

Let's, however, consider instead heating from below, so that $h \equiv 0$.

Summary: the Boussinesq equations are

$$\boxed{\begin{array}{ll} \nabla\cdot\underline{u} = 0 & \text{continuity} \\ D_t\underline{u} = -\nabla(P/\rho_0) + \nu\nabla^2\underline{u} - \underline{g}\,\alpha\,\Delta\theta & \text{momentum} \\ D_t\,\Delta\theta = \underset{\uparrow K}{k}\,\nabla^2\Delta\theta & \text{energy} \end{array}}$$

The unknowns are now $\underline{u}$, P/ρ_0, $\Delta\theta$: 5 eqns for 5 unknowns. Gravity $\underline{g} = -g\hat{z}$ is known and a particular fluid is now characterized by 3 constants ν, $\overset{K}{k}$ and α.

Note: Boussinesq approximation is not a linearization: $D_t = \underline{u}\cdot\nabla\underline{u} + \partial_t\underline{u}$ and, more important, $D_t\,\Delta\theta = \partial_t\,\Delta\theta + \underbrace{\underline{u}\cdot\nabla\Delta\theta}_{\uparrow}$

this term, advection of temperature, is fundamental to the convection process.

3 material constants K, ν, α

There is also a form of the Boussinesq approximation applicable to compressible fluids; this required in geophysical and astrophysical problems (since in mantle ρ varies by factor of 2 from top to bottom due to pressure, not temperature). Note: we have assumed all deviations away from $\rho_0 =$ const. are due to thermal expansion.

For more general case, see Spiegel and Veronis, Astrophys. J. 131, 442–7 (1960) and Jarvis and McKenzie, J. Fluid Mech., 96, 515 (1980).

Generally speaking, only change is that $\Delta\theta$ is replaced by the potential temperature $= \theta - \theta_{\text{adiabatic}}$, but circumstances under which this approx. is adequate require careful justification.

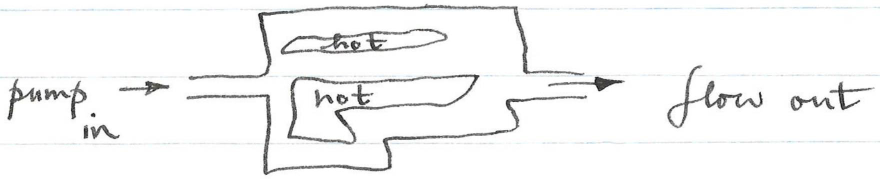
Aside: forced, as opposed to free convection: the flow $\underline{u}$ is driven by forces other than buoyancy and the buoyancy term can be neglected. Then solve

$$\left.\begin{aligned} &\nabla\cdot\underline{u} = 0 \\ &D_t\underline{u} = -\nabla(P/\rho_0) + \nu\nabla^2\underline{u} \end{aligned}\right\} \text{ for } \underline{u}, P$$

This is just NS eqns. Then once $\underline{u}$ is known, solve

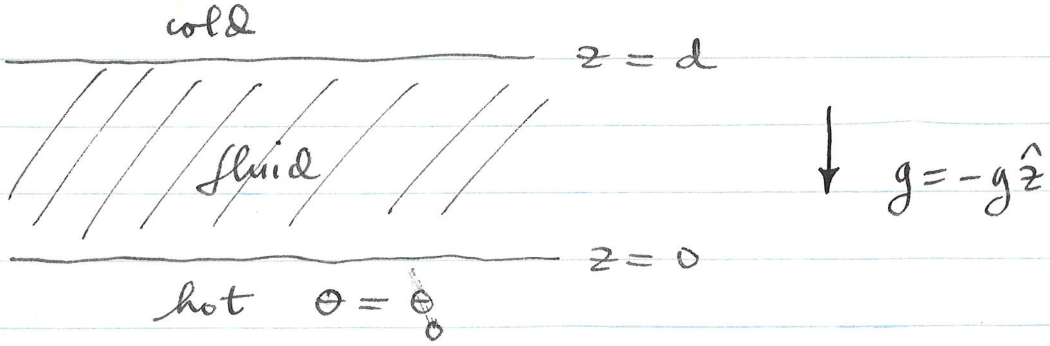
$$(\partial_t + \underline{u}\cdot\underline{\nabla})\,\Delta\theta = k\,\nabla^2\Delta\theta \quad \text{for temp. fluctuations.}$$

Example : convective cooling of, say, a nuclear power plant, other engineering applications.



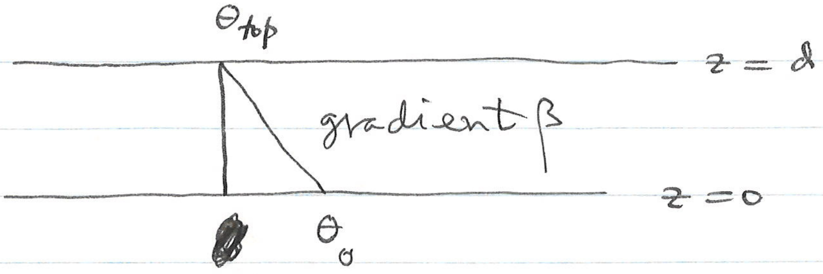
Linearized Rayleigh – Benard convection :

Let us now consider the free convection of a body of fluid *heated from below*.



Consider first fluid *at rest* in this situation.

The steady state temp. distribution is governed by $\nabla^2\theta = 0$. A solution is



$$\boxed{\begin{array}{lll} \theta = \theta_0 - \beta z & & \text{linear profile} \\ & \beta = \dfrac{\theta_0 - \theta_{top}}{d} & \text{temp. gradient} \\ & & \Delta\theta = \beta z \end{array}}$$

Corresponding steady state density profile is

lighter (since hotter) at bottom

$$\boxed{\rho = \rho_0 (1 + \alpha\beta z) \text{ , also linear}}$$

The corresponding pressure in the rest state satisfies

$$\begin{aligned} \partial p/\partial z &= -\rho g \\ &= -\rho_0 g \, (1 + \alpha\beta z) \end{aligned}$$

[redacted]

$$p = p_0 - \rho_0 g z - \tfrac{1}{2} \rho_0 g \, \alpha\beta z^2$$

↗ pressure at bottom

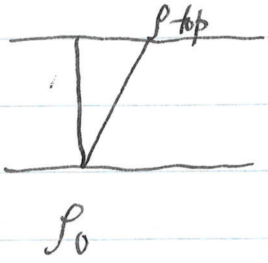
modified pressure

$\hookrightarrow P = p - \frac{1}{2} \rho_0 g \, \alpha\beta z^2$

We shall only consider <u>small</u> <u>perturbations</u> away from this initial state.

<u>Linearized</u> <u>convection</u>. An analysis of the <u>stability</u> of this initial state characterized by a temp gradient.

Note density profile looks like:



heavy fluid over light

<u>Question</u>: how large an adverse temp gradient can be maintained without overturning or convection occurring?

Let $\delta\Theta$ and δP be the additional perturbations in temp. and pressure associated with the motion. Then

$\underline{u}$ = perturbation velocity

$\Delta\Theta = -\beta z + \delta\Theta$

$P = - \frac{1}{2} \rho_0 g \alpha \beta z^2 + \delta P$

Substitute in and neglect <u>all</u> second-order terms

$\nabla\cdot\underline{u} = 0$ linear already

$$\partial_t\underline{u} + \overset{\text{neglect}}{\cancel{\underline{u}\cdot\nabla\underline{u}}} = -\frac{1}{\rho_0}\nabla\Big(\overset{\text{zero}}{\text{[illegible]}} \cancel{+ \frac{1}{2}\rho_0 g_0\alpha\beta z^2}\Big)$$

$$\boxed{\text{cancel}}$$

$$-\frac{1}{\rho_0}\nabla\delta P + \nu\nabla^2\underline{u} - \cancel{g\alpha\beta\underline{z}} + g\alpha\,\delta\theta\,\hat{z}$$

or

$$\partial_t\underline{u} = -\frac{1}{\rho_0}\nabla\delta P + \nu\nabla^2\underline{u} + g\alpha\,\delta\theta\,\hat{z}$$

[redacted]

this advection term most important neglected quantity; limits analysis to marginally unstable convection

$$\overset{\text{zero}}{\cancel{\partial_t(-\beta z}} + \delta\theta) + \overset{\boxed{\text{neglect}}}{\overbrace{\underline{u}\cdot\nabla(-\beta z + \delta\theta)}}$$

$$= \underset{\overset{\nearrow}{K}}{k}\,\nabla^2(\overset{}{\underset{\text{zero}}{\cancel{-\beta z}}} + \delta\theta)$$

Let $\boxed{w \equiv \hat{z}\cdot\underline{u}\text{, upward velocity}}$

$$\partial_t\delta\theta \overset{-}{\cancel{+}} \beta w = \underset{\overset{\uparrow}{K}}{k}\,\nabla^2\delta\theta$$

<u>Summarizing</u>: linear convection equations

$$\nabla\cdot\underline{u} = 0$$

$$\partial_t \underline{u} = -\frac{1}{\rho_0}\nabla\delta P + \nu\nabla^2\underline{u} + g\alpha\,\delta\theta\,\hat{z}$$

$$\partial_t \delta\theta \overset{-}{\cancel{+}} \beta w = \underset{\underset{\underline{K}}{\uparrow}}{k}\,\nabla^2 \cancel{\theta}\,\delta\theta$$

5 eqns for 5 unknowns: $\underline{u}$, δP, $\delta\theta$

Convenient to eliminate δP from eqns by taking curl. Recall

$$\underline{\omega} \equiv \nabla\times\underline{u} \equiv \text{vorticity}$$

$$\nabla\times\partial_t\underline{u} = \nabla\times\left(-\frac{1}{\rho_0}\nabla\delta P + \nu\nabla^2\underline{u} + g\alpha\,\delta\theta\,\hat{z}\right)$$

curl (grad) = 0. Get

$$\partial_t\underline{\omega} = \nu\nabla^2\underline{\omega} - g\alpha\,\hat{z}\times\nabla\delta\theta$$

(∂_t and $\nabla\times$ commute and so do ∇^2 and $\nabla\times$)

$$\boxed{\text{Let } \xi \equiv \hat{z}\cdot\underline{\omega},\ \hat{z} \text{ comp. of vorticity}}$$

Then $\hat{z}$ comp. of this eqn. is

$$\partial_t\xi = \nu\nabla^2\xi \qquad (\text{since } \hat{z}\times\nabla\delta\theta \perp \hat{z})$$

Finally, take curl again.

$$\partial_t \omega_k = \nu \nabla^2 \omega_k - g\alpha\, \varepsilon_{klm}\, \delta_{l3}\, \partial_m \delta\theta$$

$$\text{curl}_k = \varepsilon_{ijk}\, \partial_j \,, \quad \text{so}$$

$$\partial_t (\varepsilon_{ijk}\, \partial_j \omega_k) = \nu \nabla^2 (\varepsilon_{ijk}\, \partial_j \omega_k)$$

$$- g\alpha\, \varepsilon_{ijk}\, \varepsilon_{klm}\, \delta_{l3}\, \partial_j \partial_m (\delta\theta)$$

Now $\varepsilon_{ijk}\, \partial_j \omega_k = \varepsilon_{ijk}\, \varepsilon_{klm}\, \partial_j \partial_l u_m$

$$= (\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl})\, \partial_j \partial_l u_m$$

$$= \partial_i (\underset{\uparrow}{\nabla \cdot \underline{u}}) - \nabla^2 u_i$$

zero by cont. eqn (this where we use it)

Likewise $\varepsilon_{ijk}\, \varepsilon_{k3m}\, \partial_j \partial_m (\delta\theta)$

$$= (\delta_{i3}\,\delta_{jm} - \delta_{im}\,\delta_{j3})\, \partial_j \partial_m (\delta\theta)$$

$$= \delta_{i3}\, \nabla^2 \delta\theta - \partial_i \partial_3\, \delta\theta$$

$$\partial_t \nabla^2 u_i = \nu \nabla^4 u_i + g\alpha\, (\delta_{i3}\, \nabla^2 \delta\theta - \partial_i \partial_3\, \delta\theta)$$

Now set $i=3$, i.e. take $\hat{z}$ comp. of this.

$$\partial_t \nabla^2 w = \nu \nabla^4 w + g\alpha \left(\frac{\partial^2 \delta\theta}{\partial x^2} + \frac{\partial^2 \delta\theta}{\partial y^2} \right)$$

<u>Summary</u>: we have 3 eqns for $\delta\theta$, $\zeta \equiv \hat{z} \cdot \underline{\omega}$ and $w \equiv \hat{z} \cdot u$.

$$\partial_t \zeta = \nu \nabla^2 \zeta$$

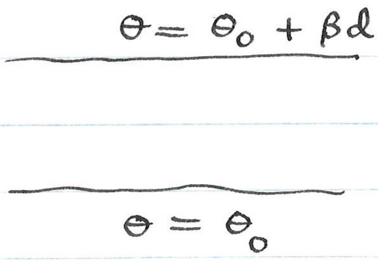
$$\partial_t \nabla^2 w = \nu \nabla^4 w + g\alpha \left(\frac{\partial^2 \delta\theta}{\partial x^2} + \frac{\partial^2 \delta\theta}{\partial y^2} \right)$$

$$\partial_t \delta\theta = \beta w + \underset{\substack{\uparrow \\ \text{better to call } \underline{K}}}{k} \nabla^2 \delta\theta$$

<u>Linear</u> system of 3 eqns, 3 unknowns governing marginally stable Rayleigh-Benard convection; 3 unknowns $\delta\theta$, $w \equiv \hat{z} \cdot \underline{u}$, $\zeta \equiv \hat{z} \cdot (\underline{\nabla \times u})$

<u>Boundary conditions</u>:

<u>Regardless</u> of the nature of the boundaries we must have



<u>uniform</u> top and bottom temps

Thus $\delta\theta = 0$ on $z = 0$, $z = d$

Also $w = 0$ on $z = 0$, $z = h$: no vertical flow (fluid is confined between the two planes)

For further b.c. necessary to distinguish 2 types of bdries.

1. <u>rigid</u> : <u>no slip</u> occurs ; $u = v = 0$

Since $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$

$\Rightarrow$ $dw/dz = 0$ on rigid surface.

Also $\xi = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0$ on rigid surface.

2. free surface : <u>no shear stress</u>

$V_{xz} = V_{yz} = 0$ so

$$\mu\left(\frac{\partial u}{\partial z} + \underset{\text{zero}}{\cancel{\frac{\partial w}{\partial x}}}\right) = 0 \quad \text{and} \quad \mu\left(\frac{\partial v}{\partial z} + \underset{\text{zero}}{\cancel{\frac{\partial w}{\partial y}}}\right) = 0$$

$du/\partial z = \partial v/\partial z = 0$

from $\dfrac{\partial}{\partial z}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) = 0$ get $\dfrac{\partial^2 w}{\partial z^2} = 0$

from $\xi = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ ~~$= 0$~~ get $\dfrac{\partial \xi}{\partial z} = 0$.

<u>Summary</u>: b.c. in terms of $\delta\theta$, ξ, w.

$$\left.\begin{aligned} \delta\theta &= 0 \\ w &= 0 \end{aligned}\right\} \text{ all boundaries}$$

$$\left.\begin{aligned} \xi &= 0 \\ \partial w/\partial z &= 0 \end{aligned}\right\} \text{ rigid boundary}$$

$$\left.\begin{aligned} \frac{\partial \xi}{\partial z} &= 0 \\ \frac{\partial^2 w}{\partial z^2} &= 0 \end{aligned}\right\} \text{ free boundary}$$

Look for <u>normal mode</u> solutions periodic in x, y and t:

solns: $\delta\theta, w, \xi \sim e^{i(k_x x + k_y y) + pt}$

Then
$$\partial_t \to p$$
$$\partial^2/\partial x^2 + \partial^2/\partial y^2 \to -k^2 \qquad (k^2 = k_x^2 + k_y^2)$$
$$\nabla^2 = d^2/dz^2 - k^2$$

This leads to the system.

$$p\,\varphi = \nu\left(\frac{d^2}{dz^2} - k^2\right)\varphi$$

$$p\left(\frac{d^2}{dz^2} - k^2\right)w = -\,g\alpha k^2\,\delta\theta + \nu\left(\frac{d^2}{dz^2} - k^2\right)^2 w$$

$$p\,\delta\theta = \beta w + \underset{\substack{\uparrow\\ \text{sorry about this; was calling this (thermal diffusivity) } k.}}{\underline{K}}\left(\frac{d^2}{dz^2} - k^2\right)\delta\theta$$

It can be shown, we shall not, that p <u>must be real</u>, i.e. solutions are either exponentially decaying or growing for any given horizontal wavenumber k. If $p < 0$, the initial steady state is <u>stable</u>, any small perturbations w, $\delta\theta$, φ will <u>decay</u>. If $p > 0$, they will instead grow → instability.

The condition of <u>marginal stability</u> which characterizes the <u>onset</u> of convection thus corresponds to $p \equiv 0$ (this called the principle of exchange of stabilities).

The marginal conditions thus satisfy

$$\left(\frac{d^2}{dz^2} - k^2\right)\zeta = 0$$

$$\zeta = Ae^{kz} + Be^{-kz}$$

this makes physical sense: no vertical vortices.

There is no non-zero ζ of this form satisfying b.c. $\zeta = 0$ or $d\zeta/dz = 0$ on $z = 0, d$.

Thus in the marginal state $\zeta = 0$,
Consider the 2 remaining eqns.

$$\boxed{\begin{aligned} \left(\frac{d^2}{dz^2} - k^2\right)^2 w &= \frac{g\alpha k^2}{\nu}\,\delta\theta \\ \left(\frac{d^2}{dz^2} - k^2\right)\delta\theta &= -\frac{\beta}{K}\,w \end{aligned}}$$

or, upon eliminating the temp. variation $\delta\theta$

$$\boxed{{}^{*}\quad \left(\frac{d^2}{dz^2} - k^2\right)^3 w = -\frac{g\alpha\beta k^2}{K\nu}\,w}$$

We must solve this <u>sixth order</u> ODE subject to the <u>boundary conditions</u>:

<u>rigid</u>:

$$w = 0$$
$$dw/dz = 0$$
$$(d^2/dz^2 - k^2)^2 w = 0 \qquad (\text{from } \delta\theta = 0)$$

<u>free</u>:

$$w = 0$$
$$d^2w/dz^2 = 0$$
$$(d^2/dz^2 - k^2)^2 w = 0$$

six b.c.: three at each boundary.
Cannot expect to <u>always</u> find a solution (Sturm–Liouville)

Easiest to set up in lab: rigid – rigid

——— rigid
——— rigid

Or, for a pan on the stove (or the mantle)

——— free
——— rigid

Easiest to solve, however, is rather artificial

——— free
——— free

↑ could possibly be floating on a heavier liquid

Consider free-free b.c. Then

$$\left.\begin{array}{r} w = 0 \\ d^2w/dz^2 = 0 \\ d^4w/dz^4 = 0 \end{array}\right\} \text{ at both bdries}$$

Then from eqn $\star$: $\dfrac{d^6w}{dz^6} = 0$

From $\star$ differentiated twice : $\dfrac{d^8w}{dz^8} = 0$, etc.

All even derivatives vanish at both boundaries. Then it follows the solution to $\star$ satisfying the b.c. will be of the simple form

$$\boxed{w = A \sin \frac{n\pi z}{d}} \qquad n = 1, 2, \ldots$$

Plugging into $\star$:

$$\left(\frac{n^2\pi^2}{d^2} + k^2\right)^3 = \frac{g\alpha\beta k^2}{\mathbb{K}\nu} \qquad \text{or}$$

$$\frac{g\alpha\beta d^4}{\mathbb{K}\nu} = \frac{(n^2\pi^2 + k^2d^2)^3}{k^2d^2}$$

The dimensionless product on the left is called the Rayleigh number, after Rayleigh who first analyzed this problem.

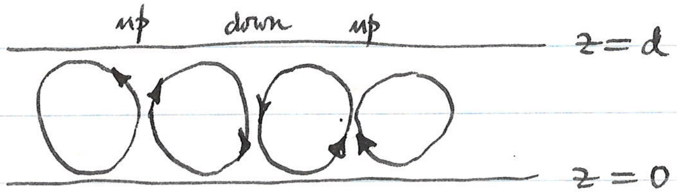
$$\boxed{Ra \equiv \frac{g\alpha\beta d^4}{\kappa\nu}}$$

Note it has the destabilizing influences in the numerator (β: the thermal gradient, g, α and d) and the stabilizing influences in the denominator.

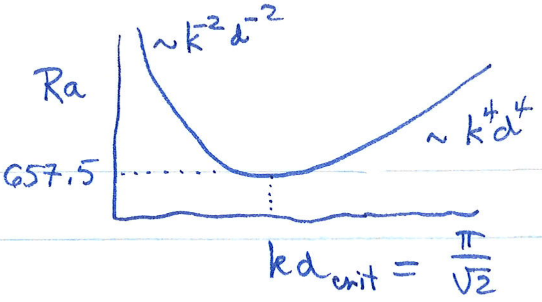
To find when instability will set in we must find the minimum value of Ra for a given k. The minimum for any fixed k is attained for $n=1$

$$\boxed{w = A \sin\frac{\pi z}{d}}$$

single layer or single cell convection is most unstable (makes sense)



$$Ra = \frac{(\pi^2 + k^2d^2)^3}{k^2d^2}$$

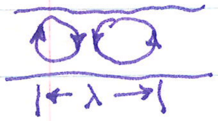


The critical Ra is determined by

$$\frac{d\,Ra}{d(kd)^2} = 0$$

$$\frac{3(\pi^2 + k^2d^2)^2}{k^2d^2} - \frac{(\pi^2 + k^2d^2)^3}{(k^2d^2)^2} = 0$$

$$k^2d^2 = \pi^2/2 \qquad kd = \frac{\pi}{\sqrt{2}}$$



$$\lambda = \frac{2\pi}{k} = 2\sqrt{2}\,d$$

$\frac{\lambda}{2}$ is size of a cell: aspect ration is $\sqrt{2}:1$ $= 1.4:1$

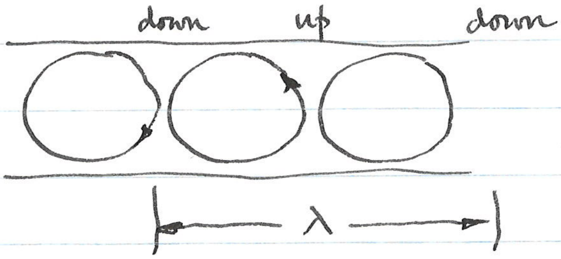
$$\boxed{\lambda_{crit} = 2\sqrt{2}\,d \sim 2.8\,d \quad \text{for free - free bdries}}$$

$$Ra_{crit} = \frac{(3/2\,\pi^2)^3}{\frac{1}{2}\pi^2}$$

$$\boxed{Ra_{crit} = \frac{27}{4}\pi^4 = 657.5 \quad \text{free - free}}$$

We thus have a prediction of exactly where and in what form instability will set in. If $Ra > Ra_{crit}$, the initial state will be unstable.

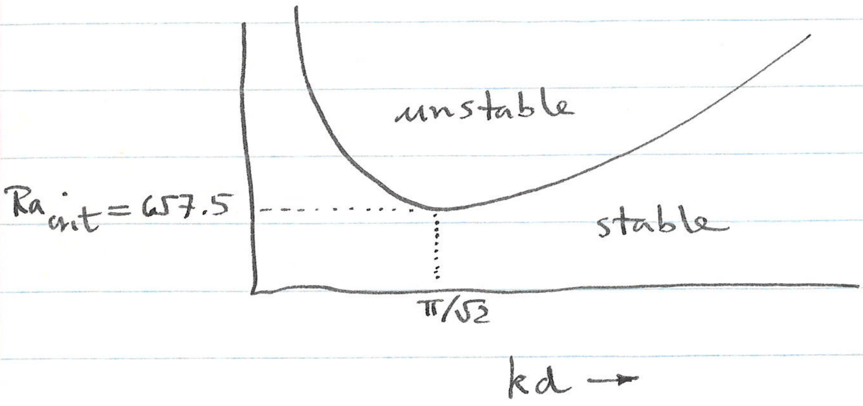
Most unstable mode has $\lambda_{crit} = 2\sqrt{2}\, d$



each cell is roughly 1:1 aspect ratio

(Actually, more like $\sqrt{2}:1$ for free-free, but see below)

If Ra is plotted vs kd :



The constraining effect of rigid bdries makes the flow more stable

More generally: Chandrasekhar, p. 43

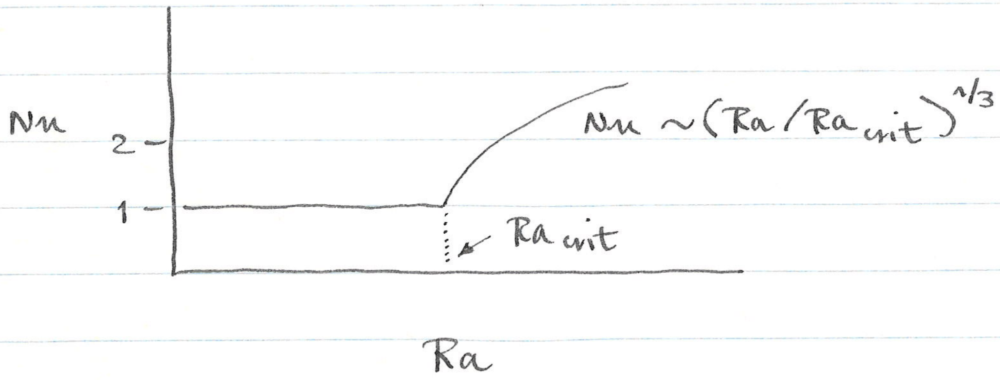
bdrys	Ra_{crit}	kd_{crit}	λ_{crit}/d	
free-free	657.5	2.22	2.83	(1.4:1)
rigid-rigid	1707.8	3.12	2.02	(1:1)
rigid-free	1100.7	2.68	2.34	(1.2:1)

swap order

Other cases more complicated to solve

Very careful experiments have been done to verify this theory, e.g. Silveston (1958) : measure temperatures and also heat flux ≡ total heat ($I^2 R$ energy) supplied to heater at base. Plot Nu vs Ra

$$Nu \equiv \frac{\text{heat transport}}{\text{heat conduction}}$$



For $Ra < Ra_{crit}$ or $\frac{g\alpha\beta d^4}{K\nu}$, heat conduction is adequate to remove heat supplied to bottom and viscosity ν keeps fluid from moving. For $Ra > Ra_{crit}$, too much heat is being supplied and conduction is insufficient to remove it so $Nu > 1$.

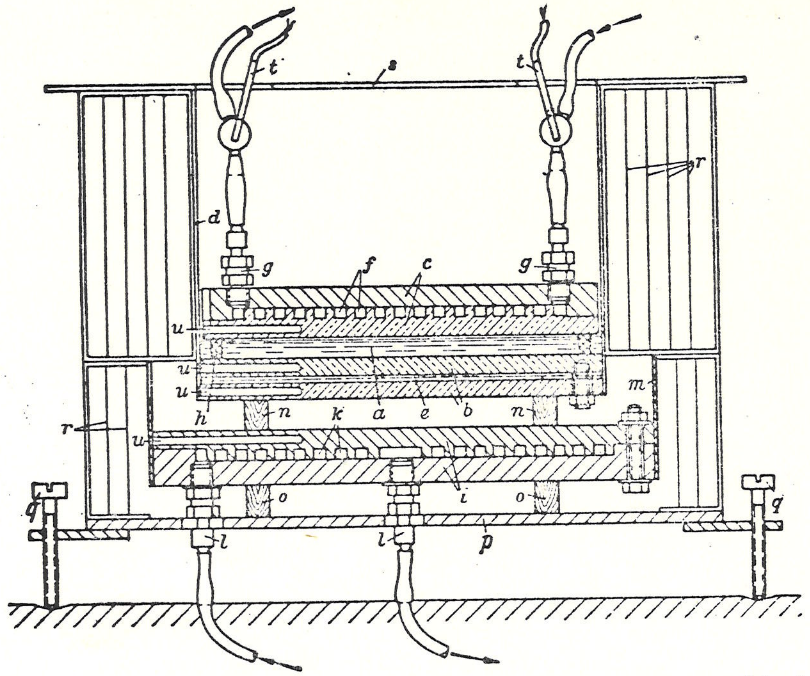


Fig. 12a. Silveston's experimental arrangement for measurements of heat transfer.

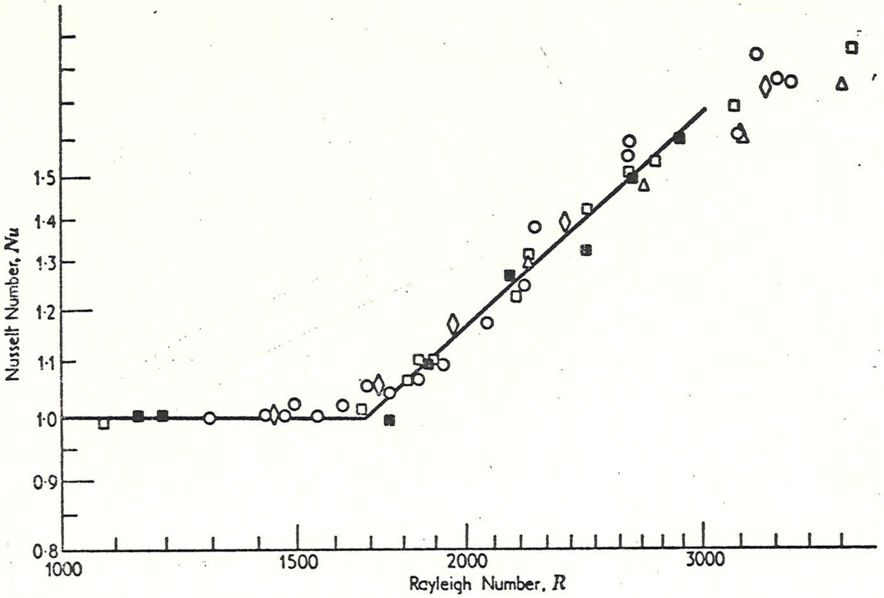


Fig. 14. Silveston's experimental results in the neighbourhood of instability in various liquids (■ silicone oil AK 350; ◇ silicone oil AK 3; □ ethylene glycol; △ heptane; ○ water). These data consistently indicate that in all of these liquids instability sets in at the Rayleigh number 1700±51 (to be compared with the theoretical value 1708).

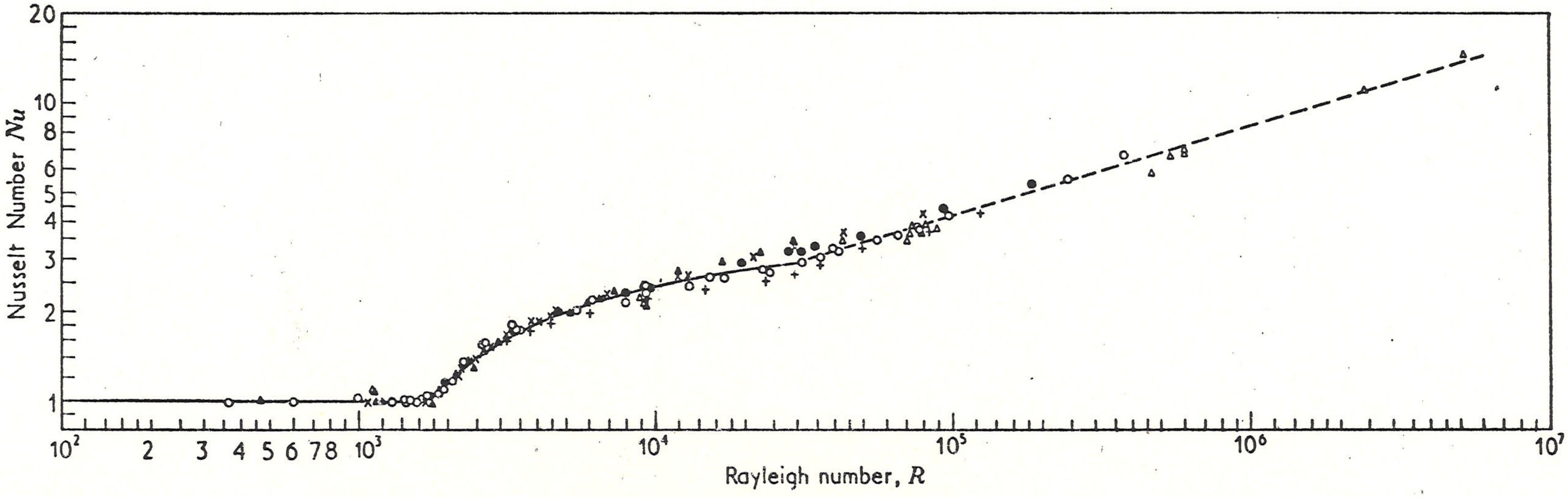
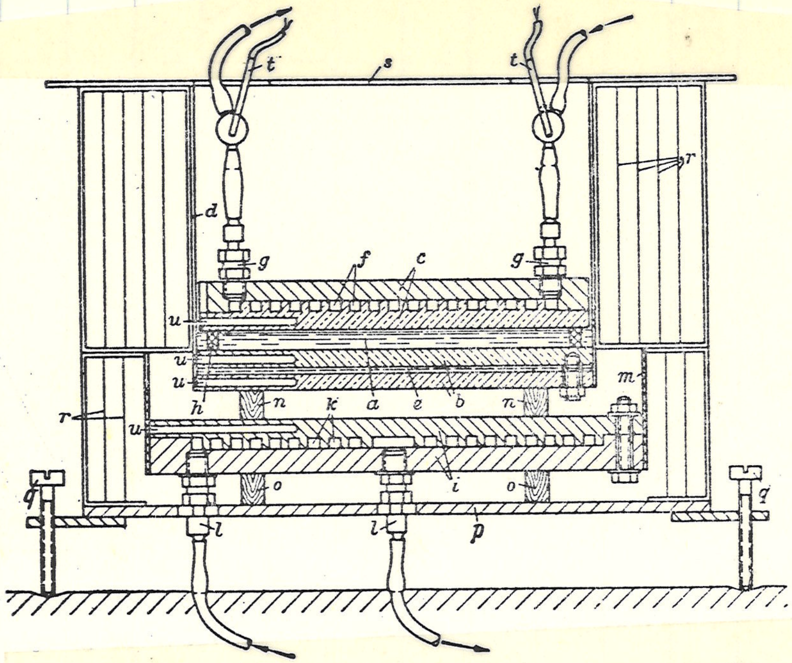


Fig. 13. Silveston's experimental results on the heat transfer in various liquids (○ water; + heptane; × ethylene glycol; ● silicone oil AK 3; ▲ silicone oil AK 350; △ air data of Mull and Reiher). The Nusselt number is plotted against the Rayleigh number.



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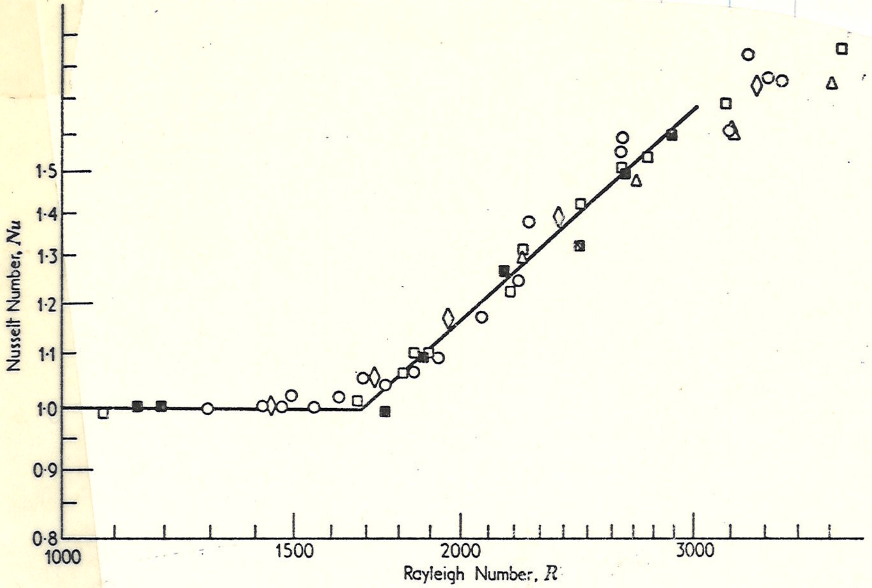

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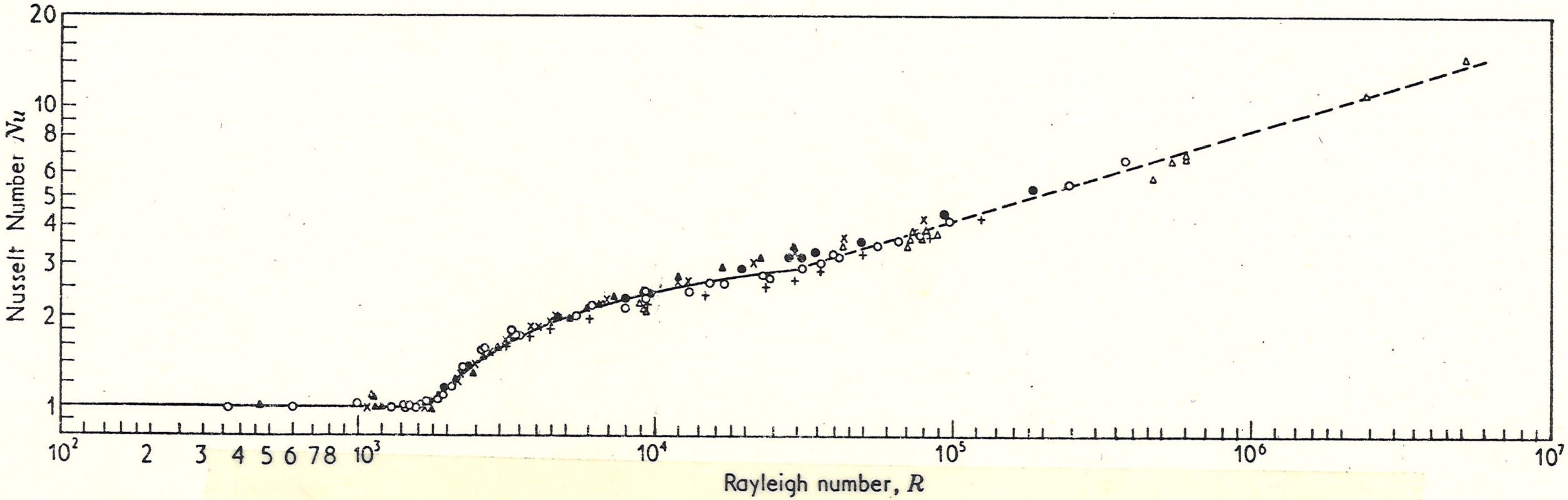

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3. (The attenuation of sound waves). A fluid of constant density ρ_0, constant pressure p_0, constant temperature θ_0 and constant sound speed c_0 fills all of space.

(a). Show that if a propagating infinitesimal sound wave has a velocity field of the form

$$\underline{u} = U_0 \hat{x} \cos(kx - \omega t),$$

the associated density, pressure and temperature fluctuations are given by

$$\rho_1 = (\rho_0 / c_0) U_0 \cos(kx - \omega t)$$
$$p_1 = (\rho_0 c_0) U_0 \cos(kx - \omega t)$$
$$\theta_1 = (\alpha c_0 \theta_0 / c_p) U_0 \cos(kx - \omega t),$$

where α is the coefficient of thermal expansion and c_p is the specific heat at constant pressure.

(b). Using the above expressions, calculate the time-averaged rate of energy dissipation (per cm^3 per sec) $\langle \dot{E} \rangle$ in the fluid, due to both thermal conduction and viscosity. Show that

$$\langle \dot{E} \rangle = -\tfrac{1}{2} k^2 U_0^2 \left[\left(\tfrac{4}{3}\eta + \gamma \right) + \kappa \left(c_v^{-1} - c_p^{-1} \right) \right],$$

where η and γ are the shear and bulk viscosities, κ is the thermal conductivity and c_v is the specific heat at constant volume. You will need to use, and should verify, the thermodynamic identity

$$c_p - c_v = \theta \alpha^2 c_0^2 c_v / c_p .$$

(c). Show that the amplitude of the sound wave will be attenuated by viscosity and heat conduction by the amount $v_0 = v_0 e^{-\beta x}$, where the absorption coefficient β is given by

$$\beta = \frac{\omega^2}{2\rho_0 c_0^3} \left[\left(\tfrac{4}{3}\eta + \gamma \right) + \kappa \left(c_v^{-1} - c_p^{-1} \right) \right].$$

The above method, which calculates $\langle \dot{E} \rangle$ and β using the <u>dissipation-free</u> fields $\underline{u}$, ρ_1, p_1 and θ_1, will be valid as long as the dissipation per wavelength is small, i.e. provided $\beta k^{-1} \ll 1$. For ~~[illegible]~~ common fluids at ordinary pressures and temperatures, this is demonstrably so.

3. Sound attenuation.

(a). The velocity field is $\underline{u} = u_0 \hat{x} \cos(kx - \omega t)$. For the density, pressure and temperature, we write $\rho = \rho_0 + \rho_1$, $p = p_0 + p_1$, $\theta = \theta_0 + \theta_1$.

From problem 6 of problem set # 5, we have the linearized relations

$$1. \quad \partial_t \rho_1 + \rho_0 \underline{\nabla} \cdot \underline{u} = 0$$

$$2. \quad \partial_t p_1 = c_0^2 \, \partial_t \rho_1 .$$

We have in addition used the fact that $\underline{\nabla} \rho_0 = 0$ to simplify the above two relations. Equation 1 implies that $\rho_1 = (\rho_0 k / \omega) u_0 \cos(kx - \omega t)$, or since $c_0 = \omega / k$,

$$\rho_1 = (\rho_0 / c_0) u_0 \cos(kx - \omega t).$$

Equation 2 then implies that

$$p_1 = (\rho_0 c_0) u_0 \cos(kx - \omega t).$$

Since the sound wave is propagating isentropically, θ_1 is given by

$$\theta_1 = (\partial \theta / \partial p)_s \, p_1 .$$

But by one of Maxwell's relations for a fluid:

$$(\partial \theta / \partial p)_s = -(\partial s / \partial p)_\theta \,/\, (\partial s / \partial \theta)_p$$
$$= (\partial \tau / \partial \theta)_p \,/\, (\partial s / \partial \theta)_p$$
$$= (\theta / c_p)(\partial \tau / \partial \theta)_p$$
$$= \alpha \tau \theta / c_p \quad \text{where}$$

$c_p = \theta (\partial s / \partial \theta)_p$ is the specific heat at constant pressure, and $\alpha = -\frac{1}{\tau}\left(\frac{\partial \tau}{\partial \theta}\right)_p$

is the isobaric coefficient of thermal expansion. This leads to the relation

$$\theta_1 = (\alpha c_0 \theta_0 / c_p)\, u_0 \cos(kx - \omega t).$$

(b). The rate of production of entropy in a Newtonian viscous fluid is

$$\frac{h}{\theta} + \frac{\kappa}{\theta^2}|\nabla\theta|^2 + \frac{\gamma}{\theta}(\mathrm{tr}\,\underline{\underline{\varepsilon}})^2 + \frac{2\eta}{\theta}\mathrm{tr}\,(\underline{\underline{D}}\cdot\underline{\underline{D}}).$$

Call this quantity P. Then the rate of dissipation of mechanical energy is $\dot{E} = -\theta P$. In our case, $h = 0$. Neglecting terms higher than quadratic in the perturbed quantities, we obtain

$$\dot{E} = -\left[\frac{\kappa}{\theta_0}|\nabla\theta_1|^2 + \gamma(\mathrm{tr}\,\underline{\underline{\varepsilon}})^2 + 2\eta\,(\underline{\underline{D}}:\underline{\underline{D}})\right].$$

Now $(\mathrm{tr}\,\underline{\underline{\varepsilon}})^2 = (\partial u/\partial x)^2$ and

$$\underline{\underline{D}}:\underline{\underline{D}} = \tfrac{2}{3}(\partial u/\partial x)^2.$$

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}\frac{\partial u}{\partial x}$$

$$\underline{\underline{d}} = \begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix}\frac{\partial u}{\partial x}$$

We thus find

$$\dot{E} = -\left[\kappa\theta_0(\alpha c_0/c_p)^2 + (\gamma + \tfrac{4}{3}\eta)\right] k^2 u_0^2 \sin^2(kx - \omega t).$$

$$\underline{\underline{d}}:\underline{\underline{d}} = \left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right)\left(\frac{\partial u}{\partial x}\right)^2$$

The time average of $\sin^2(kx - \omega t)$ is 1/2, so

$$\langle\dot{E}\rangle = -\tfrac{1}{2}k^2 u_0^2\left[(\tfrac{4}{3}\eta + \gamma) + \kappa\theta_0(\alpha c_0/c_p)^2\right].$$

Making use of the thermodynamic identity $c_p - c_v = \theta\alpha^2 c_0^2 c_v/c_p$, we obtain the desired result

$$\langle\dot{E}\rangle = -\tfrac{1}{2}k^2 u_0^2\left[(\tfrac{4}{3}\eta + \gamma) + \kappa\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right].$$

To verify the identity, consider the

two equations of state $p = p(\tau, S)$ and $\theta = \theta(\tau, S)$. Differentiating these, let us define X, Y and Z by

$$dp = -X\,d\tau + Y\,dS$$
$$d\theta = -Y\,d\tau + Z\,dS.$$

The quantity $X = -(\partial p/\partial\tau)_S = \rho^2(\partial p/\partial\rho)_S = \rho^2 c_0^2$ and $Z = (\partial\theta/\partial S)_\tau = \theta/c_v$. To find Y, we use the definition of c_p, viz.

$$c_p = \theta(\partial S/\partial\theta)_p.$$

When $dp = 0$, we have $d\tau = (Y/X)\,dS$ and therefore $d\theta = (1 - Y^2/XZ)\,Z\,dS$. Thus c_p is given by $c_p = (1 - Y^2/XZ)^{-1} c_v$ or

$(*)$ $Y^2 = (1 - c_v/c_p)\,X Z = (1 - c_p/c_v)\,\rho^2 c_0^2\theta/c_v$.

We may also write, when $dp = 0$, $d\theta = -(1 - XZ/Y^2)\,Y\,d\tau$, and using the definition $\alpha = -\frac{1}{\tau}(\partial\tau/\partial\theta)_p$, we see that we may also write $Y = (1 - XZ/Y^2)^{-1}\rho/\alpha$ $= (c_p/c_v - 1)\,\rho/\alpha$ $(**)$. Upon equating $*$ and $(**)^2$ we find the required identity $c_p - c_v = \theta^2\alpha^2 c_0^2\, c_v/c_p$.

(c). The rate of decay of wave energy per unit distance will be $dE/dx = \langle\dot{E}\rangle/c_0$. In the absence of dissipation, the wave energy density is $E = \frac{1}{2}\rho_0\mathcal{U}_0^2$. Since $c_0 = \omega/k$, we have $dE/dx = -E\,\{(\omega^2/\rho_0 c_0^3)[(\gamma + \frac{4}{3}\eta) + \kappa(c_v^{-1} - c_p^{-1})]$. Since $E \propto \mathcal{U}_0^2$, the <u>amplitude</u> decay law will be

$$\mathcal{U} = \mathcal{U}_0 \exp\{-(\omega^2/2\rho_0 c_0^3)[(\tfrac{4}{3}\eta + \gamma) + \kappa(\tfrac{1}{c_v} - \tfrac{1}{c_p})]x\}.$$

Alex → Tony

Problem set numero 0 : Attenuation in the SOFAR zone

We neglect the attenuation due to viscous shear and diffusion of heat. Under those circumstances, the decay scale is given by

$$\delta = \frac{2\rho c^3}{\omega^2 \gamma} \tag{1}$$

Using

ω	$2\pi\ kHz$
γ	$10^{-1}\ Pa\ s$
ρ	$1000\ kg\ m^{-3}$
c	$1400\ m\ s^{-1}$

(c is relatively insensitive to changes in Pressure/Salinity), one gets

$$\delta \simeq 1,500\ km. \tag{2}$$

More info on sound velocity in seawater is available on http://freespace.virgin.net/sd.richards/speed.html.