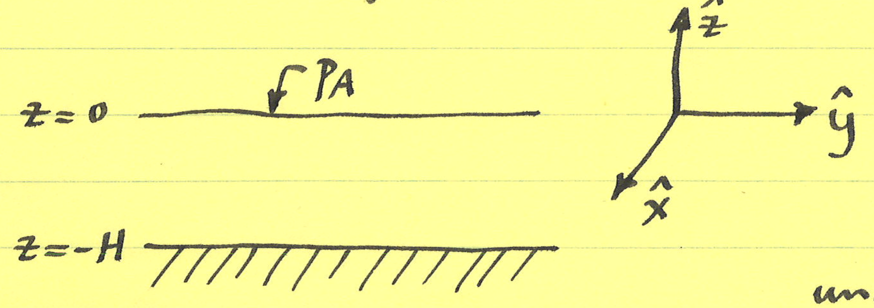
A special lecture on surface gravity waves:

We consider the propagation of surface gravity waves on an infinite plane ocean of depth H, and consisting of homogeneous incompressible fluid of constant density ρ

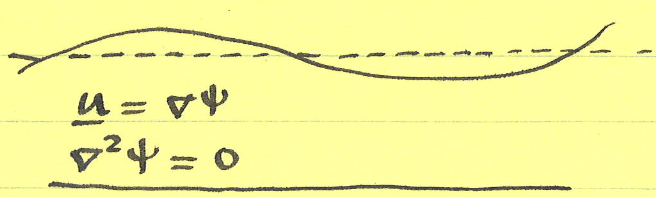


undeformed state

$\underline{u} = 0$ no motion.
$\rho = \rho =$ constant
$\phi = gz$, $g =$ constant

$p/\rho = p_A/\rho - gz$ hydrostatic pressure
$p_A =$ surface atmospheric pressure

Consider all possible motions which are <u>potential</u> <u>flows</u>.



$\underline{u} = \nabla\psi$
$\nabla^2\psi = 0$

$\underline{u} = u\hat{x} + v\hat{y} + w\hat{z}$
$u = \partial_x\psi$, $v = \partial_y\psi$, $w = \partial_z\psi$

<u>Inside</u> we have : $-H \leq z \leq h(x,y,t)$

$$\underline{u}(x,y,z,t) = \nabla\psi(x,y,z,t)$$

$$\nabla^2\psi(x,y,z,t) = 0$$

And by <u>Bernoulli's theorem</u>

$$\partial_t\psi + \frac{1}{2}|\underline{\nabla}\psi|^2 + gz + \frac{p}{\rho} + f(t) = \text{const.}$$

Now in the <u>absence</u> of motion $\psi(x,y,z,t) = 0$ and we have

$$gz + \frac{p}{\rho} = \frac{p_A}{\rho}, \quad \text{hence}$$

Bernoulli theorem is

$$\boxed{\partial_t\psi + \frac{1}{2}|\underline{\nabla}\psi|^2 + gz + \frac{p}{\rho} = \frac{p_A}{\rho}}$$

Now what are the b.c.?

At $z = -H$: $\hat{n}\cdot\nabla\psi = 0$ or

$$\partial_z\psi(x,y,z,t) = 0$$

At $z = h(x,y,z,t)$, what is the relevant <u>kinematic</u> b.c.?

The wavy upper bdry always consists of the same fluid particles.

The eqn for the upper surface is

$$z - h(x, y, t) = 0$$

We may consider the fcn $\chi(x, y, z, t) \equiv z - h(x, y, t)$ to be defined throughout the fluid. The particles on the upper surface must see

$$D_t \chi(x, y, z, t) \equiv 0 \quad \text{or}$$

$$D_t [z - h(x, y, t)] = 0 \quad \text{or}$$

$$[\partial_t + \underline{u} \cdot \nabla][z - h(x, y, t)] = 0$$

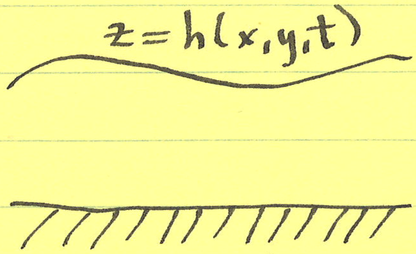
$$\partial_z \psi - \partial_t h - (\partial_x \psi)(\partial_x h) - (\partial_y \psi)(\partial_y h) = 0$$

on $z = h(x, y, t)$. Can write as $w(h) = D_t h$, for later

This is the <u>kinematic</u> b.c.

The dynamic b.c. is, in the absence of surface tension, $p = p_A$

In summary, we have



$$\underline{u}(x,y,z,t) = \nabla\psi(x,y,z,t)$$

Inside: $\nabla^2\psi = 0$, a linear eqn

b.c. on $z = -H$: $\partial_z\psi = 0$

b.c. on top $z = h(x,y,t)$:

$$\left.\begin{aligned} &\partial_z\psi = \partial_t h + (\partial_x\psi)(\partial_x h) + (\partial_y\psi)(\partial_y h) \\ \longrightarrow\ &\partial_t\psi + \tfrac{1}{2}|\underline{\nabla}\psi|^2 + gh = 0 \end{aligned}\right\} *$$

eqn $\rightarrow$ can be considered a defn of $h(x,y,t)$ in terms of $\psi(x,y,h(x,y,t),t)$.

The eqn governing the motion is easy and linear $\nabla^2\psi = 0$, but the top b.c. is non-linear. Also they are specified on a surface $h(x,y,t)$ which is unknown a priori.
The solution has never been obtained in closed form.

We will linearize the problem, considering only infinitesimal waves.

The linearized problem:

Inside: $\nabla^2 \psi = 0$

Bottom: $\partial_z \psi = 0$ at $z = -H$

Top: we neglect non-linear terms $\underline{u} \cdot \underline{\nabla h}$

furthermore, to the same accuracy, we may apply the b.c. on the level unperturbed bdry $z = 0$

Consider a typical term in the b.c. *

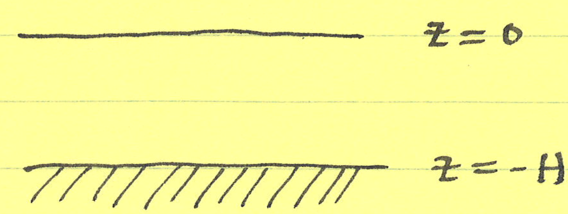
e.g. $\partial_z \psi$

Taylor series

$$f(z+h) = f(z) + \cancel{h \left(\frac{\partial f}{\partial z}\right)_{z=0}}$$

second order

Hence our linear problem is



Inside: $\nabla^2 \psi = 0$ in $-H \le z \le 0$

Bottom: $\partial_z \psi = 0$ on $z = -H$

Top:
$$\left.\begin{aligned} \partial_t h &= \partial_z \psi \\ \partial_t \psi + g h &= 0 \end{aligned}\right\} \text{ on } z = 0$$

Eliminating h from the top b.c.

$$\boxed{\partial_t^2 \psi + g\,\partial_z \psi = 0 \quad \text{on} \quad z = 0}$$

We look for wavelike solutions

$$h(x,y,t) = A\, e^{i(lx + my - \omega t)}$$

$\leftarrow$ don't write this at first — solve for ψ first

$$\psi(x,y,z,t) = Q(z)\, e^{i(lx+my-\omega t)}$$

$\nabla^2 \psi = 0$: $\quad (\partial_z^2 - l^2 - m^2) Q = 0$ in $-H < z < 0$ $\star$

$\partial_z \psi = 0$: $\quad \partial_z Q = 0$ at $z = -H$

top: $\quad (-\omega^2 + g\,\partial_z) Q = 0$ at $z = 0$

Let $k^2 \equiv l^2 + m^2$. The soln (general) to $\star$ is

$$Q(z) = B e^{kz} + C e^{-kz}$$

Now make use of two b.c.

$$\partial_z Q = 0 \text{ at } z = -H$$
$$k\left(Be^{-kH} - Ce^{kH}\right) = 0$$

Let $B = \frac{1}{2} e^{kH} \cancel{A} D$

$C = \frac{1}{2} e^{-kH} \cancel{A} D$

Then $Q(z) = \cancel{A}^{D} \cosh k(z+H)$ $\qquad \frac{d \cosh x}{dx} = \sinh x$

Now from the top b.c.

$$-\omega^2 + gk \tanh kH = 0$$

This is the dispersion relation for surface gravity infinitesimal waves

$$\boxed{\omega^2 = gk \tanh kH}$$

Now what is the associated $h(x,y,t)$?

$$h = -\frac{1}{g}\left(\partial_t \psi\right)_{z=0}$$

$$h = \frac{i\omega}{g} \cancel{A} D \cosh kH \; e^{i(lx+my-\omega t)}$$

To get a real solution, we take the real part. Let's renormalize.

Say the surface displacement is

$$\boxed{h(x,y,t) = A\cos(lx+my-\omega t)}$$

Then

$$\boxed{\psi(x,y,z,t) = \frac{g}{\omega}A\,\frac{\cosh k(z+H)}{\cosh kH}\,\sin(lx+my-\omega t)}$$

We can find the associated pressure fluctuations from

$$\frac{p}{\rho} = \frac{p_A}{\rho} - gz - \partial_t \psi$$

$$\boxed{\text{Define} \quad \Delta p(x,y,t) \equiv p(x,y,z,t) - p_A + \rho g z}$$

Incremental pressure fluctuation: Then

$$\boxed{\Delta p(x,y,z,t) = \rho g A\,\frac{\cosh k(z+H)}{\cosh kH}\,\cos(lx+my-\omega t)}$$

In phase with surface elevation.

Let's reorient the x, y axes so that wave propagates in $\hat{x}$ direction

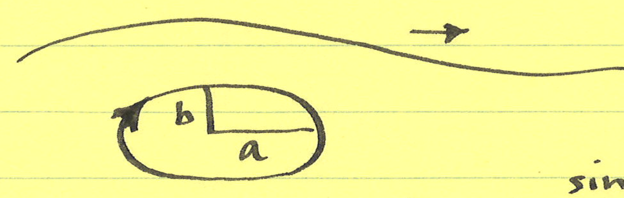
$$\underline{u}(x,z,t) = \nabla\psi(\underline{x},z,t)$$
$$= u(x,z,t)\,\hat{x} + w(x,z,t)\,\hat{z}$$

Then

$$\boxed{\begin{aligned} u(x,z,t) &= \omega A \frac{\cosh k(z+H)}{\sinh kH} \cos(kx-\omega t) \\ w(x,z,t) &= \omega A \frac{\sinh k(z+H)}{\sinh kH} \sin(kx-\omega t) \end{aligned}}$$

Particle trajectories are <u>ellipses</u>

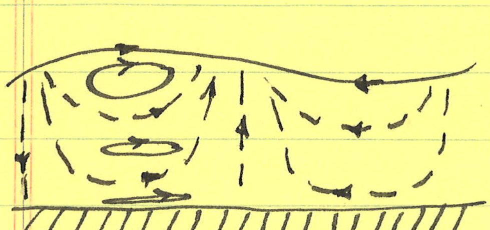
Streamlines $\neq$ trajectories



$$b = A \frac{\sinh k(z+H)}{\sinh kH}$$

$$a = A \frac{\cosh k(z+H)}{\sinh kH}$$

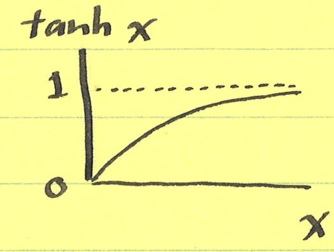
streamlines dotted below:



u is in phase with h,
w is out of phase with h.

Let us examine in more detail the dispersion relation

$$\omega^2 = gk \tanh kH$$



The phase velocity of surface gravity waves is

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh kH} = \frac{g}{\omega}\tanh kH$$

The group velocity is

$$G = \frac{d\omega}{dk} = \frac{g}{2\omega}\,\frac{\sinh kH \cosh kH + kH}{\cosh^2 kH}$$

There are two interesting limiting cases

Deep water $kH \gg 1$ water depth $\gg$ wavelength

$$\omega^2 = gk$$
$$c^2 = g/k \quad \text{or} \quad c = g/\omega = gT/2\pi$$

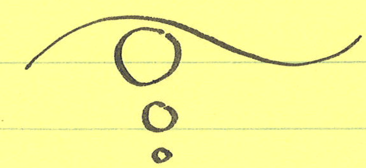
if c is in knots $c \approx 3T$, T in sec.
speed of deep water swell
in deep water, long period waves
go faster.

1 nautical mile $\cong$ 1.15 miles

In deep water $G = \frac{1}{2} c$

$$u(x,z,t) = \omega A e^{kz} \cos(kx - \omega t)$$
$$w(x,z,t) = \omega A e^{kz} \sin(kx - \omega t)$$

Particle trajectories are circles
Exponential decay with depth



Pressure fluctuation also decays exponentially with depth

$$\Delta p(x,z,t) = \rho g A e^{kz} \cos(kx - \omega t)$$

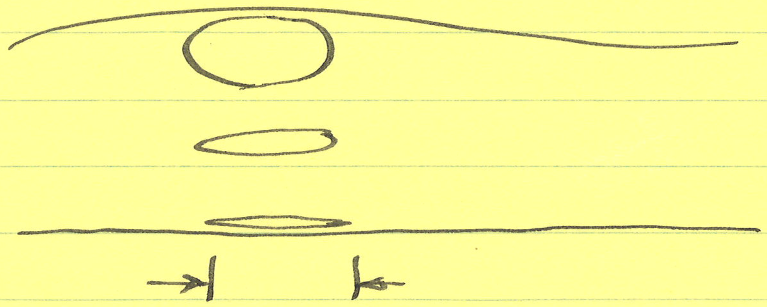
Shallow water $kH \ll 1$ water depth $\ll$ wavelength

phase velocity $c = \sqrt{gH}$ ← speed of tsunamis
non-dispersive (shallow water waves)

Orbital velocities are, in shallow H_2O

$$u(x,z,t) = \sqrt{\frac{g}{H}}\, A \cos(kx - \omega t)$$
$$w(x,z,t) = \omega A \left(1 + \frac{z}{H}\right) \sin(kx - \omega t)$$

Particle trajectories



Pressure fluctuations:

$$\Delta p(x,z,t) = \rho g A \cos(kx-\omega t)$$
$$= \rho g h(x,z,t)$$

as if in hydrostatic equilibrium

Let us now go back and examine more carefully the criteria for the neglect of the non-linear terms.

We had $\partial_t h = \partial_z \psi - (\partial_x \psi)(\partial_x h) - (\partial_y \psi)(\partial_y h)$ on $z = h(x,y,t)$

and also $\partial_t \psi + \frac{1}{2}|\nabla\psi|^2 + gh = 0$ on $z = h(x,y,t)$

We neglected $\left(\frac{\partial \psi}{\partial x}\right)\left(\frac{\partial h}{\partial x}\right)$ w.r.t. $\frac{\partial \psi}{\partial z}$ near $z = 0$

$$\left(\frac{\partial \psi}{\partial x}\right)\left(\frac{\partial h}{\partial x}\right) \sim kA \, \frac{kg}{\omega} A$$

$$\frac{\partial \psi}{\partial z} \sim \frac{kg}{\omega} A \tanh kH$$

in shallow water $\coth kH \sim 1/kH$

$$\boxed{\text{This is small if } A/H \ll 1}$$

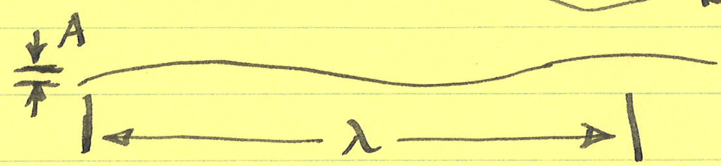
Ratio $\sim kA \coth kH$

$$\boxed{\text{This is small if } kA \ll 1}$$

amplitude $\ll$ depth

In deep water $\coth kH \simeq 1$

$$\frac{2\pi A}{\lambda} \ll 1$$



$kA \cong$ slope of wave $\ll 1$

We also neglect $\frac{1}{2}|\underline{\nabla}\psi|^2$ w.r.t. $\partial_t \psi$

$$|\underline{\nabla}\psi|^2 \sim (u^2 + w^2)$$
$$\sim \omega^2 A^2 (1 + \coth^2 kH)$$

$$\partial_t \psi \sim gA$$

$$\text{Ratio} \sim \frac{\omega^2}{g} A \,(1 + \coth^2 kH)$$

$$= kA\,(\tanh kH + \coth kH)$$

In <u>shallow</u> water $\quad \tanh kH \sim 0$

$\coth kH \sim 1/kH$

require $A/H \ll 1$, amplitude $\ll$ depth

In <u>deep</u> water, again reduces to $kA \ll 1$,
since both $\tanh kH$ and $\coth kH \sim 1$.

One further aspect of linearization is evaluation of upper b.c. at $z=0$ instead of $z=h$

Look at next term in Taylor series.
Leads to same criteria.

Summary: criteria for applicability of linearization

$kA \ll 1$ small slope

$A/H \ll 1$ small ampl. compared to depth

Let us now examine the <u>pressure fluctuations</u> associated with surface gravity waves a little more closely.

It turns out that with a little trickery, we can deduce some very interesting information about $p(x, y, z, t)$ <u>correct to order $(kA)^2$</u>.

We consider the vertical component of the <u>momentum</u> equation

$$\rho D_t \underline{u} = \text{[redacted]} - \nabla p - \rho g \hat{z}$$

$\hat{z}$ component is recall $\underline{u} \equiv (u, v, w)$

$$\rho \partial_t w + \rho \underline{u} \cdot \underline{\nabla w} + \partial_z p + \rho g = 0$$

$$\rho \partial_t w + \rho \nabla \cdot (w \underline{u}) - \rho w \cancel{\nabla \cdot \underline{u}} + \partial_z p + \rho g = 0$$

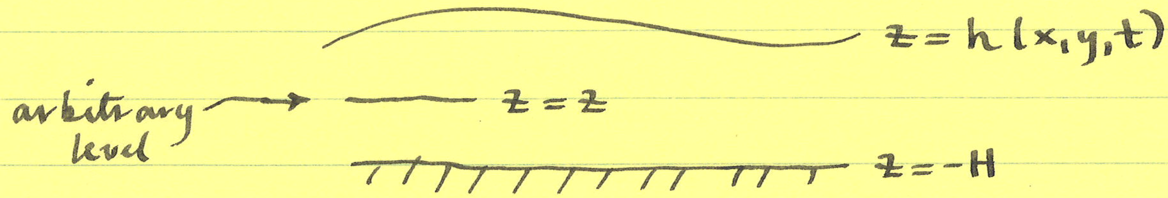
zero by cont eqn.

$$\partial_z p = -\rho g - \rho \partial_t w - \rho \nabla_h \cdot (w \underline{q}) - \rho \partial_z (w^2)$$

here we have let $\underline{u} \equiv \hat{z} w + \underline{q}$

$\nabla_h \equiv \hat{x} \partial_x + \hat{y} \partial_y$ horizontal grad.

Now we integrate this eqn from $z = z$ to $z = h(x, y, t)$



$$\int_z^h \partial_z p \, dz = -\int_z^h \left[\rho g + \rho \partial_t w + \rho \nabla_h \cdot (w \underline{q}) + \rho \partial_z (w^2) \right] dz$$

$$p(z) = p_A + \overbrace{\int_z^h \rho g \, dz}^{\rho g (h-z)} + \int_z^h \rho \partial_t w \, dz$$

$$+ \int_z^h \rho \nabla_h \cdot (w \underline{q}) \, dz$$

$$+ \rho w(h)^2 - \rho w(z)^2$$

d/dt, not $\partial/\partial t$

Now consider $\dfrac{d}{dt} \displaystyle\int_z^h \rho w \, dz =$

$$\int_z^h \rho \partial_t w \, dz + \rho w(h) \partial_t h$$

Consider also $\nabla_h \cdot \int_z^h \rho w \underline{q} \, dz$

$$= \int_z^h \rho \, \nabla_h \cdot (w \underline{q}) \, dz \quad + \quad \rho w(h) \, \underline{q}(h) \cdot \nabla_h h$$

Now from the exact kinematic b.c. we have

$$w(h) = \partial_t h + \underline{q}(h) \cdot \nabla_h h = D_t h$$

Hence

$$\boxed{\begin{aligned} p(z) = \; & p_A + \rho g (h - z) \\ & + \frac{d}{dt} \int_z^h \rho w \, dz \\ & + \nabla_h \cdot \int_z^h \rho w \underline{q} \, dz \quad - \rho w(z)^2 \end{aligned}}$$

This equation is exact, for a fluid of constant density ρ.

mean dynamical pressure

Now let us consider the horizontal spatial mean of the pressure.

Denote the horizontal spatial mean by $\langle\ \rangle$.

We will now neglect terms of order $(kA)^3$. Don't mention this until below.

$$\left\langle \nabla_h \cdot \int_z^h \rho w \underline{q}\, dz \right\rangle \equiv 0$$ if the waves are spatially periodic

$\langle \rho g h \rangle \equiv 0$ as well

$$\left\langle \rho \frac{d}{dt} \int_z^h w\, dz \right\rangle = \rho \frac{d}{dt} \left\langle \int_z^h w\, dz \right\rangle$$

$$= \rho \frac{d}{dt} \left\langle \int_z^0 w\, dz \right\rangle + \rho \frac{d}{dt} \left\langle \int_0^h w\, dz \right\rangle$$

now $\left\langle \int_z^0 w\, dz \right\rangle \equiv 0$

next term is $\left\langle \int_0^h w'(0)\, z\, dz \right\rangle = \left\langle w'(0) \frac{h^2}{2} \right\rangle$

so we have remaining

$$\rho \frac{d}{dt} \left\langle \int_0^h w\, dz \right\rangle = \rho \frac{d}{dt} \langle w(0)\, h \rangle + O(kA)^3$$

expand $w(z)$ in Taylor series

here for the first time we neglect $O(kA)^3$.

$$\boxed{\begin{aligned}&\text{Correct to second order (ignoring } (kA)^3\text{):}\\ &\langle p(z)\rangle = p_A - \rho g z - \rho \frac{d}{dt}\langle h w(0)\rangle\\ &\qquad\qquad - \rho \langle w(z)^2\rangle\end{aligned}}$$

I can write this in form on bottom of next page also.

The first term is clearly just the hydrostatic contribution. Rest is mean dynamical pressure.

Now we must consider two cases:

$$h = A \cos(\underline{k}\cdot\underline{x} - \omega t)$$

1. progressive waves $\Rightarrow$ $\langle\,\rangle$ spatial mean is independent of time $\Rightarrow$ Third term vanishes. Note $\frac{d}{dt}\langle\cdot\rangle = 0$

if you don't believe this, see the standing wave derivation below.

Consider fourth term

$$\boxed{\langle p(z)\rangle = p_A - \rho g z - \frac{1}{2}\rho\omega^2 A^2 \frac{\sinh^2 k(z+H)}{\sinh^2 kH}}$$

this $\frac{1}{2}$ is from the $\langle\,\rangle$ horiz. ave.

angular freq.

In deep water $kH \gg 1$

$$\boxed{\langle p(z)\rangle = p_A - \rho g z - \tfrac{1}{2}\rho\omega^2 A^2 e^{2kz}}$$

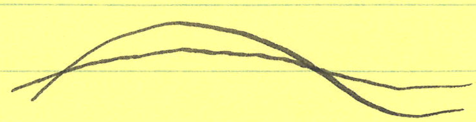
The mean dynamical pressure attenuates rapidly with depth. This is not very interesting.

2. standing waves

$$h(\underline{x}, t) = A \cos \underline{k}\cdot\underline{x} \cos \omega t$$

for example may be set up in a closed basin, or at a coastline

In this the third term is not zero
$\partial_t \langle\ \rangle \neq 0$ for a standing wave



Consider the [redacted] third term

$$-\rho\, \partial_t \langle h w(0)\rangle = -\tfrac{1}{2}\rho \frac{d^2}{dt^2}\langle h^2\rangle + O(kA)^3$$

by the linearized kinematic b.c.

$$= -\frac{1}{2}\rho\,\partial_t^2 \langle A^2 \cos^2 \underline{k}\cdot\underline{x} \; \cos^2\omega t \rangle$$

$$= -\frac{1}{2}\rho\,\partial_t^2 \left[\frac{1}{2}A^2 \cos^2\omega t \right]$$

$$= -\frac{1}{2}\rho\omega^2 A^2 \cos 2\omega t$$

Hence in a <u>standing wave</u> in deep H_2O

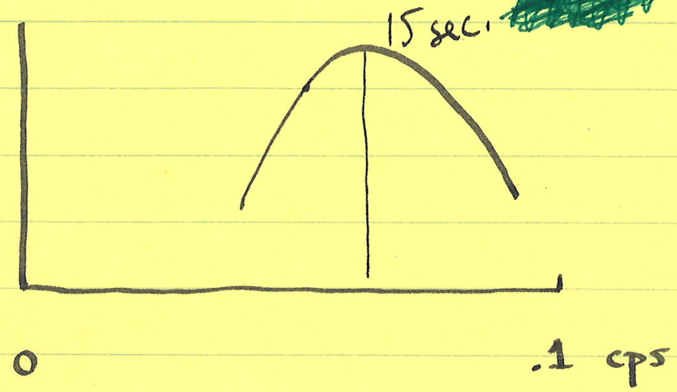
$$\langle p(z)\rangle = \underbrace{p_A - \rho g z}_{\text{hydrostatic}} - \underbrace{\tfrac{1}{2}\rho\omega^2 A^2 e^{2kz}}_{\text{decays rapidly with depth}} - \underbrace{\tfrac{1}{2}\rho\omega^2 A^2 \cos 2\omega t}_{\text{depth independent! reaches all the way to bottom}}$$

This final term is a second order pressure pulsation which reaches all the way to bottom even in very deep H_2O. The freq. of this pressure pulsation = twice that of the standing wave.

This pressure fluctuation is the <u>origin of coastal microseisms</u>. (Longuet-Higgins 1950)

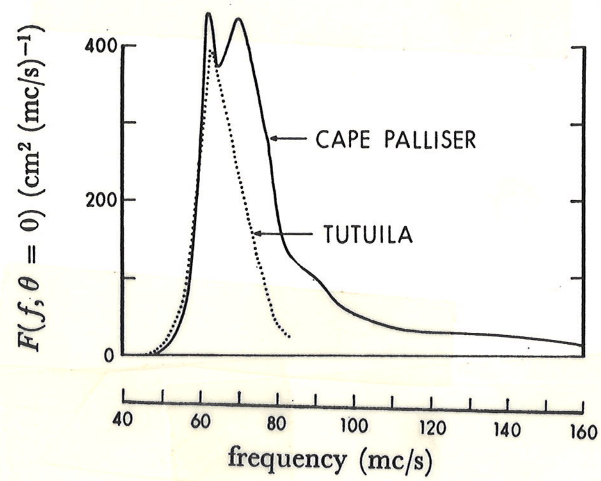
Standing waves produced by reflection at a shoreline. Microseisms generated near shore.

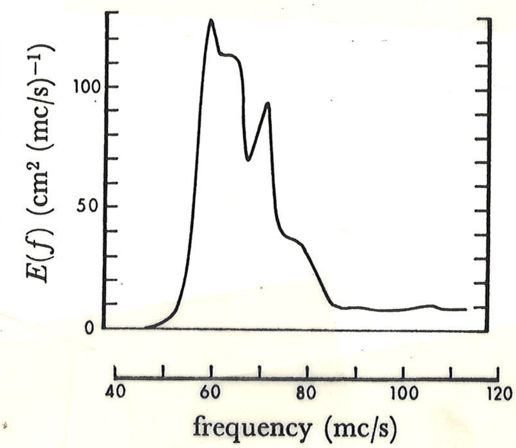
Typical oceanic swell spectrum far away from storm which generates looks like



Peak at about 15 sec period
This produces <u>coastal</u> microseisms of period ~ 7 seconds.
These make coastal or island seismographic stations very noisy, e.g. Iceland

Typical swell spectra peak at about
0.06 – 0.07 Hz = 60 – 70 mHz
This about 15 seconds.
Typical microseismic noise peaks at about <u>7 seconds</u>.





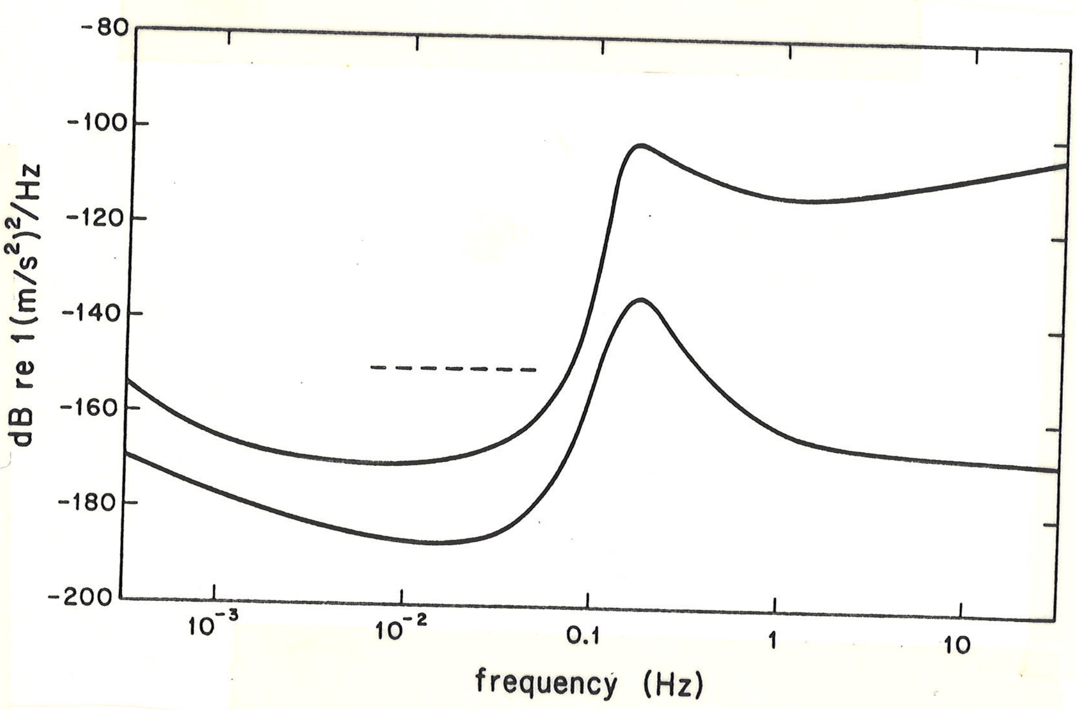
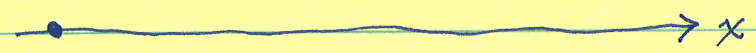


Figure 4.1. Ground noise reported at seismic stations. The dashed line indicates the ground noise to be expected on horizontal instruments at the surface or in shallow installations. Few sites achieve the lowest noise levels, particularly at the higher frequencies. (Agnew, with permission)

Dispersive waves: $\omega = \omega(k)$

Let's do a 2D analysis for simplicity



source
near $x=0$

$$h(x,t) = \mathrm{Re}\int_0^\infty dk\, A(k)\, \exp i\,[kx - \omega(k)t]$$

$$h(x,0) = \mathrm{Re}\int_0^\infty dk\, A(k)\, \exp ikx$$

$\boxed{\text{* Aside}}$

initial Fourier spectrum

$$h(x,t) = \mathrm{Re}\int_0^\infty dk\, A(k)\, e^{i[kx - \omega(k)t]}$$

$$= \mathrm{Re}\int_0^\infty dk\, A(k)\, e^{it\phi(k)}$$

$$\phi(k) = k\frac{x}{t} - \omega(k)$$

fixed

Consider $t \to \infty$ limit for <u>fixed</u> x/t.

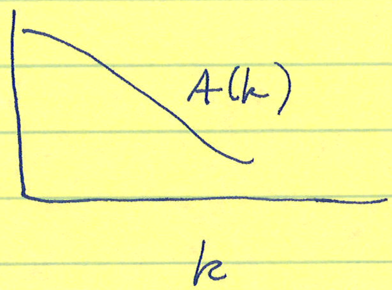
$\boxed{\text{* Aside}}$

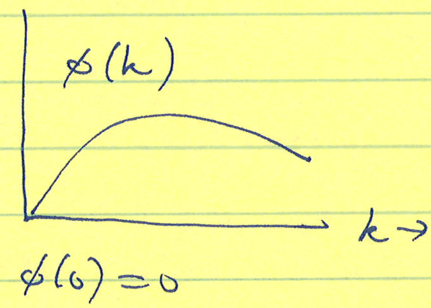
$$A(k) = 2\pi \int_{-\infty}^{\infty} h(x,0)\, e^{-ikx}\, dx$$

$= \pi \times$ FT (initial displacement)

$h(x,0)$ confined in space $\Rightarrow$ $A(k)$ broad & smooth in k-space integrand $\quad A(k)\, e^{it\phi(k)}$ ← large

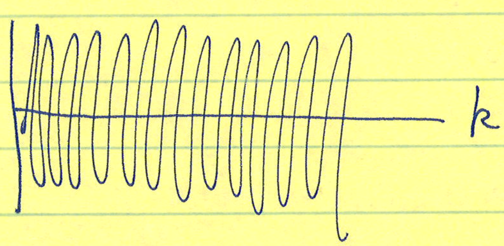
Both $A(k)$ and $\phi(k) = k\frac{x}{t} - \omega(k)$ smooth



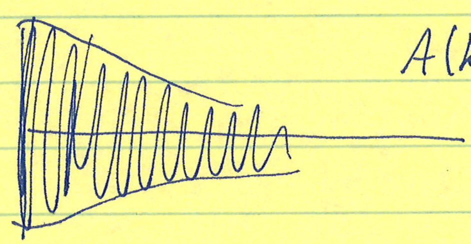


$$e^{it\phi(k)} = \cos t\phi(k) + i \sin t\phi(k)$$

very wiggly in limit $t \to \infty$



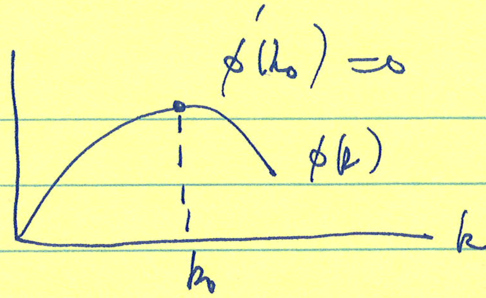
$\cos t\phi(k)$



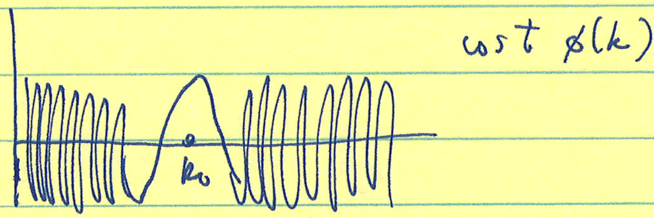
$A(k) \cos t\phi(k)$

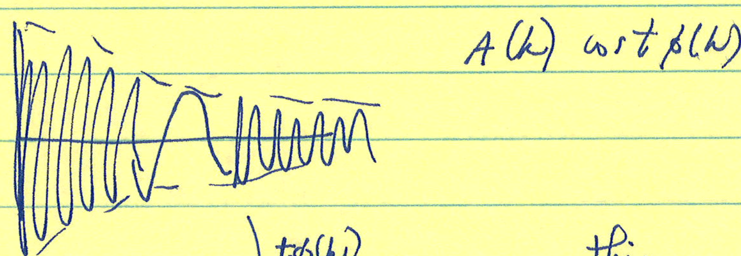
neg. cancels pos.
$\Rightarrow$ small

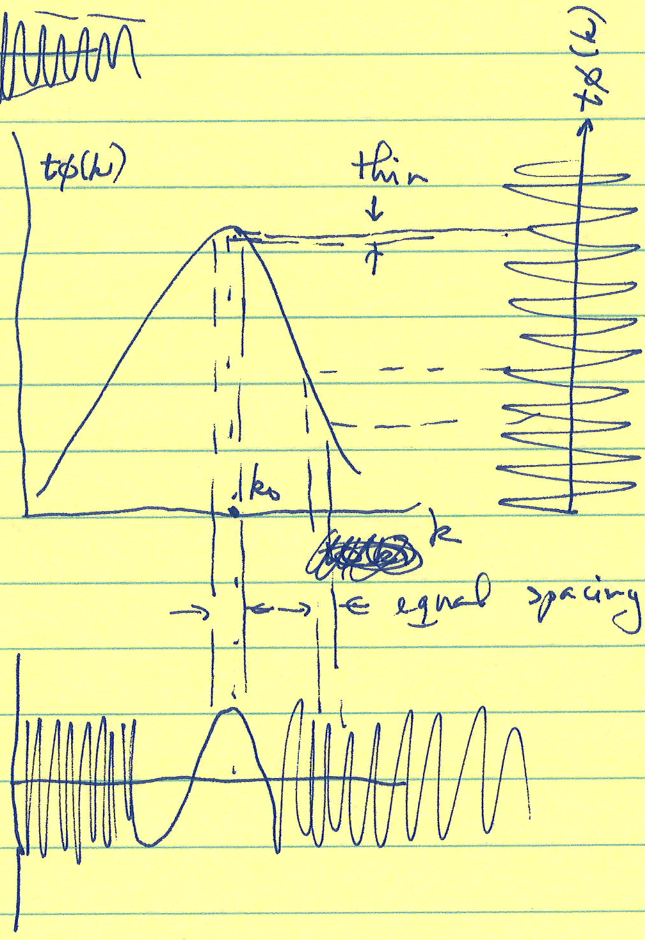
Max contr. from stationary phase points $\frac{d\phi}{dk} = 0$



k_0 = stationary point







$$\phi'(k_0) = 0 \quad \Rightarrow \quad \frac{x}{t} = \frac{d\omega}{dk}(k_0) = C(k_0)$$

$$\boxed{\frac{x}{t} = C(k_0)}$$ group velocity

Dominant contr. from stationary phase points.
Expand.

$$A(k) = A(k_0) + \cdots$$
$$\phi(k) = \phi(k_0) + \cancel{(k-k_0)\phi'(k_0)} + \tfrac{1}{2}\phi''(k_0)(k-k_0)^2 + \underset{\text{ignore}}{\cdots}$$

note $\phi''(k_0) = -\omega''(k_0)$.

$$h(x,t) \approx \mathrm{Re} \sum_{k_0} \int_0^\infty dk\, A(k_0)\, e^{it[\phi(k_0) + \frac{1}{2}\phi''(k_0)(k-k_0)^2]} \, dk$$

$$= \mathrm{Re} \sum_{k_0} A(k_0)\, e^{it\phi(k_0)}$$
$$\times \int_0^\infty dk\, e^{\frac{1}{2} it \phi''(k_0)(k-k_0)^2}$$

only dependence on k

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}} \quad \text{error integral}$$

$$\int_0^\infty dk\, e^{-\frac{1}{2} it\omega''(k_0)(k-k_0)^2}$$
$$\simeq \int_{-\infty}^{\infty} dk\, e^{-\frac{1}{2} it\omega''(k_0)(k-k_0)^2}$$

$$\alpha = \tfrac{1}{2} t\, |\omega''(k_0)|\, e^{-i\pi/2 \,\mathrm{sgn}\, \omega_0''(k_0)}$$

$$\mathrm{sgn} = \begin{cases} +1 & \text{pos} \\ -1 & \text{neg} \end{cases}$$

give physical interpretation
$C(k)$ = velocity of locus of waves of wavenumber k

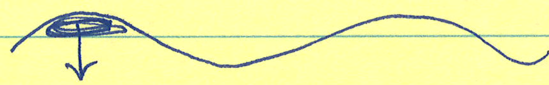
$$\boxed{h(x,t) = \mathrm{Re} \sum_{k_0} A(k_0) \left(\frac{2\pi}{t\,|\omega''(k_0)|} \right)^{1/2} \exp i \left[k_0 x - \omega(k_0) t - \frac{\pi}{4} \,\mathrm{sgn}\, \omega''(k_0) \right]}$$

$$\frac{x}{t} = C(k)$$

sum over all stationary phase points

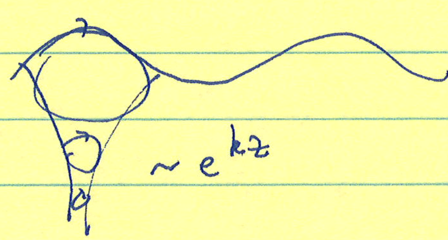
Apply to water waves — one change — surface tension

Restoring force considered so far — gravity



$$\omega = \sqrt{gk \tanh kH}$$

$$\simeq \sqrt{gk} \quad \text{deep water}$$



$$c = \sqrt{g/k}$$

$$\boxed{c(\lambda) = \sqrt{\frac{g\lambda}{2\pi}}}$$

Use λ as most easily "observed" parameter in pond experiment.

$$C = \frac{1}{2} c$$

Surface tension — additional restoring force important for <u>short</u> waves

b.c. that is changed $p = p_A \rightarrow$

$$\boxed{p = p_A + T\,\nabla_h \cdot \hat{n}}$$

Answer : $g \rightarrow g + \dfrac{Tk^2}{\rho} \leftarrow k^2 \Rightarrow$ curvature

ρ of water

$T_{water-air} = 74$ dyne/cm

$$\boxed{\omega = \sqrt{(g + Tk^2/\rho)k}}$$

$$\boxed{c = \frac{\omega}{k} = \sqrt{\frac{g + Tk^2/\rho}{k}}}$$

Minimum where $dc/dk = 0$

$$k_{min} = \sqrt{\rho g / T}$$

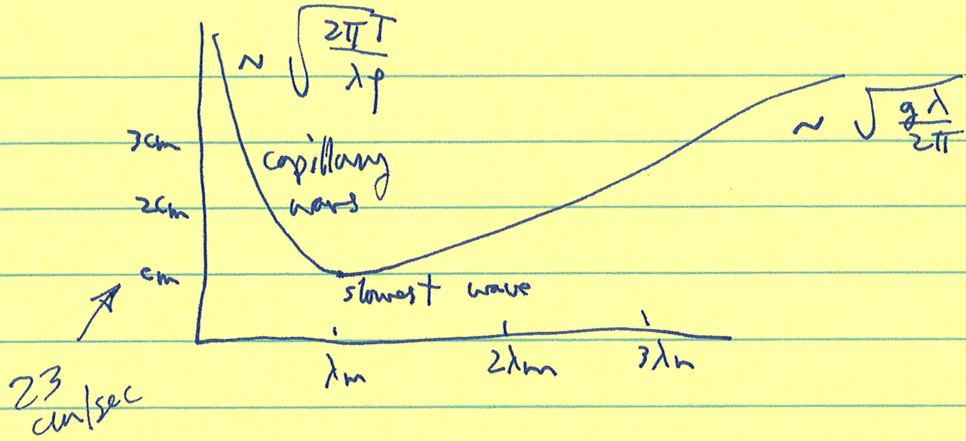
why? c^2 has same min $\Rightarrow -gk^{-2} + T/\rho = 0$

$$\lambda_{min} = \frac{2\pi}{k_m} = 2\pi\sqrt{\frac{T}{\rho g}} = 1.7 \text{ cm}$$

less than 1"

$$c_{min} = c(\lambda_{min}) = \sqrt{\frac{2g}{k_{min}}} = \underline{23 \text{ cm/sec}}$$

$C \sim \frac{3}{2} c$



$C = \frac{1}{2} c$

$\lambda \gg \lambda_m \gg$ a few inches $\quad T$ negligible

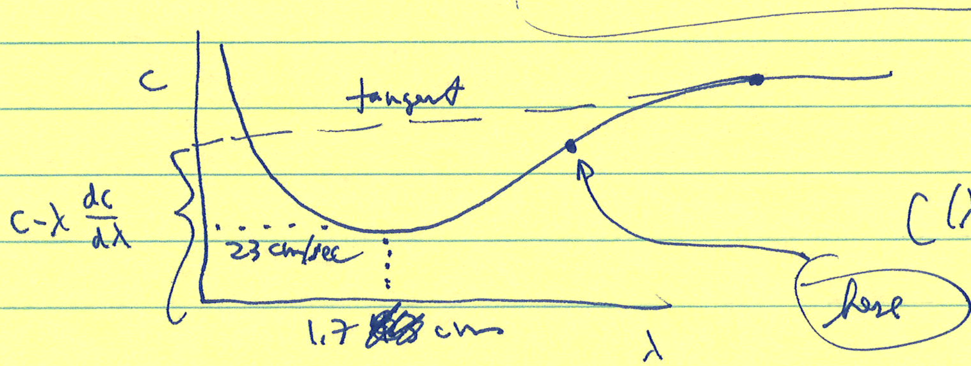
Pure capillary waves $\quad \omega = \sqrt{\frac{T k^3}{\rho}}$

$$c(\lambda) = \sqrt{\frac{2\pi T}{\lambda \rho}} \qquad C = \frac{d\omega}{dk} = \frac{3}{2}\frac{\omega}{k} = \frac{3}{2} c$$

Graphical construction

$$C = \frac{d\omega}{dk} = \frac{d}{dk}(ck) = c + k\frac{dc}{dk}$$

$$\boxed{C = c - \lambda \frac{dc}{d\lambda}}$$

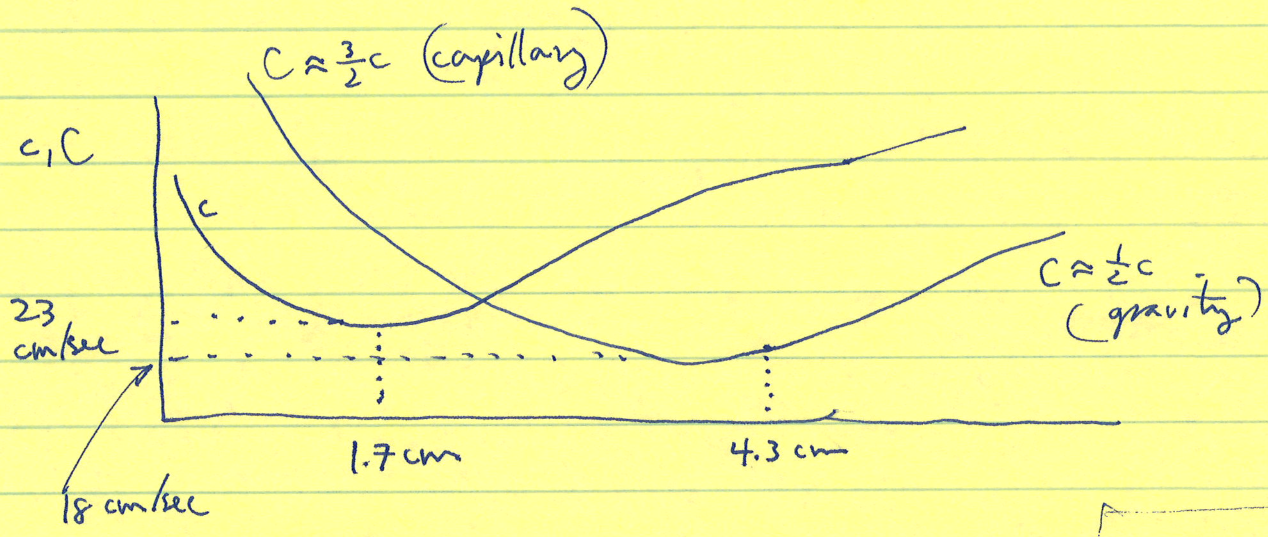


$C(\lambda)$ has min at inflection point

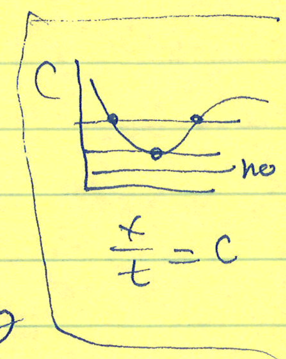
$C = C_{min}$ at $\lambda = 2.54\, \lambda_{min} = $ 4.3 cm

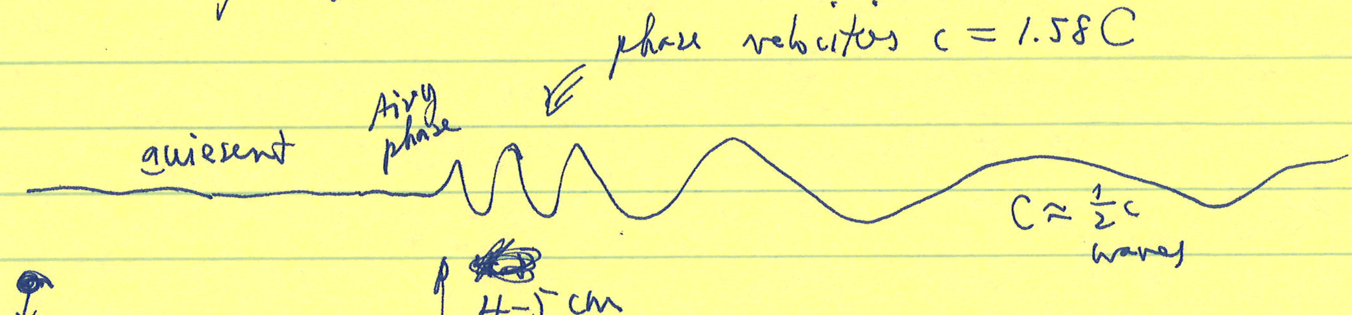
$C_{min} = 18$ cm/sec 4.3 cm waves

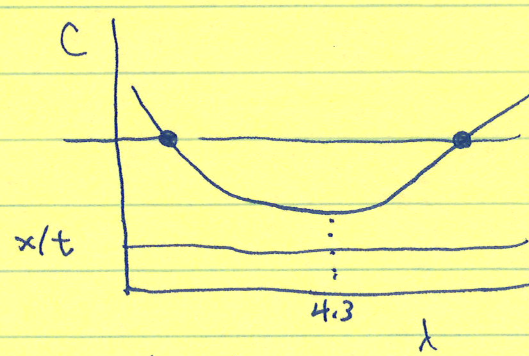
$C = 0.64c$ at min ($c = 28$ cm/sec)



Throw stone in water — in practice capillary waves, $\lambda \ll 4$ cm, attenuate rapidly, hard to observe — eye only sees gravity waves.







Switch here to pp. 12–14 of xeroxed notes

Typically λ_{max} generated $\simeq 10 \times$ size of stone

Chapter 3 analyses isotropic systems where (as we shall see) energy is propagated at right angles to the crests. If the crests form a pattern of concentric circles, as when a large stone is thrown into a pond deep enough (section 3.2) for the resulting waves to be unaffected by non-uniformities of depth, the energy of the disturbance must be travelling outwards radially from the centre. Then, as stated earlier, the longest waves produced (typically, a few times bigger than the stone) appear at the outside of the expanding concentric pattern.

It is extremely instructive to watch those outside waves. Anyone observing, however carefully, the progress of one of the crests will suddenly lose sight of it! It seems an optical illusion at first; the possible result of mistaken identity between that crest and the next one coming along behind, to which the observer's gaze is now transferred; but then this next crest, too, disappears! Meanwhile, crests are coming along thick and fast behind. Indeed, at the inside edge of the concentric pattern, new crests are appearing 'from nowhere'; that is, from the now calmed central water.

The suddenness with which sizeable crests near the outside of the pattern disappear rules out gradual attenuation (section 3.5) as the mechanism. We shall see that the true explanation is that crests travel at a wave speed c twice as big (for 'waves on deep water') as the group velocity U with which the *energy* in waves of their length is moving forwards. Each crest of every wave is thus outstripping the associated energy, so that crests can survive *only* by evolving into crests of longer waves. That, however, is impossible for waves at the outside of the group (since the original disturbance produced no energy in waves of *greater* wavelength), and so crests there can only disappear.

Waves move at $c = 2\cancel{U}c$ Thus they must increase their period and speed as they move.

Consider these long (gravity) waves on the outer edge of the pattern. If you try to follow a single wave crest with your eye you'll lose it. Let's see what the motion of a single wave crest is.

$$\boxed{\text{defn}: \quad kx - \omega t = \text{constant}}$$

A single constituent wave is

$$\cos[kx - \omega(k)t].$$

We also know that waves of wavenumbe k are found at $x/t = U(k)$.

We must solve simultaneously

$$\boxed{\begin{aligned} kx - \omega t &= \text{const} \\ x &= \overset{c}{\cancel{U}}(k)\, t \end{aligned}}$$

Now for gravity waves $\omega = \sqrt{gk}$, [redacted]

$c\ \cancel{U} = \frac{1}{2}\sqrt{g/k}$

$$\left[k \tfrac{1}{2}\sqrt{g/k} - \sqrt{gk}\,\right] t = \text{const}$$

or

$$\sqrt{gk}\; t = \text{const} \quad \text{or}$$

$$\omega t = \text{const}$$

$$\boxed{t/T = \text{constant}, \quad T = \text{period}}$$

If one could follow a single crest it's period would increase linearly with time.

Furthermore $\qquad x = \frac{1}{2}\sqrt{\frac{g}{k}}\, t$

$$kx = \frac{1}{2}\sqrt{gk}\, t = \frac{1}{2}\omega t = \text{const.}$$

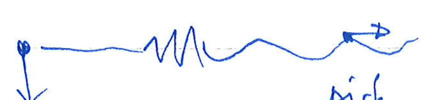
$$\boxed{\frac{x}{\lambda} = \text{const}}$$

this constant is twice as big as the first

When a wave has travelled twice as far it's twice as long.

Now suppose you pick out a single wave at some fixed time t_0. What is its subsequent motion?

$$kx = k_0 x_0 \quad , \quad \omega t = \omega_0 t_0$$



pick this and surf it

where k_0 and ω_0 are its initial wavenumber and frequency.

$$x = \frac{k_0 x_0}{k} = \frac{k_0 x_0}{\omega^2/g} = g\frac{k_0 x_0}{\omega^2}$$

$$= g\frac{k_0 x_0}{\omega_0^2 t_0^2/t^2} = \left(\frac{k_0 x_0}{\omega_0^2 t_0^2}\right) g t^2$$

$$t_0 = \frac{x_0}{v} = 2x_0\sqrt{\frac{k_0}{g}}$$ and $\omega_0^2 = g k_0$

$$\omega_0^2 t_0^2 = 4k_0^2 x_0^2$$

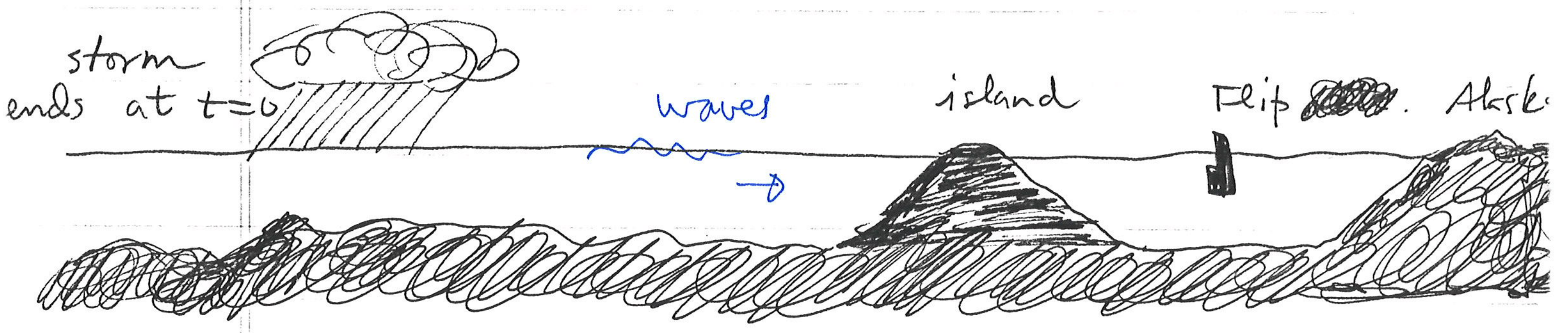
$$x = \left(\frac{g}{4k_0 x_0}\right)t^2$$ or finally in terms of wavelength

$$\boxed{x = \left(\frac{g}{8\pi}\frac{\lambda_0}{x_0}\right)t^2}$$

A given wavecrest moves with a <u>constant acceleration</u>!

In the pond experiment $\lambda_0 \sim \frac{1}{2}$ m [redacted] out on the edge and $x_0 \sim 10$ m so $\lambda_0/x_0 \sim 1/20$ and the [redacted] acceleration is about $\frac{1}{500}g$, not very much.

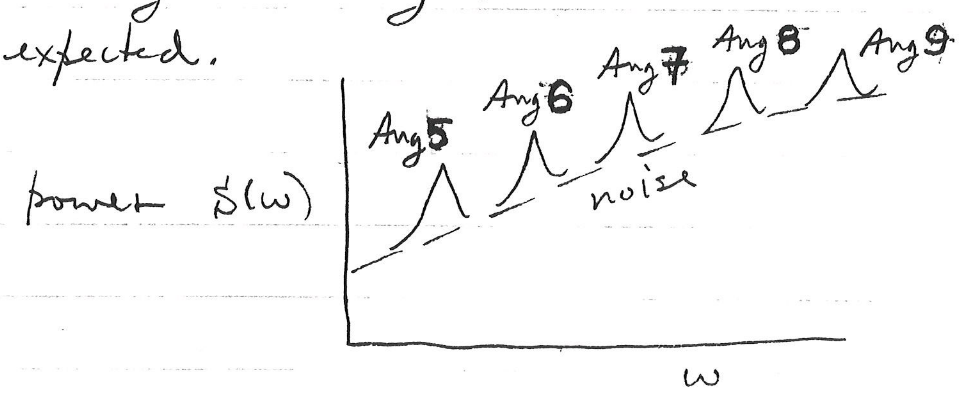
The pond experiment on a big scale: the 1966 Pacific ocean swell experiment of Munk *et al.*



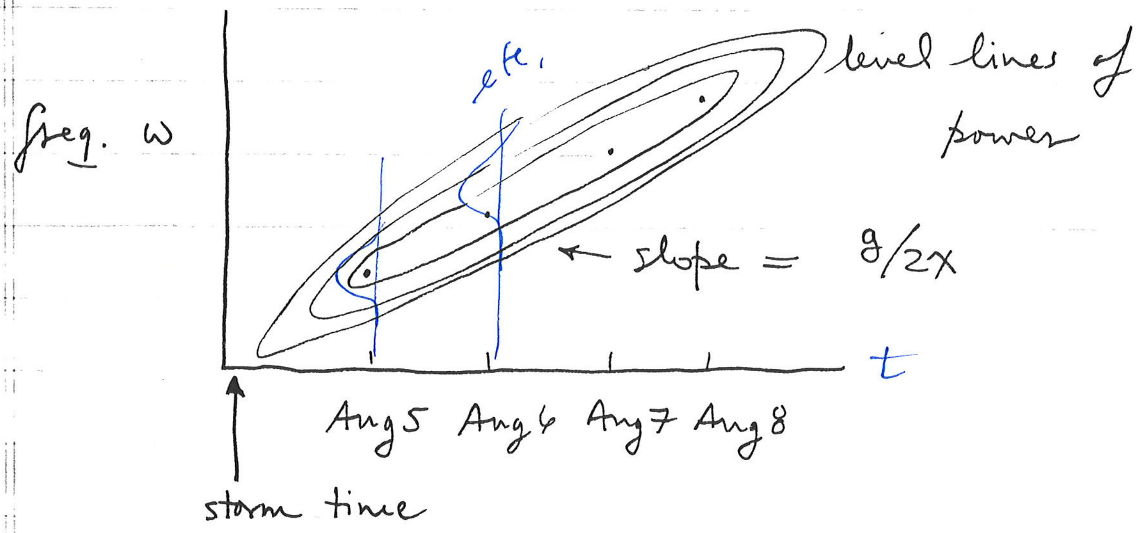
When will deep water waves of frequency ω be found a distance x away?
Answer: at $t = \frac{x}{v}$; $v = \frac{1}{2}\frac{g}{\omega}$, so

$$\boxed{\text{Waves of frequency } \omega \text{ arrive at station at } x \text{ at } t = \frac{2x}{g}\omega}$$

High frequency waves take longer. Frequency of peak in power spectrum observed to vary linearly with arrival time as expected.



Or presented differently, in three dimensions, on a frequency–time plot



Intercept on date or time axis at $\omega = 0$ is time of arrival of waves of zero frequency ($T = \infty$). This gives time of storm. ~~[illegible]~~ The slope of the "ridge lines" gives the storm distance, since $\omega = gt/2x$

fastest travelling deep water waves

$$\omega = \underbrace{(g/2x)}_{\text{slope}}t$$

Good agreement with meteorological location and time of storms was found. Farther storms have lower slopes. Wave from storms as far as 180° were detected.

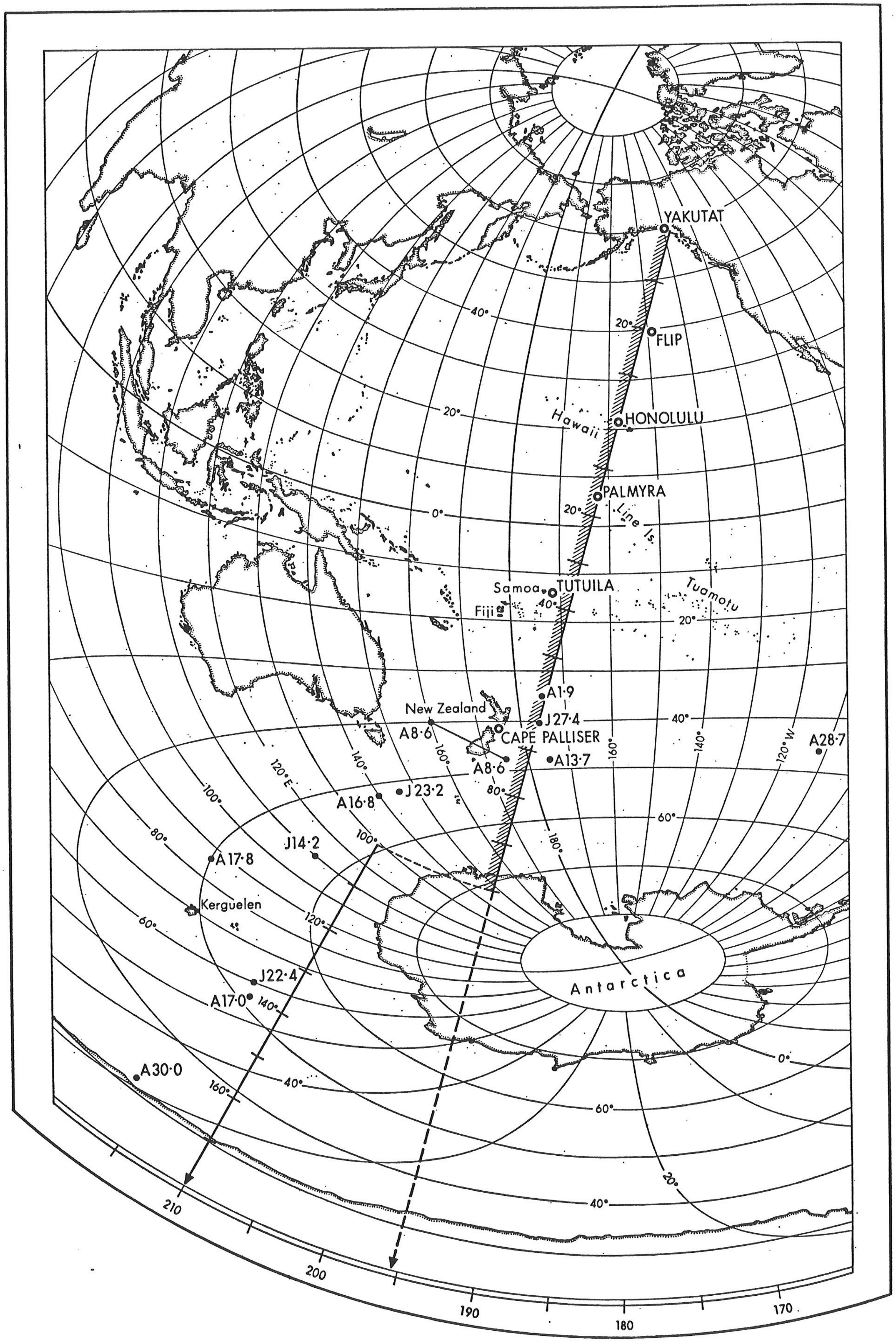
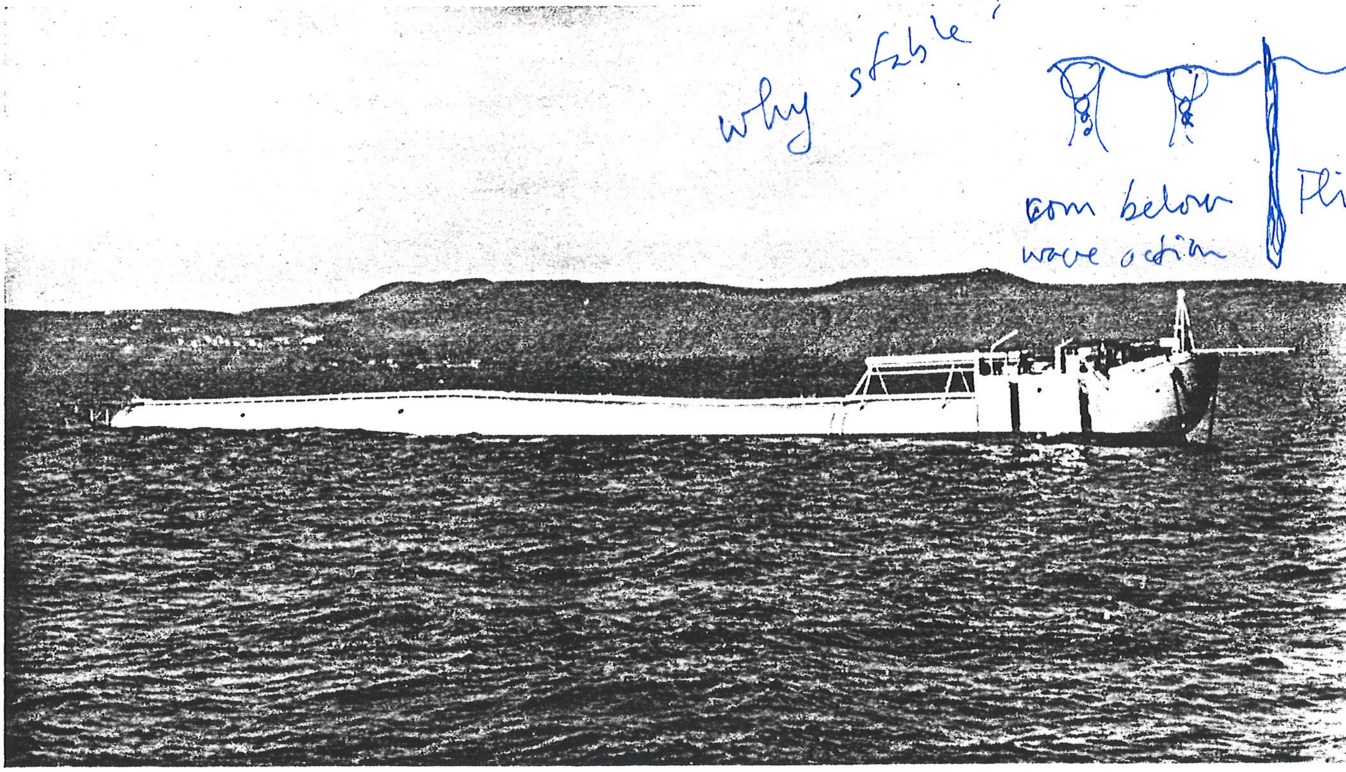


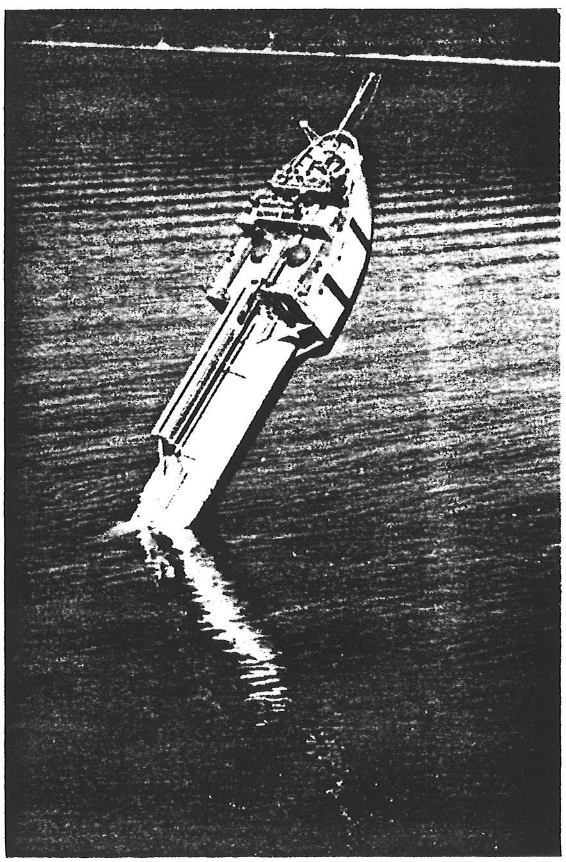
FIGURE 1. Great-circle chart based on Honolulu showing the location of the six wave instruments and of the principal storm sources. The 'reference great-circle' is in the direction 195·5° T from Honolulu; the Tasman window into the Indian Ocean bears 210° T. Distances from Honolulu are in degrees (1° = 60 nautical miles). Each storm is marked by a dot and its fractional date (J27·4 means 27 July, 9·6 h G.M.T.).



why stable?

com below
wave action

Flip



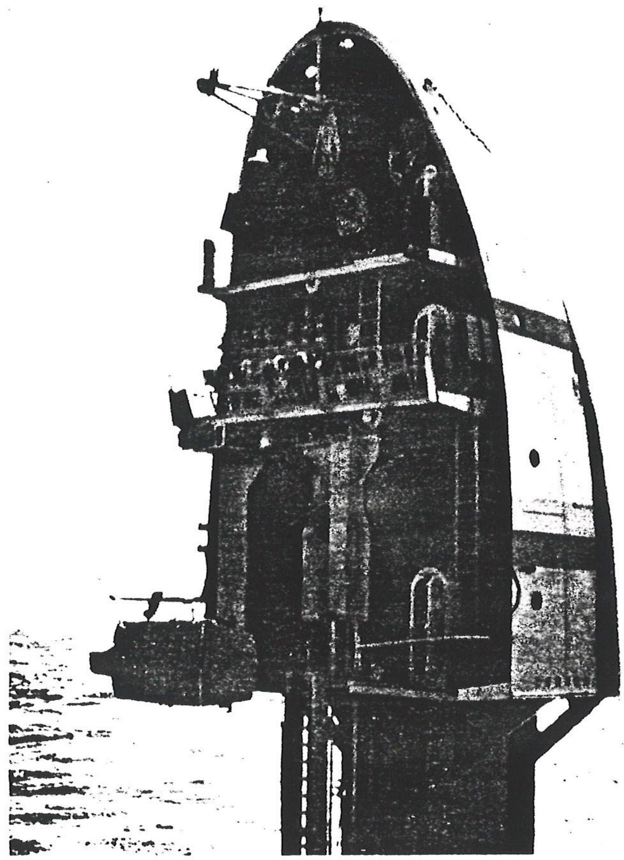
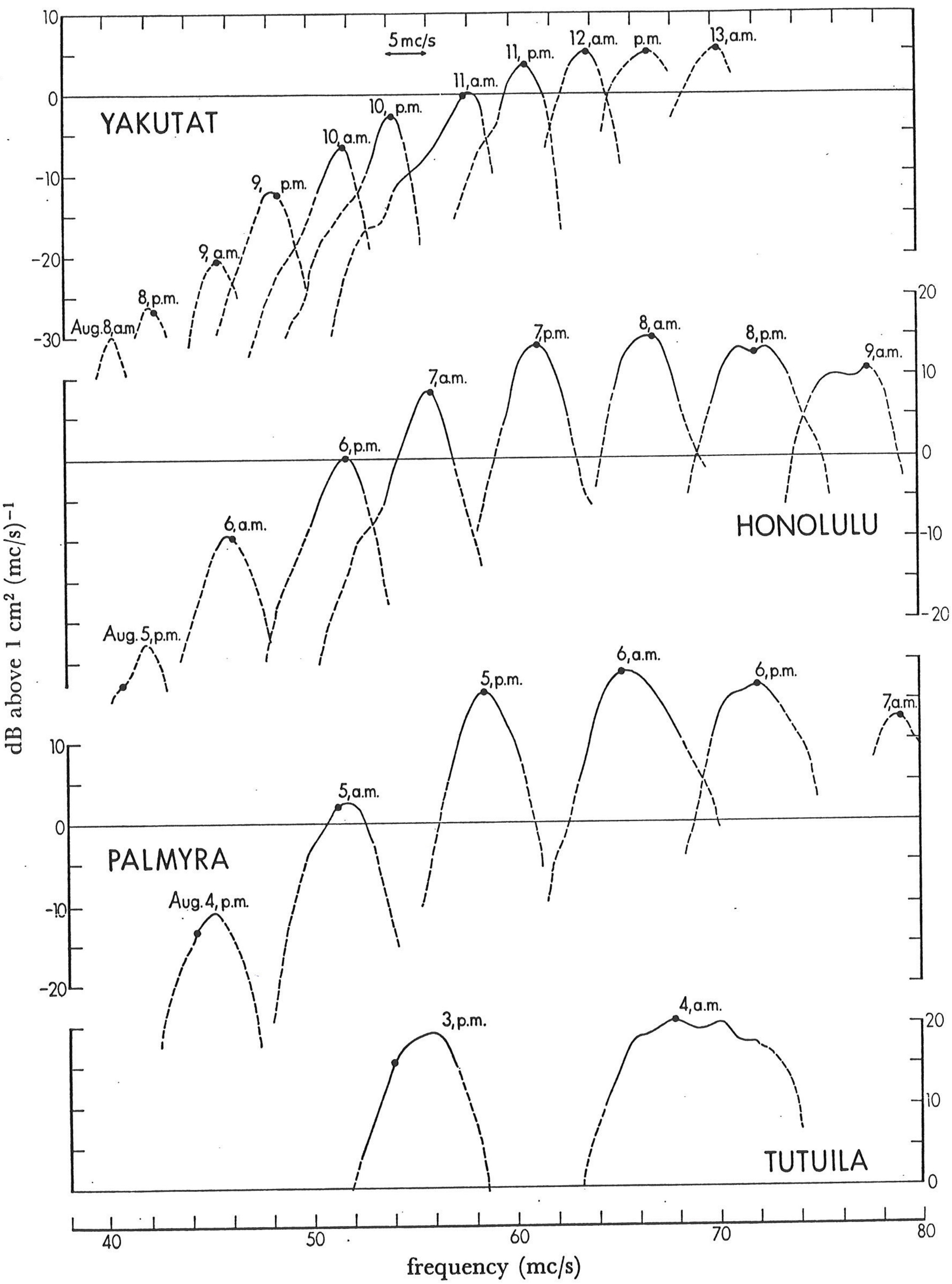


FIGURE 6. The vessel *Flip* in horizontal position, during flipping operation, and in vertical position (from Fisher & Spiess 1963).

(*Facing p.* 439)



IGURE 20. Successive spectra at the four stations for the event of 1·9 August. The dots correspond to the chosen ridge line for this particular event (see figure 16), and they are positioned relative to the bottom frequency scale. The spectra to either side are drawn on a compressed frequency scale to avoid overlap; the width of a 5 mc/s band is shown by the arrow.

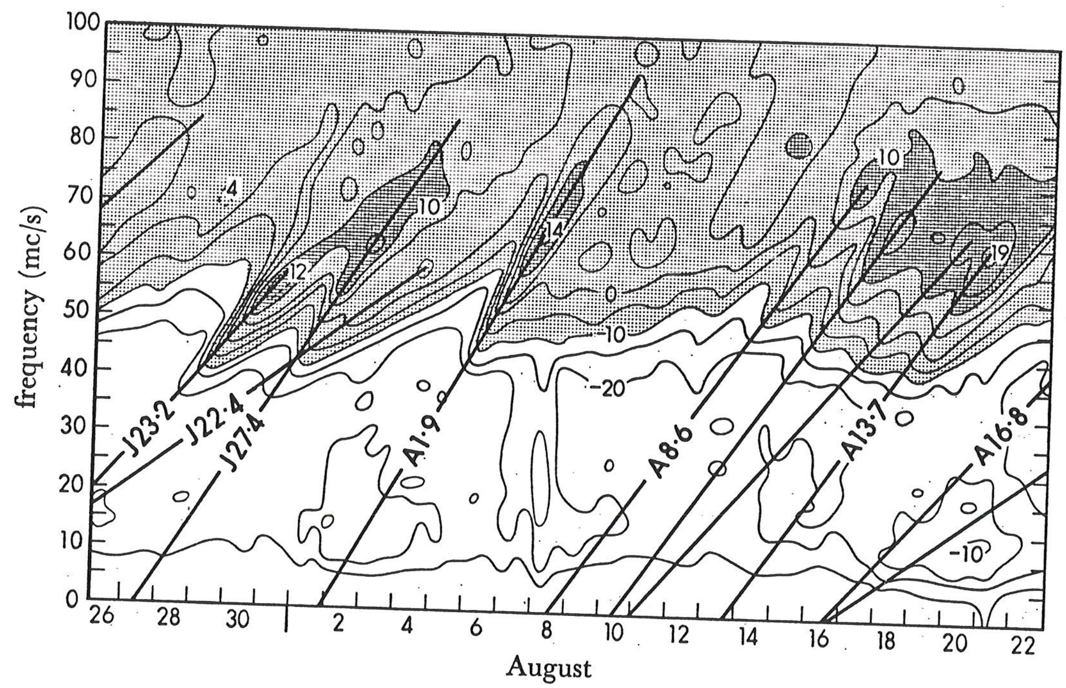


FIGURE 15. Contours of equal power density, $E(f, t)$, on a frequency–time plot for one month of Honolulu observations. Contours are drawn for -30, -25 dB, ..., $+25$, 30 dB relative to 1 $cm^2\ (mc/s)^{-1}$ (i.e. 10^{-3}, $0{\cdot}316 \times 10^{-3}$, ..., to $0{\cdot}316 \times 10^{3}$, $10^3\ cm^2\ (mc/s)^{-1}$). Additional contours are dashed and labelled. On the time axis the ticks designate midnight G.M.T. The ridges represent dispersive arrivals from the principal events, and are labelled according to the source time (J 27·4 means 27 July, 9·6 h G.M.T.).

J22.4 storm is farthest away.

Steady waves in moving fluids

Suppose the fluid is moving with a uniform velocity $\vec{U}$.

The dispersion relation relative to the fluid is $\omega = \omega(k)$, say.

Relative to a stationary observer

$$\omega(\vec{k}) = \omega(k) + \vec{U}\cdot\vec{k}$$

$$= c(k)\,k + \vec{U}\cdot\vec{k}$$

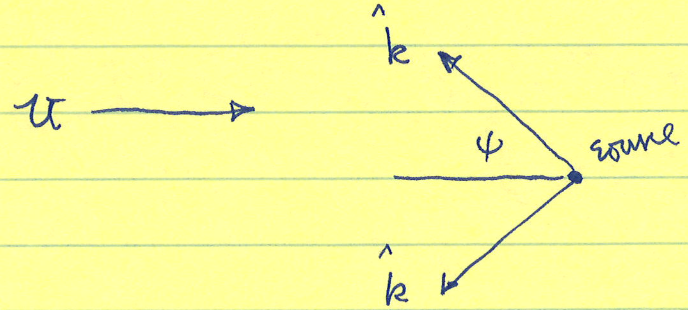
group velocity in stationary medium

The propagation is now anisotropic

$$\vec{C} = \nabla_{\vec{k}}\,\omega = c(k)\,\hat{k} + \vec{U}$$

<u>Steady</u> waves satisfy $\omega(\vec{k}) = 0$

$$c(k) = -\vec{U}\cdot\hat{k}$$



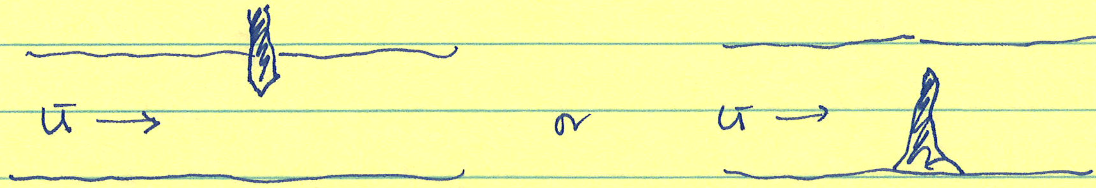
$$\boxed{c(k) = U\cos\psi}$$

condition to find a stationary phase front of wavenumber k

Consider several applications

1D waves generated by a disturbance

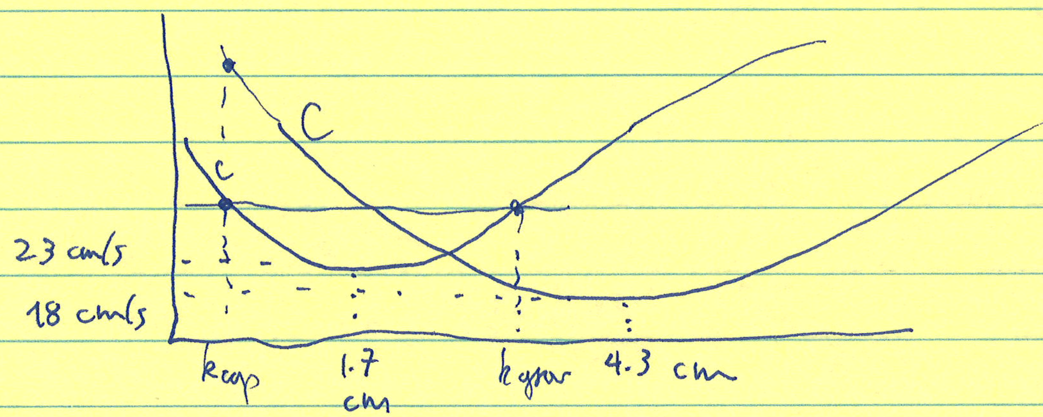
$\psi = 0$



$c(k) = U$

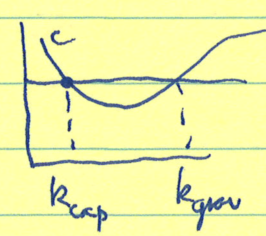
But where are waves of wavenumber k found?

Answer: the locus propogates at the group speed C



$c(k) = U$ has no solutions for $U < 23$ cm/s (slow stream)

For $U > 23$ cm/sec $\exists$ 2 solns



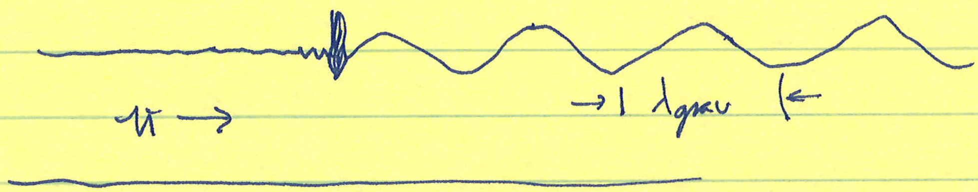
$k = k_{cap}$ or $k = k_{grav}$

k_{cap} waves have $C > c$ or $C > U$

$\Rightarrow$ upstream

k_{grav} waves have $C < c$ or $C < U$

$\Rightarrow$ downstream



front waves are small & easily dissipated

Read Robert Frost poem

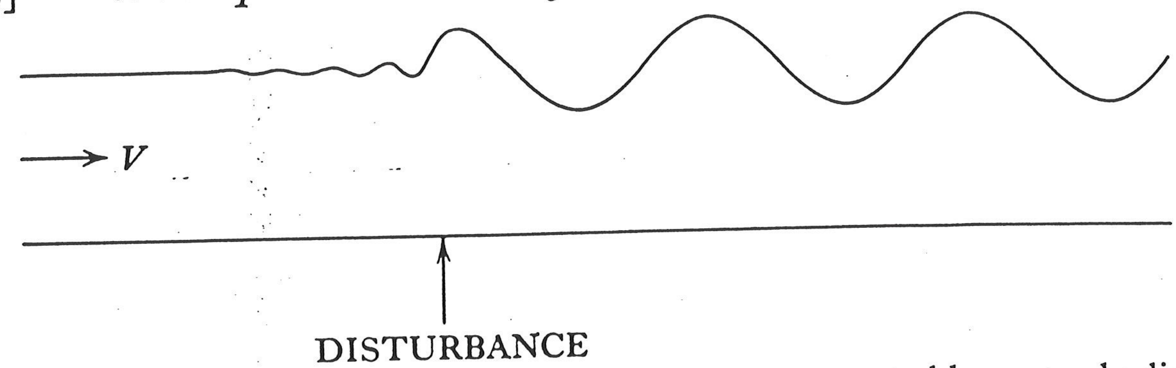


Figure 65. Schematic indication of stationary waves generated by a steady disturbance to a steady stream of velocity V. The scale for vertical displacements of the undisturbed water surface has been exaggerated so that the ripples upstream of the disturbance are clearly to be seen as well as the larger gravity waves downstream of it. The disturbance may take the form either of a step in the bed or of a cylindrical obstacle; the latter may be situated on the bed or in the midst of the water or on the surface.

Robert Frost described, in his poem 'West Running Brook', how

> The black stream, catching on a sunken rock,
> Flung backward on itself in one white wave,
> And the white water rode the black forever,
> Not gaining but not losing.

The steadiness of the flow caught his imagination:

> That wave's been standing off this jut of shore
> Ever since rivers, I was going to say,
> Were made in heaven.

He recognised how the upstream propagation of the crest could be exactly cancelling the downstream flow:

> Speaking of contraries, see how the brook
> In that white wave runs counter to itself.

He poetically explained the appeal of the phenomenon:

> It is this backward motion towards the source,
> Against the stream, that most we see ourselves in,
> The tribute of the current to the source.

Frost thought of the brook as time which sweeps individuals down to a sea of oblivion; from that standpoint, perhaps, his poetry is a sort of stationary wave pattern.

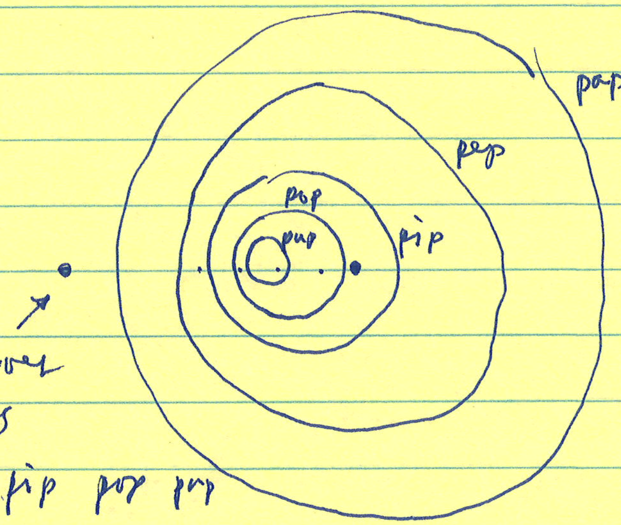
Application II — nondispersive waves (sound waves)

say $c(k) = c =$ constant

$$\omega(\vec{k}) = ck + \vec{u}\cdot\vec{k} \qquad \boxed{c = u\cos\psi}$$

Consider the sound waves emitted by a moving source — i.e. source moving through stationary air

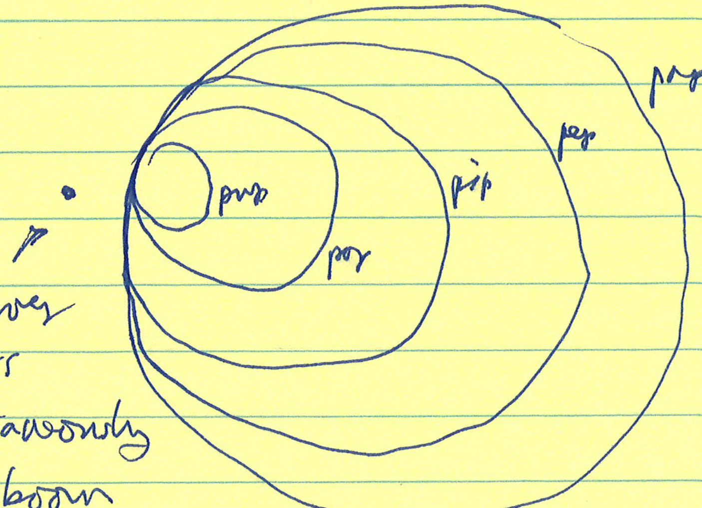
If $u < c$



observer hears

pap pep pip pop pup

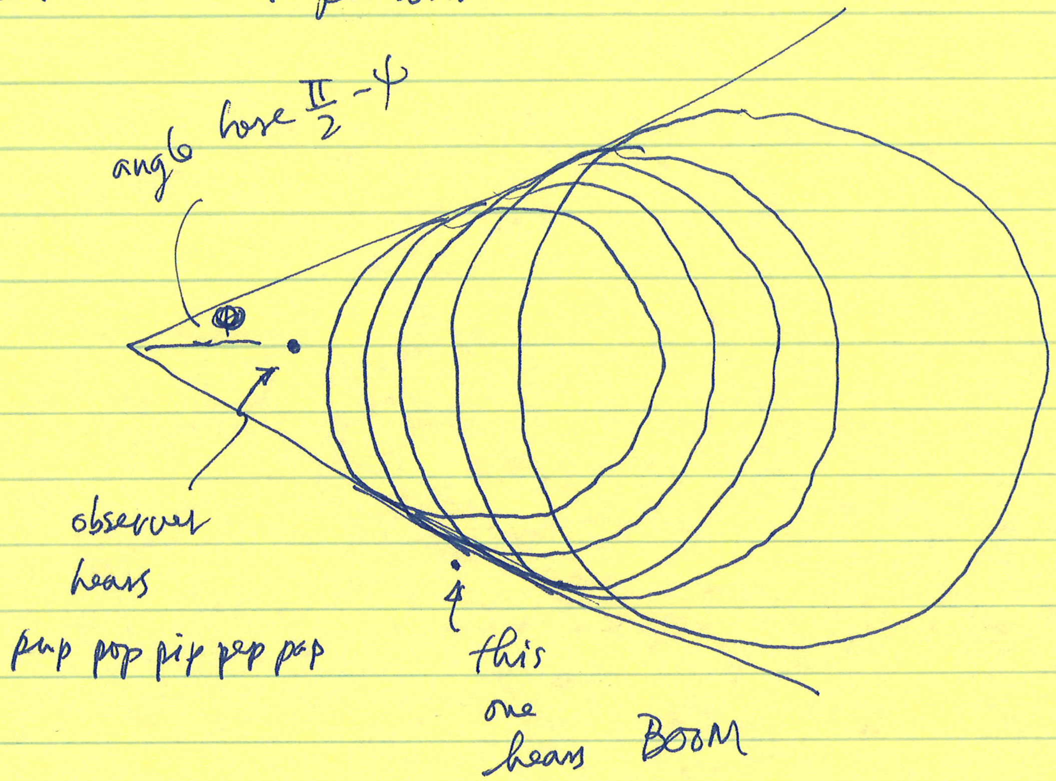
If $u = c$



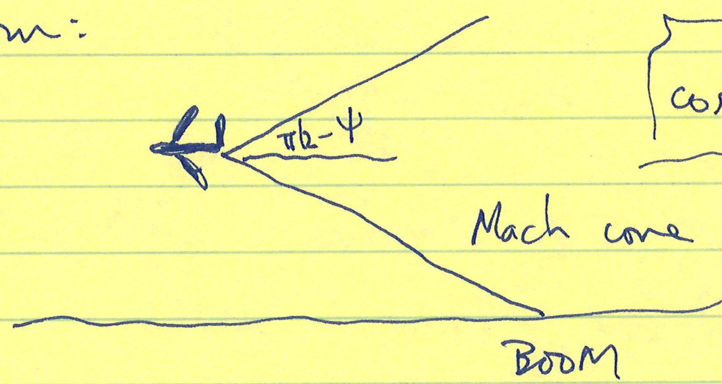
observer hears simultaneously

$\Rightarrow$ boom

$U > c$: supersonic

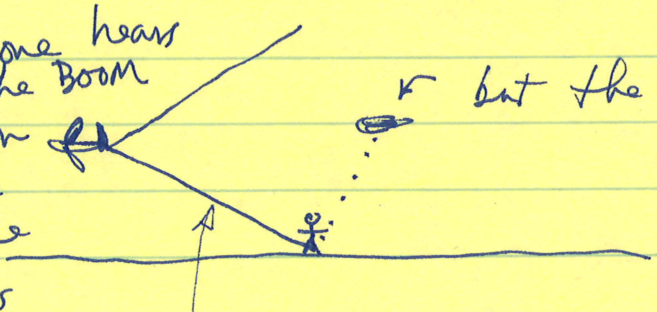


Sonic boom:



$$\boxed{\cos\psi = \frac{c}{U}}$$

one hears the BOOM when the plane is here



but the pop pep pip pop pub that comprise the BOOM were generated when the plane was here

waves along the Mach cone have $\omega = 0$ in the airplane frame

At that time the plane's velocity component toward the observer was $U \cos\psi$

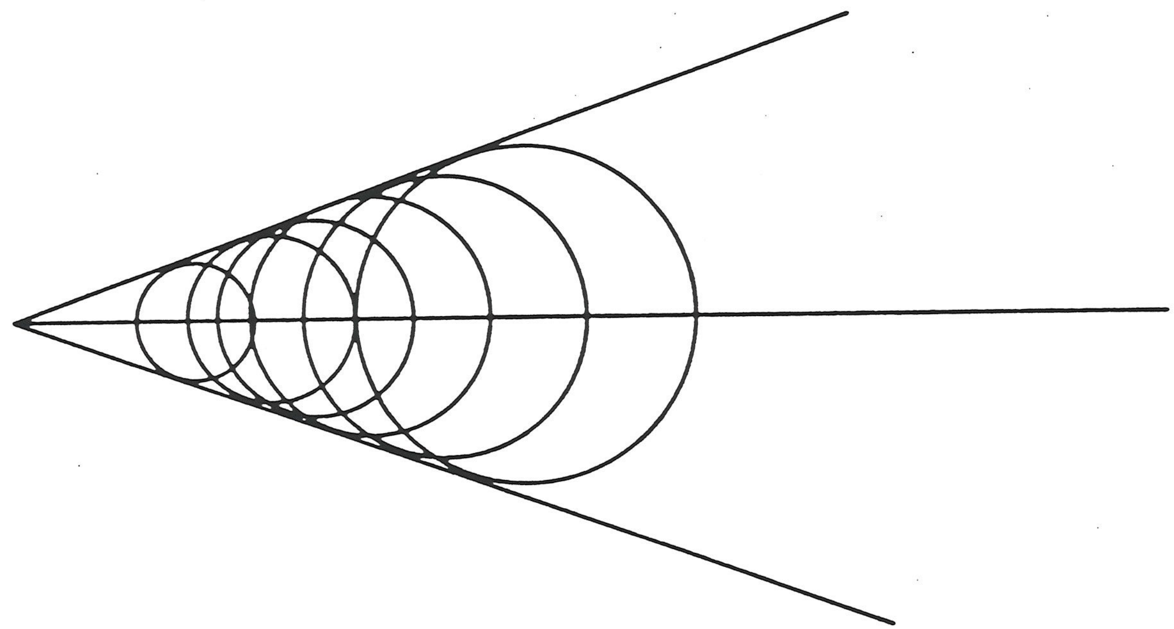
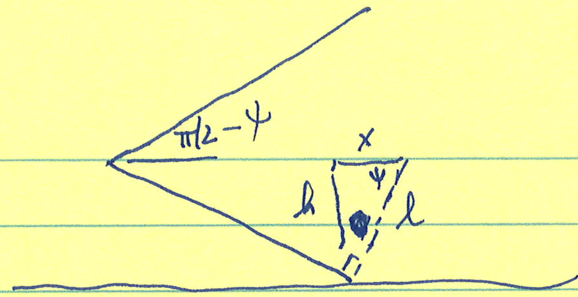


Fig. 12.3. Envelope of the disturbance emitted at successive times.



NB — ψ is the angle at which the waves leave the source

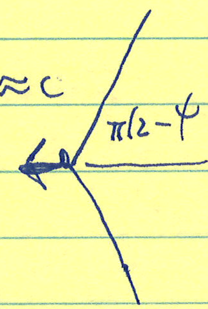
$$l^2 = h^2 + x^2$$

$$2l\dot{l} = 2x\dot{x} \quad (\dot{h}=0) \text{ level flight}$$

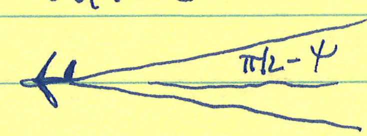
$$\dot{l} = \frac{x}{l}\dot{x} = u\cos\psi$$

$$\cos\psi = \frac{c}{u}$$

$u \approx c$



$u \gg c$

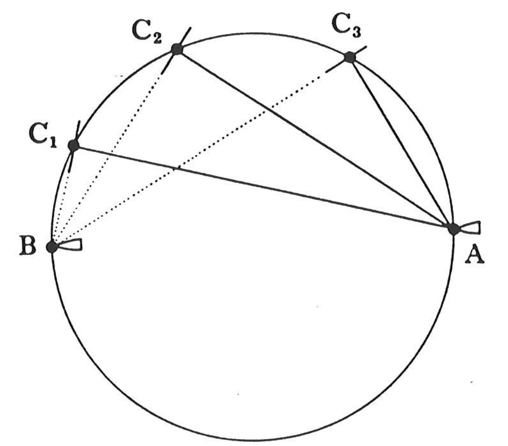


Ship waves — consider a ship moving in <u>deep</u> water:

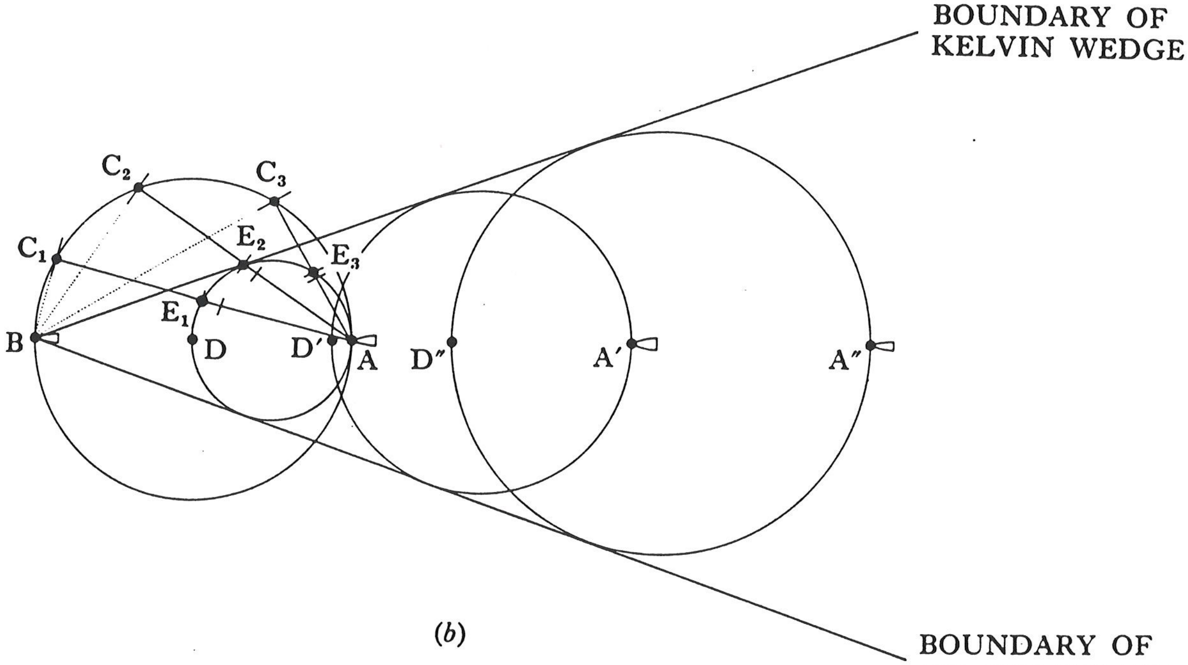
$$\boxed{\omega = \sqrt{gk} \qquad c = \sqrt{\frac{g}{k}} \qquad C = \frac{1}{2}c}$$

Refer to Lighthill Fig. 68

Shows locus of all waves generated when ship was at A.



(*a*)



(*b*)

Figure 68. Waves generated in deep water by a ship B. Case (*a*): positions C_1, C_2, C_3 of any waves generated t_g seconds ago (when the ship was at A) if their energy had travelled a distance ct_g. Case (*b*): the real positions E_1, E_2, E_3 of the same waves, taking into account that their energy has only travelled a distance $\frac{1}{2}ct_g$. At each, the dependence of wavelength on direction of emission, as inferred from equation (183), is shown. The circle with diameter AD is the locus of all such waves. Other such circles, with diameters A′D′ and A″D″, are where waves generated when the ship was at A′ and A″ are now to be found. All such circles lie within the Kelvin wedge of semi-angle (184).

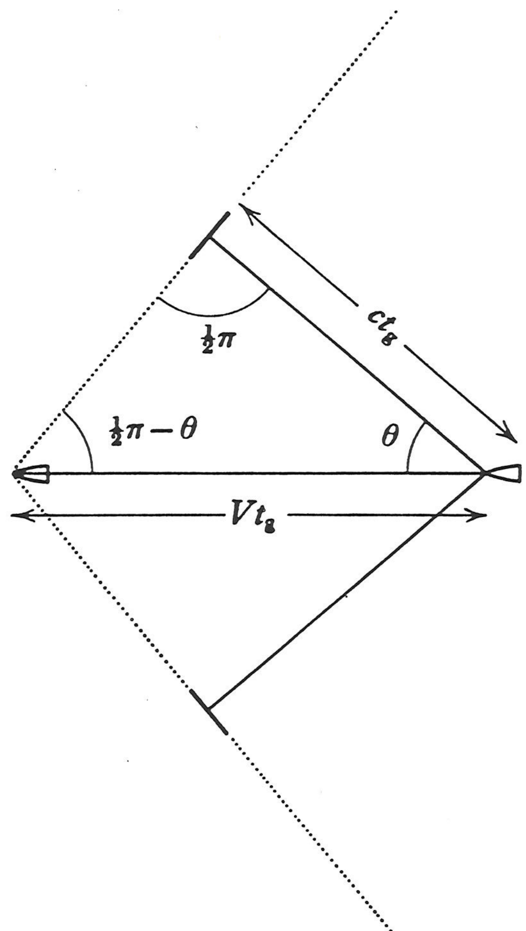
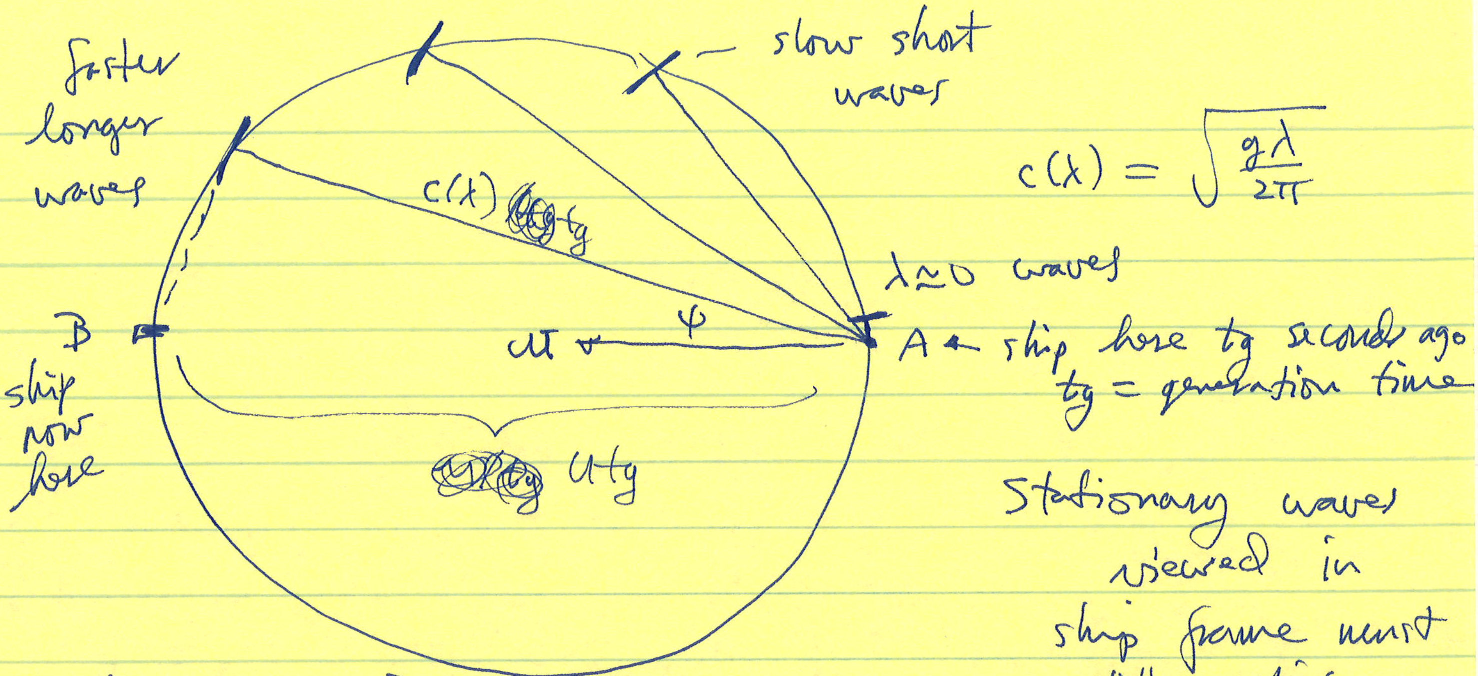


Figure 67. Waves generated in a ripple tank with constant wave speed c by a small model ship moving at velocity V. Waves are emitted at a fixed angle θ given by equation (183). Thick line: wave generated t_g seconds ago when the model was a distance Vt_g further back. Dotted line: locus of all such waves. (Note that the triangle with angles marked *must* be right-angled since (183) makes $\cos\theta$ the ratio of base to hypotenuse.)



$$c(\lambda) = \sqrt{\frac{g\lambda}{2\pi}}$$

Stationary waves viewed in ship frame must still satisfy

why a circle? see above

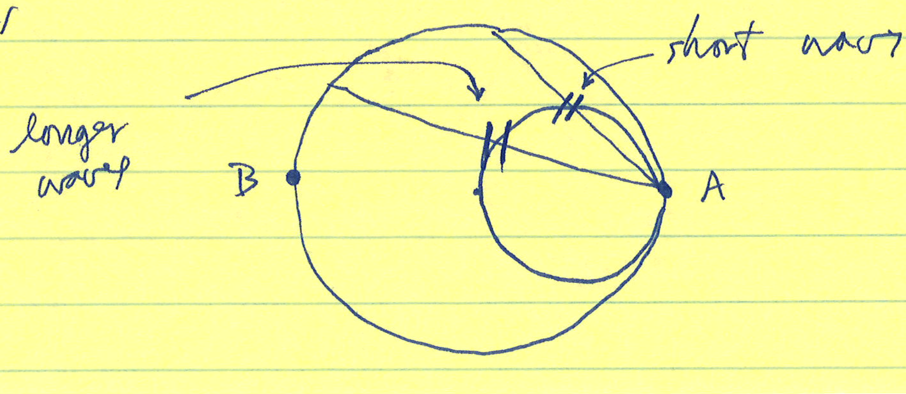
$$c(\lambda) = U \cos\psi$$

longest possible waves are at $\psi = 0$

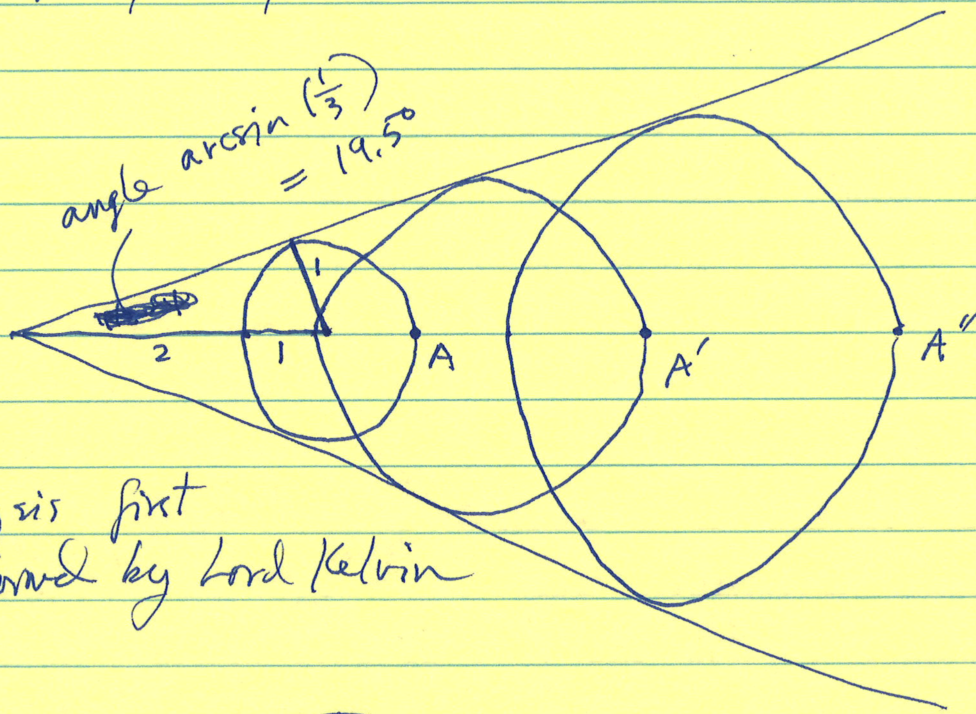
$$\lambda_{max} = \frac{2\pi U^2}{g}$$

But again we ask — where are waves of wavenumber k found? Locus propagates at $C = \frac{1}{2}c$.

Locus of all wave energy generated tg seconds ago when ship was at A is thus



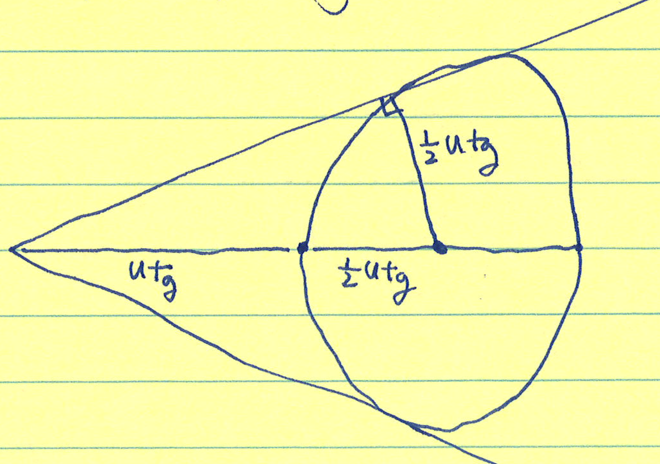
Next shows locus of wave energy at other past times when ship at A , A′, A″



Analysis first performed by Lord Kelvin

$$\boxed{\text{All wave energy within a cone at angle } \arcsin(1/3) = 19.5^\circ}$$

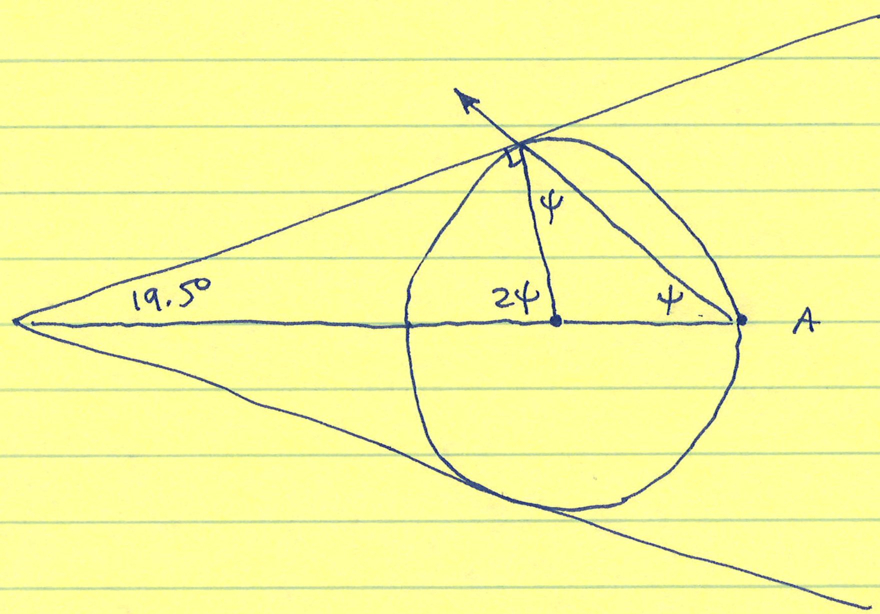
~~Waves on the wedge boundary are propagating at an angle~~



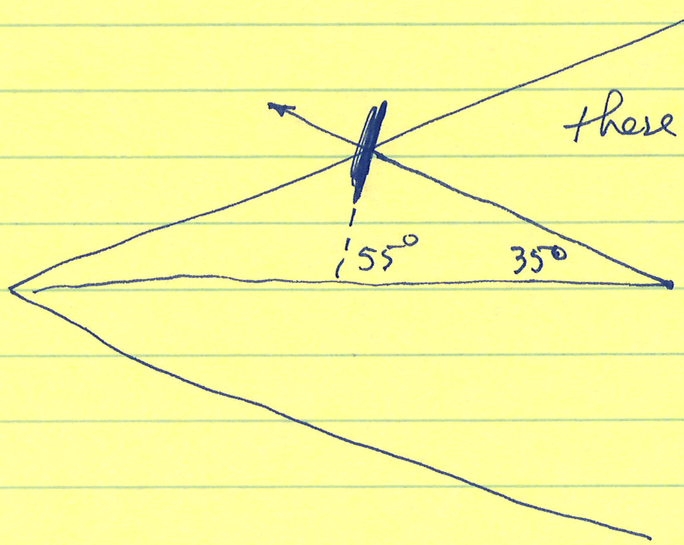
Waves on the edge of the wedge are propagating at an angle

$$\psi = \frac{1}{2}\left(90^\circ - \arcsin\frac{1}{3}\right) = 35^\circ$$ relative to the ship track

The crests are aligned at an angle $90^\circ - \psi = 55^\circ$



$2\psi = 90^\circ - 19.5^\circ$



these are the most visible waves

inside has muddled appearance due to superposition of short (E_3) and long (E_1) waves

Speed of most visible waves is

$$U = c \cos 35^\circ$$

$$\boxed{c = 0.816\, U}$$

Wavelength is $\sqrt{\dfrac{g\lambda}{2\pi}} = .816\, U$

$$\boxed{\lambda_{visible} \cong \frac{2}{3}\,\frac{2\pi U^2}{g} \equiv \frac{2}{3}\,\lambda_{max}}$$

Both Whitham & Lighthill actually solve for individual wavecrest shapes

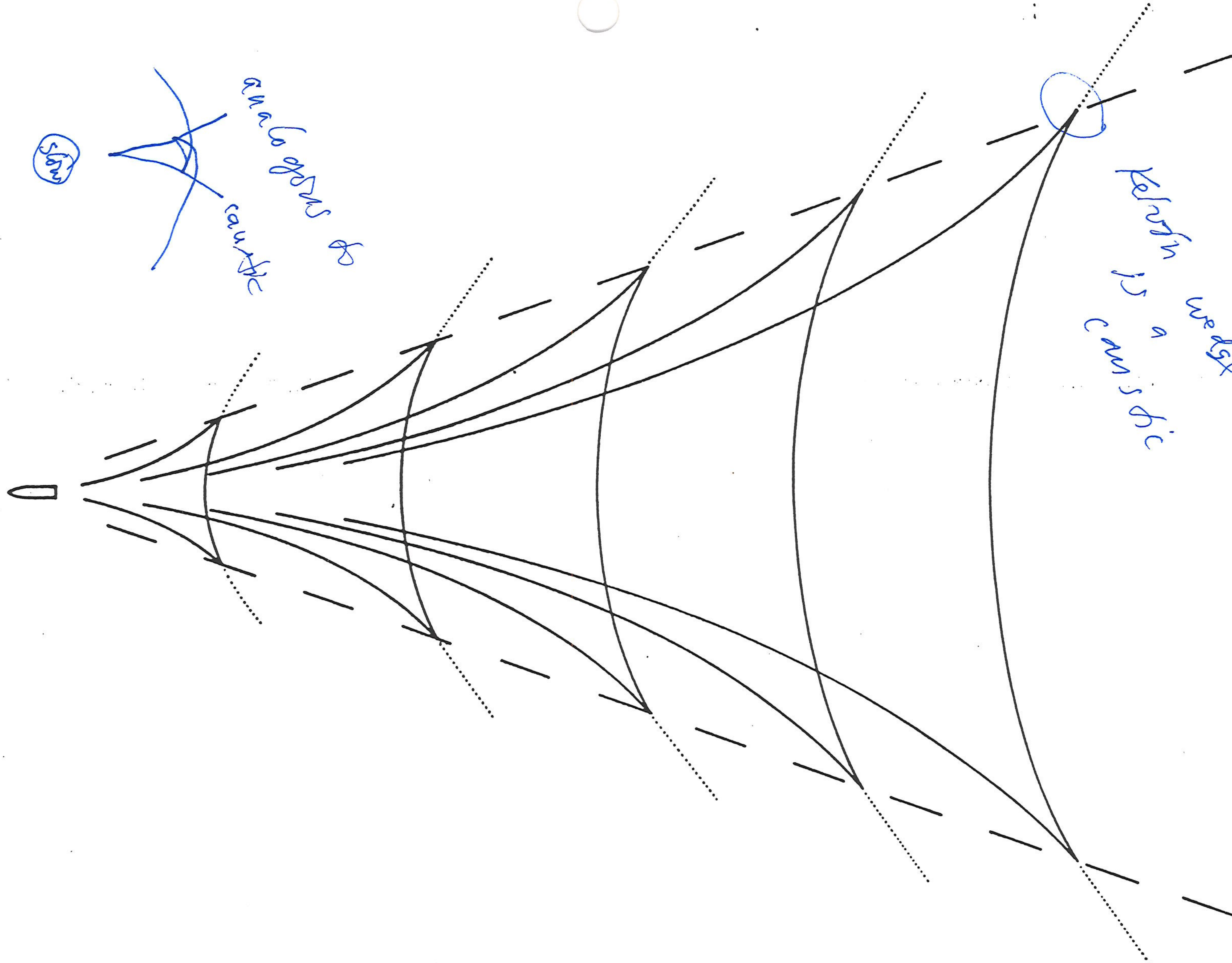
See Lighthill Fig. 70

The waves comprising an individual crest were generated ~~when the wave~~ at different instants in the past when the ship was at different positions.

analogous to caustic

slow

source



Kelvin wedge is a caustic

Figure 70. Plain lines: Kelvin ship-wave pattern. Broken lines: boundary of Kelvin wedge. Dotted lines: extension of waves beyond the Kelvin wedge indicated by the theory of sections 4.11 and 4.12.

Figure 71. An observed ship-wave pattern. [*Courtesy of Aerofilms Ltd.*]

55°

19.5°

35°

19.5°